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Research Article

Strategic Analysis of Retrial Queues with Setup Times, Breakdown and Repairs

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This paper considers a repairable M/M/1 retrial queueing model with setup times. Once the system is empty, the server will be closed down to reduce operating costs. And the system will be activated only when a new customer arrives. The customer who activates the server will enter the retrial orbit waiting to reapply for service. The server may break down during the busy period. First, the steady-state probability of the model is obtained by using the probability generating function method. And we derive performance measures of the system such as the queue length of the orbit, the numerical examples are given to show the sensitivity of the performance measures. Second, the cost function is established to find the minimum cost of the system, and we study the effects of some parameters on the cost by numerical examples. Finally, from the perspective of the customer and social planner, we construct the individual utility function and the social welfare function in the almost and fully unobservable cases, and then the optimal strategy of the customers is analyzed.

1. Introduction

When customers arrive and find the server busy, they will join the retrial orbit and wait to reapply for service, which is the feature of the retrial queueing systems. These queueing models are often used in computer systems and telecommunication networks. General models, methods, results, examples, and applications of retrial queues can be found in Artalejo and Gómez-Corral [1], Tian et al. [2]. In recent years, Phung-Duc [3] used retrial queues to model cloud computing systems and gave the steady-state probabilities of the systems.

Based on consideration of the reality, the server cannot work immediately after turning on and needs a period for buffering, which is called the setup time. Levy and Kleinrock [4] first introduced setup times to the M/M/1 queueing system. The server will be closed down when there are no customers in the system. When a new customer arrives, the server will be activated and cannot serve the customer during the setup period, so the customer has to wait in line. Due to the importance and relevance of introducing setup times, many scholars have focused on this area, see Bischof [5], Gandhi et al. [6]. Recently, Phung-Duc [7] combined setup times with retrial queues by assuming that both the service time and setup times are distributed with a general distribution function; they got the stationary distribution of the queue length. Chang and Wang [8] studied an unreliable M/M/1/1 retrial queue with setup times. Two models were considered according to whether the server can be perfectly repaired or not. In both models, they got the queue length of the system. Burnetas and Economou [9] first analyzed the queueing system with setup times from an economic perspective; they obtained equilibrium strategies of the customers in the observable and unobservable cases. Sun et al. [10] studied the Markovian queueing systems with three types of setup/closedown policies in the unobservable case. They derived the equilibrium and socially optimal strategies of customers as well as the maximal social welfare. Then, they made pricing controls to encourage customers to take the optimal strategy and maximize the profits of the server. Zhang and Wang [11] discussed an M/G/1 retrial queue with reserved idle time and setup times. The optimal pricing strategy was considered from the perspective of the social planner and the server, respectively, where both the queue length and the state of the server are unknown. Recent results on the queueing systems with setup times can be found in Yutaka et al. [12]. Wang et al. [13] studied an M/M/1 retrial queue with setup times and considered the social optimization problem from the perspective of service providers and social managers. Zhou et al. [14] analyzed the customers' strategy behavior and social optimization problems in a retrial queue with setup time and N-policy.

In real life, the server may not work normally due to some reasons during the process of system operation. Towsley and Tripathi [15] analyzed queueing system with breakdown and repairs. Li et al. [16] considered the equilibrium strategies of customers in queueing system with partial breakdowns and repairs in the observable and unobservable cases. Falin [17] studied the M/G/1 retrial queue with an unreliable server assuming that the repair time follows a general distribution, and derived performance measures for this queueing system. Kalita and Choudhury [18] considered a repairable M/G/1 queueing model with setup times and N-policy. The steady-state queue length of the model was obtained by using the supplementary variables method. Wang and Zhang [19] studied the Markovian queue with breakdowns and delayed repairs. They considered the equilibrium balking strategies of the customers in the fully observable and partially observable cases. Zhang and Wang [20] studied the unreliable retrial queue which the server may break down at different rates in the busy and idle states. They compared and analyzed the benefits of the service provider and social welfare in the observable and unobservable cases. Zhang et al. [21] studied the retrial queue with breakdowns and repairs in which arriving customers that find the server broken will not enter the system. They considered the equilibrium strategy and social benefits of customers in the partially observable and fully observable cases.

Based on the abovementioned literature, we have some new ideas about queueing models. Inspired by literature [13, 21], we consider both setup times and an unreliable server in the retrial queueing system. Different from literature [21], in the proposed paper, even if a server failure is found, the arriving customer may still enter the system. Therefore, we need to consider the customer's joining probability under the condition of server failure of almost unobservable and fully unobservable cases. According to this situation, we consider a repairable retrial queueing model with setup times, the performance measures of the system are obtained by using the generating function method. Next, the effect of the system parameters on the cost function is analyzed by constructing the cost function. Finally, we consider the customers' equilibrium strategy and the social optimization problem. The repairable retrial queueing model with setup times can be applied to wireless communication networks. The data packets in the network can be regarded as customers in the queueing system, and the wireless network node can be considered as a server, which is responsible for the transmission of the arriving data streams. When a packet arrives and finds that the node cannot deliver for it, it enters a retrial orbit to wait to reapply for delivery again. When a packet is during the process of transmission, the channel may be damaged and unusable due to some factors, such as signal interruption and line damage. We assume that the breakdown only occurs when the node is working. In information and communication systems, energy saving is a very important issue because the devices consume excessive energy when they are turned on. If there are no packets to be transmitted in the system, the network node will close down to reduce energy consumption. In the off state, only the arrival of new packets can activate the network node to turn on. According to the signaling protocol in ATM networks, the queueing system on the transformed virtual channel (SVC) often has a setup period, which is equivalent to the time used to establish a new SVC by relying on the signaling protocol. Therefore, the queueing model considered in this paper is closer to the complex wireless networks in real life, and it is of great importance to study this queueing model.

The remainder of the paper is organized as follows. We first describe the queueing system in Section 2. In Section 3, the steady-state performance analysis of the system was obtained. In Section 4, numerical experiments are explored to illustrate the effects of system parameters on performance measures and cost function. In Section 5, we consider individual equilibrium strategies and social benefits in the almost and fully unobservable cases. Finally, conclusions are given in Section 6.

2. Model Descriptions

The customers arrive according to the Poisson process with rate λ. Customers entering the system receive the service immediately when they find the server idle; otherwise, they enter the retrial orbit and wait for retrying. The retrial time is exponentially distributed with rate θ. The service time of the server is exponentially distributed with rate μ. When there are no customers in the system, the server will be closed and will not be restarted until a new customer arrives. The closed server needs a setup time to turn on, where the time is exponentially distributed with rate α. The customer who activates the server will immediately enter the retrial orbit and wait to apply for the service. The server is not completely reliable and may break down during normal operation. When the server breaks down, the server is repaired immediately. The process of server breakdown is a Poisson process with rate ξ, and the repair time is exponentially distributed with rate η. Finally, the interarrival times, setup times, service times, breakdown times, repair times, and retrial times are assumed to be mutually independent.

Let $N(t)$ be the number of customers in the retrial orbit at the time t and $I(t)$ represents the state of the server at the time t as defined in the following:

$$I(t) = \begin{cases} 0, & \text{if the server is idle,} \\ 1, & \text{if the server is busy,} \\ 2, & \text{if the server is in setup times,} \\ 3, & \text{if the server is under repair.} \end{cases} \tag{1}$$

It is easy to know that $X(t) = \{(N(t), I(t)); t \geq 0\}$ forms a Markov chain with the state space.

$$S = \{(n,i); n \geq 0, i = 0, 1, 2, 3\}. \tag{2}$$

In this paper, we assume that the customers must join the system during the idle period. While in the other state $i\,(i = 1, 2, 3)$, the customers join the system with probability $q_i\,(i = 1, 2, 3)$. Then the effective arrival rate of customers is $\lambda_i = \lambda q_i\,(i = 1, 2, 3)$ when the system is in state $i\,(i = 1, 2, 3)$, which indicates that $\lambda_i \leq \lambda$. The transition rate diagram of the model is shown in Figure 1.

3. Steady-State Performance Analysis

3.1. Steady-State Probability. Assuming that the system is stable, let p_{ni} be the steady-state probability that the system is in state (n,i).

$$p_{ni} = \lim_{t \longrightarrow \infty} P\{N(t) = n, I(t) = i\}, \ (n,i) \in S. \tag{3}$$

The following balance equations are obtained

$$\lambda p_{00} = \mu p_{01}, \tag{4}$$

$$(\lambda + \theta) p_{n0} = \alpha p_{n2} + \mu p_{n1}, n \geq 1, \tag{5}$$

$$(\lambda_1 + \mu + \xi) p_{01} = \theta p_{10} + \eta p_{03}, \tag{6}$$

$$(\lambda_1 + \mu + \xi) p_{n1} = \lambda p_{n0} + \lambda_1 p_{n-1,1} + \theta p_{n+1,0} + \eta p_{n3}, n \geq 1, \tag{7}$$

$$(\lambda_2 + \alpha) p_{12} = \lambda p_{00}, \tag{8}$$

$$(\lambda_2 + \alpha) p_{n2} = \lambda_2 p_{n-1,2}, n \geq 2, \tag{9}$$

$$(\lambda_3 + \eta) p_{03} = \xi p_{01}, \tag{10}$$

$$(\lambda_3 + \eta) p_{n3} = \xi p_{n1} + \lambda_3 p_{n-1,3}, n \geq 1. \tag{11}$$

The generating function method is used to solve the balance equation. Define the partial generating function as follows:

$$\begin{aligned} P_0(z) &= \sum_{n=0}^{\infty} p_{n0} z^n, \\ P_1(z) &= \sum_{n=0}^{\infty} p_{n1} z^n, \\ P_2(z) &= \sum_{n=1}^{\infty} p_{n2} z^n, \\ P_3(z) &= \sum_{n=0}^{\infty} p_{n3} z^n. \end{aligned} \tag{12}$$

Theorem 1. *In the repairable M/M/1 retrial queueing system with setup times, the probabilities that the server is in different states are as follows:*

The probability that the server is idle

$$p_0 = P_0(1) = \frac{(\lambda + \theta)[-\alpha(\lambda_1 \eta + \lambda_3 \xi) + \lambda_2 \mu \eta + \alpha \mu \eta] - \lambda_2 \mu \theta \eta}{\alpha[-(\lambda + \theta)(\lambda_1 \eta + \lambda_3 \xi) + \mu \theta \eta]} p_{00}. \tag{13}$$

The probability that the server is busy

$$p_1 = P_1(1) = \frac{\lambda \eta (\lambda + \theta)(\lambda_2 + \alpha)}{\alpha[-(\lambda + \theta)(\lambda_1 \eta + \lambda_3 \xi) + \mu \theta \eta]} p_{00}. \tag{14}$$

The probability that the server is in setup times

$$p_2 = P_2(1) = \frac{\lambda}{\alpha} p_{00}. \tag{15}$$

The probability that the server is under repair

$$p_3 = P_3(1) = \frac{\lambda \xi (\lambda + \theta)(\lambda_2 + \alpha)}{\alpha[-(\lambda + \theta)(\lambda_1 \eta + \lambda_3 \xi) + \mu \theta \eta]} p_{00}, \tag{16}$$

where

$$p_{00} = \frac{\alpha[-(\lambda + \theta)(\lambda_1 \eta + \lambda_3 \xi) + \mu \theta \eta]}{(\lambda + \theta)[-(\lambda_1 \eta + \lambda_3 \xi)(\lambda + \alpha) + \lambda(\lambda_2 + \alpha)(\eta + \xi) + \alpha \mu \eta] + \lambda \mu \eta (\lambda_2 + \theta)}. \tag{17}$$

Proof. Multiplying (4) and (5) by z^n respectively, and summing up over all n, we get the following equation:

$$(\lambda + \theta) P_0(z) = \alpha P_2(z) + \mu P_1(z) + \theta p_{00}. \tag{18}$$

Taking (6)–(11) in the same way as above, we obtain

$$[\lambda_1(1 - z) + \mu + \xi] P_1(z) = \left(\lambda + \frac{\theta}{z}\right)[P_0(z) - p_{00}] + \eta P_3(z), \tag{19}$$

$$[\lambda_2(1 - z) + \alpha] P_2(z) = \lambda p_{00} z, \tag{20}$$

$$[\lambda_3(1 - z) + \alpha] P_3(z) = \xi P_1(z). \tag{21}$$

Through a series of algebraic operations, we use p_{00} to express $P_i(z)\,(i = 0, 1, 2, 3)$.

$$P_0(z) = \frac{\alpha P_2(z)}{(\lambda + \theta) - [\mu F_2(z)/F_1(z)]} + \frac{\theta - [\mu F_2(z)/F_1(z)]}{(\lambda + \theta) - [\mu F_2(z)/F_1(z)]} p_{00}, \tag{22}$$

$$P_1(z) = \frac{F_2(z)[P_0(z) - p_{00}]}{F_1(z)}, \tag{23}$$

$$P_2(z) = \frac{\lambda p_{00} z}{[\lambda_2(1 - z) + \alpha]}, \tag{24}$$

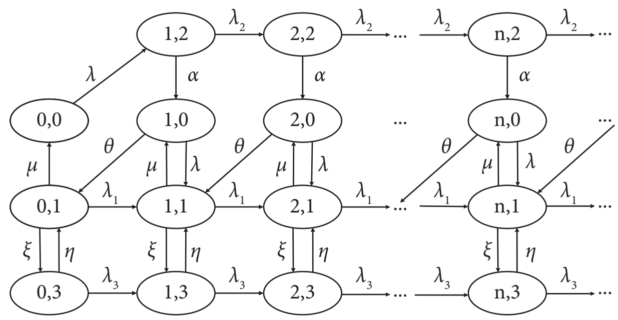

Figure 1: The transition rate diagram of the model.

$$P_3(z) = \frac{\xi P_1(z)}{[\lambda_3(1-z)+\alpha]}, \tag{25}$$

where

$$F_1(z) = \lambda_1(1-z) + \mu + \xi - \frac{\xi\eta}{\lambda_3(1-z)+\eta}, \tag{26}$$

$$F_2(z) = \lambda + \frac{\theta}{z}. \tag{27}$$

Taking (22) into (25) yields

$$P_0(z) = \frac{\lambda\alpha z F_1(z) + \theta[\lambda_2(1-z)+\alpha]F_1(z) - \mu F_2(z)[\lambda_2(1-z)+\alpha]}{[\lambda_2(1-z)+\alpha][(\lambda+\theta)F_1(z) - \mu F_2(z)]}. \tag{28}$$

Taking in $z = 1$, we can obtain $P_2(1)$. Notice that when z tends to 1, $P_0(z)$, $P_1(z)$, and $P_3(z)$ are all indeterminate forms whose numerators and denominators converge to 0, so we use L'Hospital rule to solve them and we have (13)–(16). Taking them into the normalization condition, $P_0(1) + P_1(1) + P_2(1) + P_3(1) = 1$, we derive p_{00} as given by (17).

From (17), we can obtain that the system is stable if $((\lambda+\theta)(\lambda_1\eta + \lambda_3\xi))/(\mu\theta\eta) < 1$. □

3.2. Performance Measures. Based on the above analysis, we can get some performance measures of the system.

(1) The mean queue length of the orbit in a busy period is given by

$$P_1'(1) = \frac{-\lambda\theta(\lambda_2+\alpha)}{\alpha m} p_{00} - \frac{\lambda(\lambda+\theta)(\lambda_2+\alpha)}{\alpha^2 m^2} \times \left\{-2\lambda_2 m - \alpha\left[\lambda_3^{\,2}\xi(\lambda+\theta)/\eta^2 + \mu\theta\right]\right\} p_{00}, \tag{29}$$

where

$$m = [-\alpha(\lambda+\theta) + \mu\theta]. \tag{30}$$

(2) The mean queue length of the orbit in an idle period is expressed as

$$P_0'(1) = \frac{\lambda(\lambda_2+\alpha)}{\alpha(\lambda+\theta)} p_{00} + \frac{(\lambda+\theta)}{\mu} P_1'(1). \tag{31}$$

(3) The mean queue length of the orbit in a setup period is determined as

$$P_2'(1) = \frac{\lambda(\lambda_2+\alpha)}{\alpha^2} p_{00}. \tag{32}$$

(4) The mean queue length of the orbit in a breakdown period is shown as

$$P_3'(1) = \frac{\xi}{\eta} P_1'(1) + \frac{\lambda_3\xi}{\eta^2} P_1(1). \tag{33}$$

(5) The mean queue length of the orbit is derived as

$$E(N) = P_0'(1) + P_1'(1) + P_2'(1) + P_3'(1). \tag{34}$$

(6) The mean number of customers in the system equals to the mean queue length of the orbit plus the probability that there is a customer being served. So

the mean number of customers in the system can be written as

$$E(L) = P_0'(1) + P_1'(1) + P_2'(1) + P_3'(1) + P_1(1) + P_3(1) = E(N) + p_1 + p_3. \tag{35}$$

(7) The expected waiting time in the orbit is obtained as

$$E(W) = \frac{E(N)}{\lambda_{ret}}, \tag{36}$$

where λ_{ret} is the total arrival rate in the retrial orbit, and

$$\lambda_{ret} = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3. \tag{37}$$

(8) The steady-state availability of the system is computed by

$$\begin{aligned} A &= P_0(1) + P_1(1) + P_2(1) \\ &= \frac{(\lambda+\theta)\left[-\alpha(\lambda_1\eta+\lambda_3\xi)+\lambda_2\mu\eta+\alpha\mu\eta\right]}{\alpha\left[-(\lambda+\theta)(\lambda_1\eta+\lambda_3\xi)+\mu\theta\eta\right]} p_{00} + \frac{\lambda\eta(\lambda+\theta)(\lambda_2+\alpha)-\lambda_2\mu\theta\eta}{\alpha\left[-(\lambda+\theta)(\lambda_1\eta+\lambda_3\xi)+\mu\theta\eta\right]} p_{00} + \frac{\lambda}{\alpha} p_{00}. \end{aligned} \tag{38}$$

(9) The balking rate of the customers is calculated as

$$B = (\lambda - \lambda_1)P_1(1) + (\lambda - \lambda_2)P_2(1) + (\lambda - \lambda_3)P_3(1). \tag{39}$$

4. Numerical Analysis

In this section, we give some numerical results to illustrate graphically the effects of different parameters on the performance measures and cost function of the system. In all numerical discussions, the system parameter values are chosen to satisfy the stability conditions.

4.1. Numerical Analysis of Performance Measures. From Figure 2, we can see the relationship between the probabilities of the server in different states and the arrival rate λ. As expected, when λ increases, the server gets busier and breakdown only occurs during the busy period. So the probabilities of idle period and setup period decrease, and the probabilities of busy period and breakdown period increase.

Figure 3 shows that the balking rate of the customers' decreases as α increases, and increases as ξ increases. The shortened setup time makes the server get into a working state quickly which attracts more customers to join the system. With the increase of ξ, it is intuitive that the congestion in a breakdown state reduces the willingness of customers to enter the system.

The changing trend of the mean number of customers in the system with respect to λ and θ is displayed in Figure 4. As λ increases, many customers arrive in the system, which leads to an increase in $E(L)$. The decrease of $E(L)$ is evident with the increase of the retrial rate θ, the reason is that customers in the retrial orbit can apply for service fast resulting in a small queue length of the system.

From Figure 5, the mean queue length of the orbit decreases as μ increases, and increases as α decreases. Customers in the system are served as quickly as possible, which leads to a reduction in the queue length of the orbit. Decreasing α leads to an increase in setup times, and customers pile up in the system so that the queue length of the orbit increases.

Figure 6 depicts that the expected waiting time decreases with respect to η and increases with respect to ξ. This is due to the fact that the server can be repaired in a short time and therefore the server returns to work quickly. The availability of the server decreases as ξ increases, hence, customers need to wait a long time to receive service.

4.2. Cost Analysis. In this subsection, we seek the minimum cost by establishing the operating cost function. The cost parameters are as follows.

$$\begin{aligned} c_1 &= \text{Holding cost per unit time per customer present in the retrial orbit}, \\ c_2 &= \text{Cost per unit time of providing a service rate } \mu, \\ c_3 &= \text{Cost per unit time of providing a retrial rate } \theta, \\ c_4 &= \text{Cost per unit time of providing a setup rate } \alpha, \\ c_5 &= \text{Cost per unit time of providing a repair rate } \eta. \end{aligned} \tag{40}$$

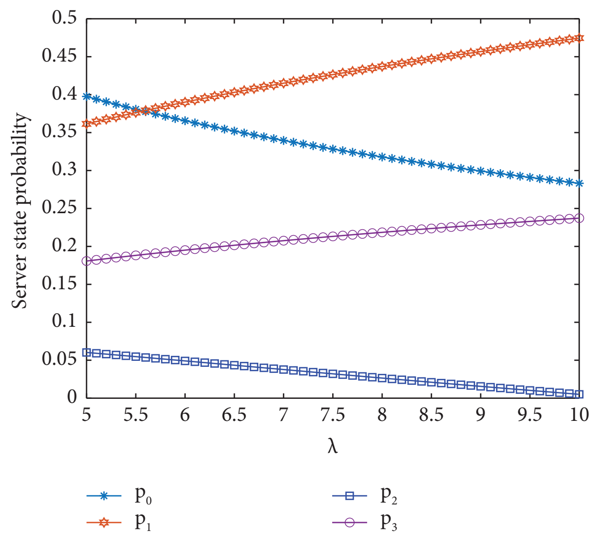

Figure 2: Server state probability vs. λ for $\lambda_1 = 3, \lambda_2 = 3, \lambda_3 = 2, \alpha = 8, \mu = 10, \theta = 7, \xi = 2, \eta = 4$.

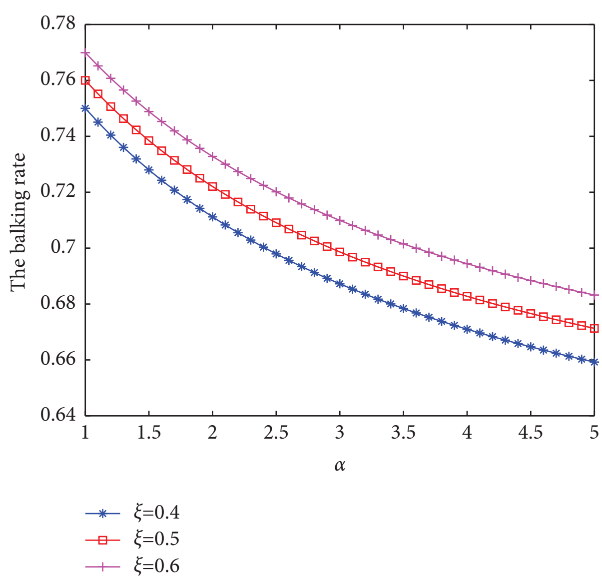

Figure 3: The balking rate vs. α for $\lambda = 3, \lambda_1 = 1.5, \lambda_2 = 2, \lambda_3 = 1, \mu = 7, \theta = 3, \eta = 4$.

Therefore, the operating cost function is given by

$$F = c_1 E(N) + c_2\mu + c_3\theta + c_4\alpha + c_5\eta. \tag{41}$$

We observe that the operating cost increases with ξ in Figure 7. It is very intuitive that the cost of the server increases due to more frequent breakdowns. The cost shows a decreasing and then increasing trend with respect to μ. It can be calculated that the cost has a minimum at $\mu = 5.3$. From Figure 8, we notice that when the setup rate α increases, the cost function first decreases and then increases. The server sets up quickly which reduces the waiting cost of the customers and increases the setup cost. Initially, the reduction of the waiting cost is more than the increase of setup cost which leads to the decline of the system cost. However, when α increases to a certain value, the increase of setup cost are dominant, hence, the system cost increases. In this case, the minimum cost is obtained at $\mu = 5.2$, $\alpha = 1.6$.

5. Individual Equilibrium and Social Optimization

In this section, we analyze individual equilibrium strategy and socially optimal strategy by establishing individual utility function and social welfare function. We assume that a customer receives a reward of R units for completing service. There is a waiting cost of C units per time unit that the customer remains in the system. To ensure that arriving customer who finds the system idle always choose to enter, we assume that $R > ((\eta + \xi)C)/(\mu\eta)$. Next, we discuss the almost and fully unobservable cases separately.

5.1. Almost Unobservable Case. In this subsection, we analyze equilibrium strategic behavior of customers in different states and use numerical examples to illustrate the effect of system parameters on the joining probabilities of customers.

We denote $T_i(j)$, $i = 0, 1, 2, 3$ as the mean waiting times of the marked customer, given that upon arrival he becomes the jth customer in the orbit and the server is at state $i\,(i = 0, 1, 2, 3)$. Then, we obtain $T_i(j)$ in the following analysis.

Theorem 2. *In the repairable M/M/1 retrial queueing system with setup times, the mean waiting times of the jth customer in the orbit at different states are respectively given by*

$$T_0(j) = j\left(\frac{1}{\theta} + \frac{\lambda + \theta}{\mu\theta} + \frac{(\lambda + \theta)\xi}{\mu\theta\eta}\right), \tag{42}$$

$$T_1(j) = j\left(\frac{1}{\theta} + \frac{\lambda + \theta}{\mu\theta} + \frac{(\lambda + \theta)\xi}{\mu\theta\eta}\right) + \frac{\eta + \xi}{\mu\eta}, \tag{43}$$

$$T_2(j) = j\left(\frac{1}{\theta} + \frac{\lambda + \theta}{\mu\theta} + \frac{(\lambda + \theta)\xi}{\mu\theta\eta}\right) + \frac{1}{\alpha}, \tag{44}$$

$$T_3(j) = j\left(\frac{1}{\theta} + \frac{\lambda + \theta}{\mu\theta} + \frac{(\lambda + \theta)\xi}{\mu\theta\eta}\right) + \frac{\eta + \xi}{\mu\eta} + \frac{1}{\eta}. \tag{45}$$

Proof. By analyzing the queueing model, we can obtain the following equation.

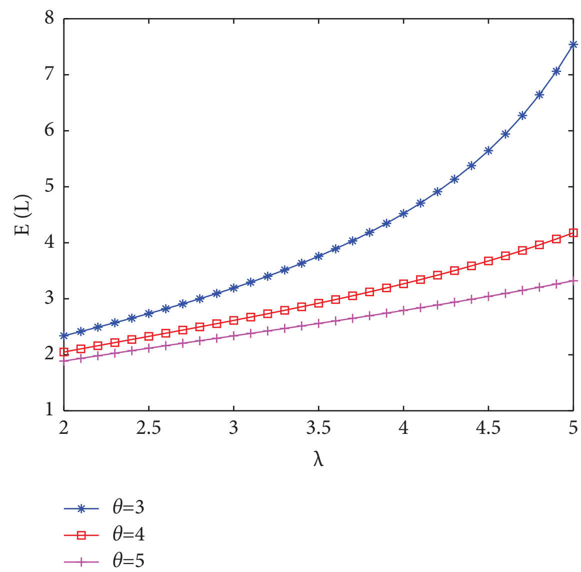

Figure 4: The mean number of customers in the system vs. λ for $\lambda_1 = 1.5, \lambda_2 = 2, \lambda_3 = 1, \alpha = 3, \mu = 5, \xi = 0.4, \eta = 4$.

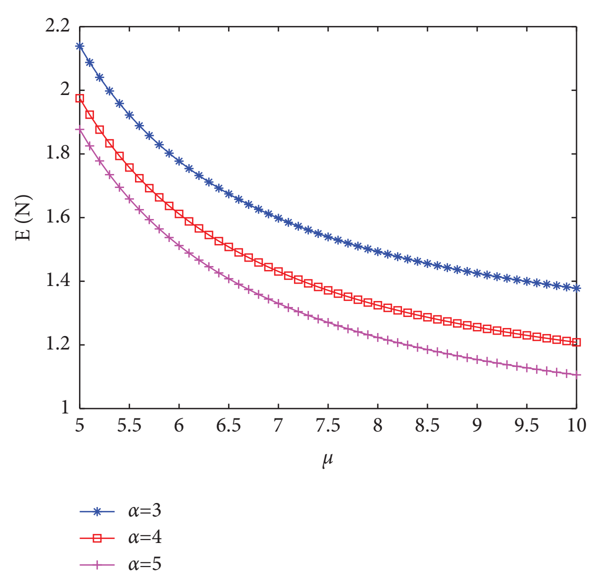

Figure 5: The mean queue length of the orbit vs. μ for $\lambda = 3, \lambda_1 = 1.5, \lambda_2 = 2, \lambda_3 = 1, \theta = 4, \xi = 0.4, \eta = 4$.

$$T_1(0) = \frac{\eta + \xi}{\mu\eta}, \tag{46}$$

$$T_0(j) = \frac{1}{\lambda + \theta} + \frac{\lambda}{\lambda + \theta} T_1(j) + \frac{\theta}{\lambda + \theta} T_1(j-1), \quad j \geq 1, \tag{47}$$

$$T_1(j) = \frac{1}{\lambda_1 + \mu + \xi} + \frac{\lambda_1}{\lambda_1 + \mu + \xi} T_1(j) + \frac{\mu}{\lambda_1 + \mu + \xi} T_0(j) + \frac{\xi}{\lambda_1 + \mu + \xi} T_3(j), \quad j \geq 1, \tag{48}$$

$$T_2(j) = \frac{1}{\lambda_2 + \alpha} + \frac{\lambda_2}{\lambda_2 + \alpha} T_2(j) + \frac{\alpha}{\lambda_2 + \alpha} T_0(j), \quad j \geq 1, \tag{49}$$

$$T_3(j) = \frac{1}{\lambda_3 + \eta} + \frac{\lambda_3}{\lambda_3 + \eta} T_3(j) + \frac{\eta}{\lambda_3 + \eta} T_1(j), \quad j \geq 0. \tag{50}$$

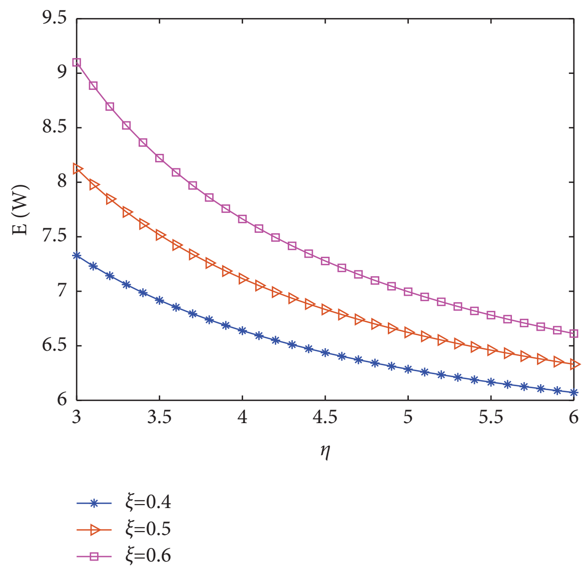

Figure 6: The expected waiting time vs. η for $\lambda = 2, \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 1, \alpha = 4, \mu = 5, \theta = 2$.

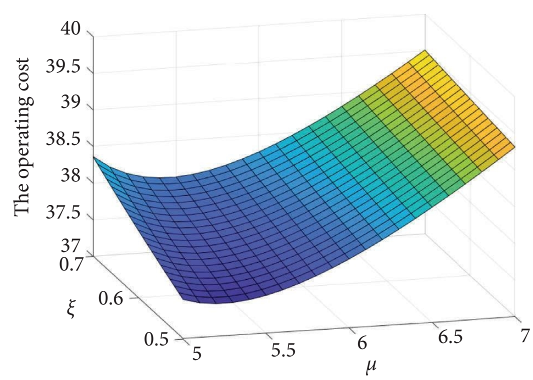

Figure 7: The operating cost vs. μ and ξ for $\lambda = 3, \lambda_1 = 2, \lambda_2 = 1.5, \lambda_3 = 1, \alpha = 5, \theta = 5, \eta = 2, c_1 = 1.5, c_2 = 2, c_3 = 1, c_4 = 2, c_5 = 4$.

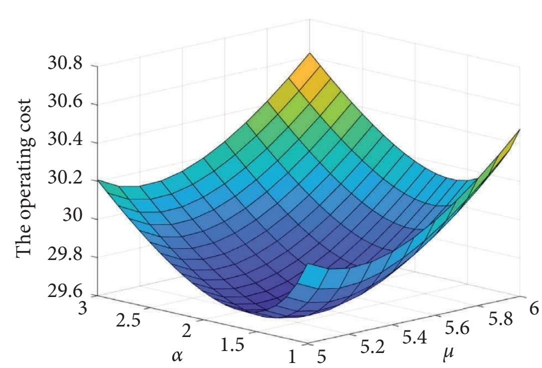

Figure 8: The operating cost vs. μ and α for $\lambda = 3, \lambda_1 = 2, \lambda_2 = 1.5, \lambda_3 = 1, \theta = 5, \xi = 0.5, \eta = 2, c_1 = 1.5, c_2 = 2, c_3 = 1, c_4 = 0.8, c_5 = 4$.

From (50), we obtain that

$$T_3(j) = T_1(j) + \frac{1}{\eta}. \tag{51}$$

From (48) and (50), we can get

$$T_1(j) = T_1(j-1) + \frac{1}{\theta} + \frac{\lambda+\theta}{\mu\theta} + \frac{(\lambda+\theta)\xi}{\mu\theta\eta}, j \geq 1, \tag{52}$$

$$T_1(j) = T_1(0) + j\left(\frac{1}{\theta} + \frac{\lambda+\theta}{\mu\theta} + \frac{(\lambda+\theta)\xi}{\mu\theta\eta}\right), j \geq 0. \tag{53}$$

From (46) and (53), we get (43). From (47), (52), and (43), we get (42). From, (42) and (49), (44) is obtained. From (43) and (51), (45) is obtained.

The situation is different for a nonmarked new arriving customer, but we can obtain the mean waiting times by Theorem 2. Denote $W_i(q_1, q_2, q_3)$, $i = 0, 1, 2, 3$ as the mean waiting times of a new arriving customer who finds $I(t) = 0, 1, 2, 3$. We come to the following conclusion. □

Theorem 3. *In the repairable M/M/1 retrial queueing system with setup times, the mean waiting times when a new customer finds the server in different states are as follows:*

$$W_0(q_1, q_2, q_3) = \frac{\eta+\xi}{\mu\eta}, \tag{54}$$

$$W_1(q_1, q_2, q_3) = a\left(\frac{\lambda_2}{\alpha} + \frac{\lambda}{\lambda+\theta} + \frac{\mu\theta\eta^2 + (\lambda+\theta)\lambda_3^2\xi}{\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda_1\eta^2 + \lambda_3\xi\eta\right)}\right) + \frac{(a+1)(\eta+\xi)}{\mu\eta}, \tag{55}$$

$$W_2(q_1, q_2, q_3) = \frac{a(\lambda_2+\alpha)+1}{\alpha} + a, \tag{56}$$

$$W_3(q_1, q_2, q_3) = a\left(\frac{\lambda_2}{\alpha} + \frac{\lambda}{\lambda+\theta} + \frac{\mu\theta\eta^2 + (\lambda+\theta)\lambda_3^2\xi}{\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda_1\eta^2 + \lambda_3\xi\eta\right)} + \frac{\lambda_3}{\eta}\right) + \frac{(a+1)(\eta+\xi)+\mu}{\mu\eta}, \tag{57}$$

where

$$a = \frac{1}{\theta} + \frac{\lambda+\theta}{\mu\theta} + \frac{(\lambda+\theta)\xi}{\mu\theta\eta}. \tag{58}$$

Proof. First, the arriving customer who finds the server idle will directly accept the service. There are two possible cases, i.e., whether the server breaks down during the working period. At this time $W_0(q_1, q_2, q_3) = (1/\mu) + (1/\mu) \times (\xi/\eta) = (\eta+\xi)/(\mu\eta)$. By using PASTA property, the probability that there are k customers in the system when the server is busy is $p(k \mid 1) = (p(k,1)/\sum_{k=1}^{\infty} p(k,1)) = (p(k,1)/p_1)$, according to the total probability formula, we derive that

$$\begin{aligned} W_1(q_1, q_2, q_3) &= \sum_{k=0}^{\infty} T_1(k+1)p(k,1) = \frac{\sum_{k=0}^{\infty} T_1(k+1)p(k,1)}{p_1} \\ &= \frac{\sum_{k=0}^{\infty} kap(k,1)}{p_1} + \frac{\sum_{k=0}^{\infty}(a + (\eta+\xi/\mu\eta))p(k,1)}{p_1} \\ &= \frac{aP'(1)}{p_1} + a + \frac{\eta+\xi}{\mu\eta}. \end{aligned} \tag{59}$$

Then, we can get (55). Following the same way we can obtain (56) and (57).

Next, we discuss individual equilibrium strategies for the arriving customer who finds the server in different states. The individual utility functions in different states are as follows:

$$S_i(q_1, q_2, q_3) = R - CW_i(q_1, q_2, q_3), i = 1, 2, 3. \tag{60}$$

□

Theorem 4. *In the repairable M/M/1 retrial queueing system with setup times, $W_i(q_1, q_2, q_3)$ increases with q_i $(i = 1, 2, 3)$, $q_i \in [0, 1]$.*

Proof. The derivative of the waiting time $W_i(q_1, q_2, q_3)$ with respect to $q_i\,(i = 1, 2, 3)$ are as follows:

$$\begin{aligned}
\frac{\partial W_1(q_1,q_2,q_3)}{\partial q_1} &= \frac{\left(\mu\theta\eta^2 + (\lambda+\theta)\lambda_3^2\xi\right)(\lambda+\theta)\lambda\eta^2 a}{\left(\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda_1\eta^2 + \lambda_3\xi\eta\right)\right)^2} > 0,\\
\frac{\partial W_1(q_1,q_2,q_3)}{\partial q_2} &= \frac{\lambda a}{\alpha} > 0,\\
\frac{\partial W_1(q_1,q_2,q_3)}{\partial q_3} &= \frac{2\lambda^2 q_3\xi(\lambda+\theta)\left(\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda_1\eta^2 + \lambda_3\xi\eta\right)\right)a + \left(\mu\theta\eta^2 + (\lambda+\theta)\lambda_3^2\xi\right)(\lambda+\theta)\lambda\xi\eta a}{\left(\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda_1\eta^2 + \lambda_3\xi\eta\right)\right)^2} > 0.
\end{aligned} \tag{61}$$

It can be concluded that $W_1(q_1, q_2, q_3)$ is strictly increasing for q_i. Similarly, the monotonicity of $W_2(q_1, q_2, q_3)$ and $W_3(q_1, q_2, q_3)$ can be proved. □

Theorem 5. *In the repairable M/M/1 retrial queueing system with setup times, the individual equilibrium strategy (q_1^e, q_2^e, q_3^e) of the customer is shown in the following:*

(1) $(R/C) \le 2a + (1/\alpha)$

$$(q_1^e, q_2^e, q_3^e) = \begin{cases}
(0,0,0), & \dfrac{R}{C} \le \dfrac{\lambda a}{\lambda+\theta} + a + \dfrac{\eta+\xi}{\mu\eta},\\[2mm]
(q_1^*,0,0), & \dfrac{\lambda a}{\lambda+\theta} + a + \dfrac{\eta+\xi}{\mu\eta} < \dfrac{R}{C} < \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + b,\\[2mm]
(1,0,0), & \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + b \le \dfrac{R}{C} < \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + b + \dfrac{1}{\eta},\\[2mm]
(1,0,q_3^*), & \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + b + \dfrac{1}{\eta} \le \dfrac{R}{C} < \dfrac{\mu\theta\eta^2 a + (\lambda+\theta)\lambda^2\xi a}{\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda\eta^2 + \lambda\xi\eta\right)} + b + \dfrac{\lambda a + 1}{\eta},\\[2mm]
(1,0,1), & \dfrac{R}{C} \ge \dfrac{\mu\theta\eta^2 a + (\lambda+\theta)\lambda^2\xi a}{\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda\eta^2 + \lambda\xi\eta\right)} + b + \dfrac{\lambda a + 1}{\eta}.
\end{cases} \tag{62}$$

(2) $2a + (1/\alpha) < (R/C) < a + (a(\lambda+\alpha) + 1/\alpha)$

$$(q_1^e, q_2^e, q_3^e) = \begin{cases}
(0,q_2^*,0), & \dfrac{R}{C} \le \dfrac{\lambda a}{\lambda+\theta} + \dfrac{\lambda ac}{\alpha} + a + \dfrac{\eta+\xi}{\mu\eta},\\[2mm]
(q_1^{**},q_2^*,0), & \dfrac{\lambda a}{\lambda+\theta} + \dfrac{\lambda ac}{\alpha} + a + \dfrac{\eta+\xi}{\mu\eta} < \dfrac{R}{C} < \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + \dfrac{\lambda ac}{\alpha} + b,\\[2mm]
(1,q_2^*,0), & \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + \dfrac{\lambda ac}{\alpha} + b < \dfrac{R}{C} < \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + \dfrac{\lambda ac}{\alpha} + b + \dfrac{1}{\eta},\\[2mm]
(1,q_2^*,q_3^{**}), & \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda+\theta)} + \dfrac{\lambda ac}{\alpha} + b + \dfrac{1}{\eta} < \dfrac{R}{C} < \dfrac{\mu\theta\eta^2 a + (\lambda+\theta)\lambda^2\xi a}{\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda\eta^2 + \lambda\xi\eta\right)} + \dfrac{\lambda ac}{\alpha} + b + \dfrac{\lambda a + 1}{\eta},\\[2mm]
(1,q_2^*,1), & \dfrac{R}{C} \ge \dfrac{\mu\theta\eta^2 a + (\lambda+\theta)\lambda^2\xi a}{\mu\theta\eta^2 - (\lambda+\theta)\left(\lambda\eta^2 + \lambda\xi\eta\right)} + \dfrac{\lambda ac}{\alpha} + b + \dfrac{\lambda a + 1}{\eta}.
\end{cases} \tag{63}$$

(3) $(R/C) \geq a + (a(\lambda + \alpha) + 1/\alpha)$

$$(q_1^e, q_2^e, q_3^e) = \begin{cases} (0,1,0), \dfrac{R}{C} \leq \dfrac{\lambda a}{\lambda + \theta} + \dfrac{\lambda a}{\alpha} + a + \dfrac{\eta + \xi}{\mu\eta}, \\ (q_1^{***}, 1, 0), \dfrac{\lambda a}{\lambda + \theta} + \dfrac{\lambda a}{\alpha} + a + \dfrac{\eta + \xi}{\mu\eta} < \dfrac{R}{C} < \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda + \theta)} + \dfrac{\lambda a}{\alpha} + b, \\ (1,1,0), \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda + \theta)} + \dfrac{\lambda a}{\alpha} + b < \dfrac{R}{C} < \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda + \theta)} + \dfrac{\lambda a}{\alpha} + b + \dfrac{1}{\eta}, \\ (1,1,q_3^{***}), \dfrac{\mu\theta a}{\mu\theta - \lambda(\lambda + \theta)} + \dfrac{\lambda a}{\alpha} + b + \dfrac{1}{\eta} < \dfrac{R}{C} < \dfrac{\mu\theta\eta^2 a + (\lambda + \theta)\lambda^2 \xi a}{\mu\theta\eta^2 - (\lambda + \theta)\left(\lambda\eta^2 + \lambda\xi\eta\right)} + \dfrac{\lambda a}{\alpha} + b + \dfrac{\lambda a + 1}{\eta}, \\ (1,1,1), \dfrac{R}{C} \geq \dfrac{\mu\theta\eta^2 a + (\lambda + \theta)\lambda^2 \xi a}{\mu\theta\eta^2 - (\lambda + \theta)\left(\lambda\eta^2 + \lambda\xi\eta\right)} + \dfrac{\lambda a}{\alpha} + b + \dfrac{\lambda a + 1}{\eta}. \end{cases} \quad (64)$$

where

$$b = a + \frac{n + \xi}{\mu\eta} - \frac{\theta a}{\lambda + \theta}, c = q_2^* = \frac{R\alpha - 2a\alpha C - C}{\lambda a C}, M = \frac{R}{Ca} - 1 - \frac{n + \xi}{\mu\eta a} + \frac{a}{\lambda + \theta}, \quad (65)$$

$$\begin{aligned} M_1 &= M - \frac{1}{\eta a} \\ M_2 &= M - \frac{\lambda}{\alpha} \\ M_3 &= M_1 - \frac{\lambda}{\alpha}, \\ M_4 &= M - \frac{\lambda c}{\alpha} \\ M_5 &= M_1 - \frac{\lambda c}{\alpha} \end{aligned} \quad (66)$$

$$\begin{aligned} q_1^* &= \frac{\mu\theta(M - 1)}{\lambda M(\lambda + \theta)}, \\ q_3^* &= \frac{\mu\theta\eta M_1 - (\lambda + \theta)\lambda\eta^3 M_1 - \mu\theta\eta^3}{\lambda\mu\theta + (\lambda + \theta)(\xi M_1 - \lambda)\lambda\eta^2}, \\ q_1^{**} &= \frac{\mu\theta(M_4 - 1)}{\lambda M_4(\lambda + \theta)}, \end{aligned} \quad (67)$$

$$q_3^{**} = \frac{\mu\theta\eta M_5 - (\lambda+\theta)\lambda\eta^3 M_5 - \mu\theta\eta^3}{\lambda\mu\theta + (\lambda+\theta)(\xi M_5 - \lambda)\lambda\eta^2},$$
$$q_1^{***} = \frac{\mu\theta(M_2 - 1)}{\lambda M_2(\lambda+\theta)}, \tag{68}$$
$$q_3^{***} = \frac{\mu\theta\eta M_3 - (\lambda+\theta)\lambda\eta^3 M_3 - \mu\theta\eta^3}{\lambda\mu\theta + (\lambda+\theta)(\xi M_3 - \lambda)\lambda\eta^2}.$$

Proof. From Theorem 4, we can see that $S_2(q_1, q_2, q_3)$ is only related to q_2 and monotonically decreases with q_2.

(1) If $(R/C) \le 2a + (1/\alpha)$, then $S_2(q_1, 0, q_3) \le 0$. Arriving customers find that the server is in the setup period, the individual utility function is always negative and no one will choose to enter the system. Then, there exists a unique equilibrium strategy, i.e., $q_2^e = 0$.

(a) If $(R/C) \le (\lambda a)/(\lambda+\theta) + a + (\eta+\xi)/(\mu\eta)$, then $S_1(0,0,0) < 0$ and $S_3(0,0,0) = S_1(0,0,0) - (C/\eta) < 0$. From Theorem 4, $S_1(q_1, 0, q_3) \le S_1(0,0,0) - (C/\eta) < 0$. The customer suffers a negative benefit. Hence, the customer's best choice would be to balk if he observes the server at state 1. Moreover, $S_3(0,0,q_3) \le S_3(0,0,0) < 0$, the best choice is balking if he finds the server at state 3. In a word, the equilibrium strategy is $(q_1^e, q_2^e, q_3^e) = (0,0,0)$.

(b) If $(\mu\theta a)/(\mu\theta - \lambda(\lambda+\theta)) + b \le (R/C) < (\mu\theta a)/(\mu\theta - \lambda(\lambda+\theta)) + b + (1/\eta)$, then $S_1(0,0,0) > 0$ and $S_3(1,0,0) < 0$. From Theorem 4, we can conclude that $S_1(q_1,0,0) > S_1(1,0,0) > 0$. The customer suffers a positive benefit. Hence, the customer's best choice would be to enter if he observes the server at state 1. Moreover, $S_3(1,0,q_3) \le S_3(1,0,0) < 0$, the best choice is balking if he finds the server at state 3. In a word, the equilibrium strategy is $(q_1^e, q_2^e, q_3^e) = (1,0,0)$.

(c) If $(R/C) \ge (\mu\theta\eta^2 a + (\lambda+\theta)\lambda^2\xi a)/(\mu\theta\eta^2 - (\lambda+\theta)(\lambda\eta^2 + \lambda\xi\eta)) + b + (\lambda a + 1/\eta)$, then $S_3(1,0,1) > 0$ and $S_1(1,0,1) = S_3(1,0,1) + (C/\eta) > 0$. From Theorem 4, $S_1(q_1,0,1) > S_1(1,0,1) > 0$. The customer's benefits are always positive, the customer's best choice would be to enter if he observes the server at state 1. Moreover, $S_3(1,0,q_3) > S_3(1,0,1) > 0$, There exists a unique equilibrium strategy when the server is at 3. In a word, the equilibrium strategy is $(q_1^e, q_2^e, q_3^e) = (1,0,1)$.

(d) If $(\lambda a)/(\lambda+\theta) + a + (\eta+\xi)/(\mu\eta) < (R/C) < (\mu\theta a)/(\mu\theta - \lambda(\lambda+\theta)) + b$, then $(q_1^e, q_2^e, q_3^e) = (q_1^*, 0, 0)$, from case (a), we have $(q_1^e, q_2^e, q_3^e) = (0,0,0)$; from case (c), we have $(q_1^e, q_2^e, q_3^e) = (1,0,0)$. From Case (a) and Case (c), we find $S_1(1,0,0) < 0 < S_1(0,0,0)$, By the zero theorem, there exists a unique q_1^* such that $S_1(q_1^*, 0, 0) = 0$. Therefore, in this case, we have $(q_1^e, q_2^e, q_3^e) = (q_1^*, 0, 0)$.

(e) If $(R/C) \ge (\mu\theta\eta^2 a + (\lambda+\theta)\lambda^2\xi a)/(\mu\theta\eta^2 - (\lambda+\theta)(\lambda\eta^2 + \lambda\xi\eta)) + b + (\lambda a + 1/\eta)$, then $(q_1^e, q_2^e, q_3^e) = (q_1^*, 0, 0)$, from case (c), we have $(q_1^e, q_2^e, q_3^e) = (1,0,0)$; from case (e), we have $(q_1^e, q_2^e, q_3^e) = (1,0,1)$. From Case (a) and Case (e), we find $S_3(1,0,1) < 0 < S_3(1,0,0)$, By the zero theorem, there exists a unique q_3^* such that $S_3(1,0,q_3^*) = 0$. Therefore, in this case, we have $(q_1^e, q_2^e, q_3^e) = (1,0,q_3^*)$.

(2) If $2a + (1/\alpha) < (R/C) < a + (a(\lambda+\alpha) + 1/\alpha)$, then $S_2(q_1, 1, q_3) < 0 < S_2(q_1, 0, q_3)$. Since $S_2(q_1, q_2, q_3)$ decreases monotonically with q_2, there exists a unique equilibrium strategy q_2^e such that equation $S_2(q_1, q_2, q_3) = 0$. We solve for $q_2^e = q_2^* = (R\alpha - 2a\alpha C - C)/(\lambda a C)$. The proof method of the subcases is similar to case (1).

(3) If $(R/C) \ge a + (a(\lambda+\alpha) + 1/\alpha)$, then $S_2(q_1, 1, q_3) > 0$. Arriving customers find that the server is in the setup period, the individual utility function is always positive and customers will choose to enter the system, there exists a unique equilibrium strategy, i.e., $q_2^e = 1$. The proof method of the subcases is similar to case (1). □

Remark 1. We considered a special case. Taking $\xi \longrightarrow 0$, in our results, we get the M/M/1 retrial queue with setup times. The results of individual equilibrium strategies are consistent with the literature [13].

Next, we find the maximum social benefit by using the social welfare function. The social welfare function is defined as

$$S(q_1, q_2, q_3) = \lambda\left(R - \frac{(\eta+\xi)C}{\mu\eta}\right)p_0 + \lambda q_1(R - CW_1)p_1 + \lambda q_2(R - CW_1)p_2 + \lambda q_3(R - CW_1)p_3. \tag{69}$$

The purpose of social planners is to maximize $S(q_1, q_2, q_3)$ by searching for the optimal joining probability of customers q_1^s, q_2^s and q_3^s. The expression of $S(q_1, q_2, q_3)$ is too complex to obtain its analytical solution. We can use the particle swarm algorithm (PSO algorithm) to search for the numerical solution of this problem. The most significant advantage of the particle swarm algorithm is that it does not require too much analyticity of the objective function. The algorithm is based on the theory of collective intelligence. In each iteration of the search process, the particles in the swarm can dynamically adjust their positions and velocities by tracking the two extremes in the swarm and finding the global optimal solution through multiple iterations. Specific numerical examples are given in the following:

We analyze the effects of system parameters on individual equilibrium joining probabilities and social welfare through numerical examples.

As can be seen in Figure 9, q_i increase with service rate μ. Intuitively, customers have a greater incentive to enter the system when the server can serve more customers. From Figures 10 and 11, it is easy to understand that the arrival rate λ and the failure rate ξ have the same effect. As the waiting time increases, the equilibrium joining probability of customers decreases. It can be known from Figure 12 that there exists the opposite tendency of q_i with regard to α. It is reasonable that with the decreasing of the setup time, customers finding the server at state 2 are more willing to enter the system. On the other hand, as long as the individual benefit is positive, selfish customers will always choose to enter, which results in the congestion of the system. Customers are reluctant to enter the system at busy state or breakdown state.

As for the socially optimal strategy, it can be seen from Table 1 that with the increase of arrival rate, many customers join the system resulting in congestion which leads to new customers are no longer willing to enter. However, due to the increase in the number of customers arriving at the system, more social welfare has been brought. From Table 2, it is found that q_1^s and q_3^s decrease with increasing α, q_2^s increases with α. At this time, the number of customers entering the system increases, which makes the social benefits increase. From Table 3, the optimal social joining probability of each state increases with θ. Customers' willingness to join the system leads to many social welfare.

5.2. Fully Unobservable Case. In this subsection, we study individual equilibrium strategy and socially optimal strategy in the fully unobservable case. Since there are many states of the server, the arriving customers choose to enter the system with different probabilities depending on the state of the server. To avoid the complexity of different probabilities, we consider the special case where the joining probability of customers is the same $(q_1 = q_2 = q_3 = q)$.

The individual utility function is given by

$$U(q) = R - CE(W). \tag{70}$$

The customers' equilibrium joining probability, defined as q_e, where q_e is given by

(i) $q_e = 0$ if only if $U(0) < 0$.

(ii) Similarly, $q_e = 1$ if only if $U(1) > 0$.

(iii) In addition, a necessary and sufficient condition, for $q_e \in (0, 1)$ to be an equilibrium joining probability is that $U(q_e) = 0$.

The social welfare function is given by

$$S(q) = \lambda^* R - CE(N), \tag{71}$$

where λ^* is the effective arrival rate of customers, and

$$\lambda^* = \lambda p_0 + \lambda q(p_1 + p_2 + p_3) = \frac{\lambda\mu\eta(\lambda+\theta)(\lambda q+\alpha)}{\alpha[\mu\theta\eta - \lambda q(\lambda+\theta)(\eta+\xi)]}p_{00}. \tag{72}$$

From the individual perspective, when customers arrive at the system, they judge whether to enter the system based on their profit gain or loss. According to the individual utility function, we can find the customers' equilibrium joining probability q_e. From the social perspective, the social planner aims to maximize social welfare by finding the customers' optimal joining probability q_s. According to the individual utility function and social welfare function given in the paper, $U(q)$ and $S(q)$ are complex functions on q. We have difficulty in deriving the analytic solution for q_e and q_s. Therefore, some numerical analysis is given in the following figures.

Figure 13 indicates that q_e and q_s gradually decrease with the arrival rate λ. The reason is that many customers join the system and the orbit becomes crowded, which leads to a decline in customers' enthusiasm to join the system. It can be seen from Figure 14 that q_e and q_s show an upward trend with the increase of θ. This is because customers in the retrial orbit receive service fast, which leads to less waiting time and more willingness of customers to join the system. In Figure 15, q_e and q_s are increasing with respect to μ. The reason is that the server can complete the service quickly; customers are inclined to enter the system. Figure 16 examines the influence of η on q_e and q_s. As the repair rate η increases, the customers reduce the waiting time when the server is fast to change state from breakdown to the normal working level, hence customers prefer to enter the system.

From Figure 17, the social welfare gradually increases with λ and decreases with ξ. This is because as λ increases, many customers enter the system leading to an increase in the maximum of social welfare. The increase of ξ shortens the available time of the server, which leads to a decrease in the number of customers entering the system and the social welfare is reduced. The trend of social welfare with respect to R and C is depicted in Figure 18. It is intuitive that many rewards attract customers to join the system, leading to greater social welfare. The higher cost per unit time leads to higher waiting costs for customers, which reduces the enthusiasm of customers to enter the system and leads to lower social welfare. The relationship between the social welfare and the setup rate α for different service rates μ is shown in Figure 19. It presents that the social welfare increases with respect to the setup rate and service rate. The shortened setup time and service time

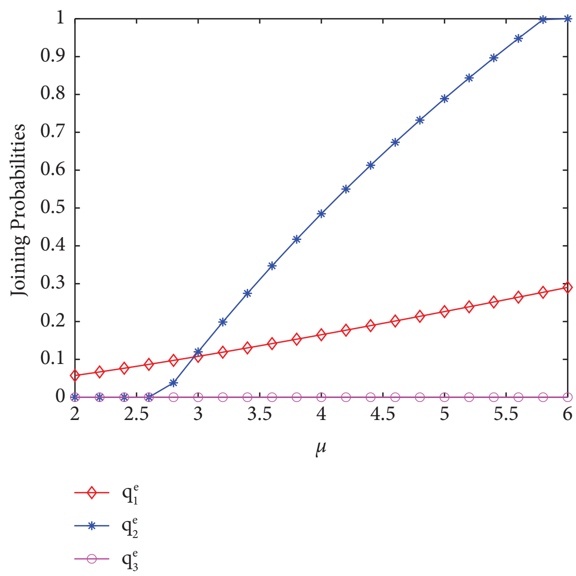

FIGURE 9: The joining probabilities vs. μ for $\lambda = 3, \theta = 2, \xi = 0.8, \alpha = 2.4, \eta = 2, R = 20, C = 5$.

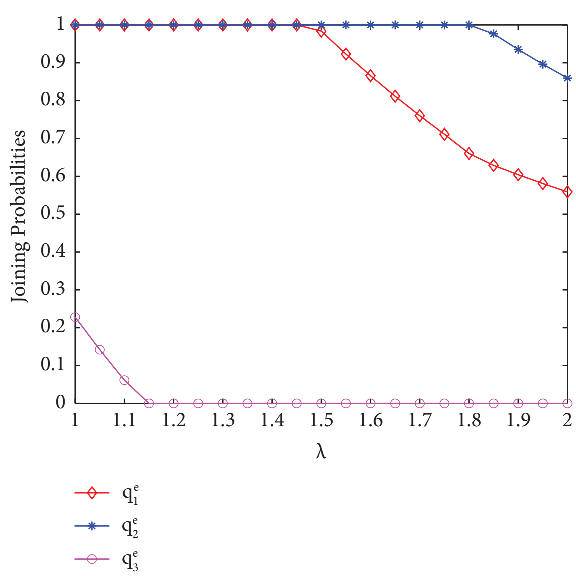

FIGURE 10: The joining probabilities vs. λ for $\theta = 3.5, \mu = 3.6, \xi = 0.5, \alpha = 1.4, \eta = 1, R = 15, C = 4$.

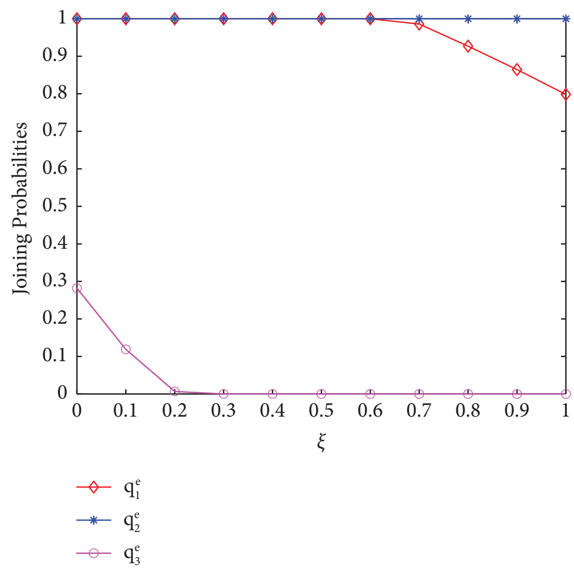

Figure 11: The joining probabilities vs. ξ for $\lambda = 1.3, \mu = 3.9, \theta = 2.6, \eta = 1, \alpha = 2.4, R = 15, C = 4$.

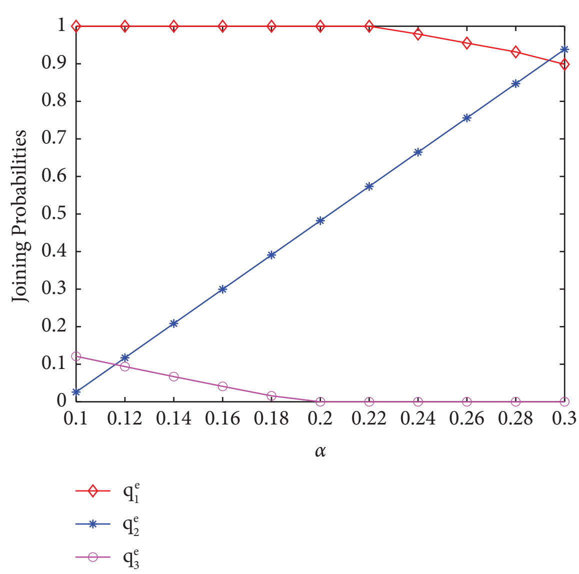

Figure 12: The joining probabilities vs. α for $\lambda = 2.45, \mu = 5.5, \theta = 2.9, \xi = 0.8, \eta = 1, R = 15, C = 1.2$.

Table 1: (q_1^s, q_2^s, q_3^s) and $S(q_1^s, q_2^s, q_3^s)$ vs. λ for $\mu = 3.6, \xi = 0.5, \alpha = 1.4, \eta = 1, R = 15, C = 4$.

	$\lambda = 0.7$	$\lambda = 0.8$	$\lambda = 0.9$	$\lambda = 1$	$\lambda = 1.1$	$\lambda = 1.2$
q_1^s	0.9996	0.9992	0.9991	0.9969	0.9951	0.9072
q_2^s	0.8353	0.6345	0.4744	0.3298	0.2491	0.1969
q_3^s	0.0166	0.0026	0.0011	0.0010	0.0002	0
$S(q_1^s, q_2^s, q_3^s)$	6.2379	6.7708	7.2505	7.6781	8.0494	8.3686

attract more customers to enter the system which promotes the growth of social welfare. Figure 20 displays that the growth of both η and θ can increase the maximum social welfare. In this case, the decrease of the waiting time encourages customers to join the system; this contributes to the growth of social welfare.

Table 2: (q_1^s, q_2^s, q_3^s) and $S(q_1^s, q_2^s, q_3^s)$ vs. α for $\lambda = 1.6, \mu = 4.5, \xi = 0.8, \theta = 2.9, \eta = 1, R = 20, C = 1.2$.

	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 1.2$
q_1^s	1	1	1	0.9990	0.9984	0.9982
q_2^s	0.4198	0.7143	0.9681	0.9960	0.9987	0.9992
q_3^s	0.4045	0.2961	0.2311	0.2215	0.2180	0.2155
$S(q_1^s, q_2^s, q_3^s)$	15.6170	18.5100	19.8904	20.6842	21.1638	21.4821

Table 3: (q_1^s, q_2^s, q_3^s) and $S(q_1^s, q_2^s, q_3^s)$ vs. θ for $\lambda = 1.6, \mu = 5, \xi = 0.8, \alpha = 2, \eta = 2, R = 20, C = 3$.

	$\theta = 1.5$	$\theta = 1.8$	$\theta = 2.1$	$\theta = 3$	$\theta = 3.5$	$\theta = 4$
q_1^s	0.6597	0.8001	0.9254	0.9990	0.9999	1
q_2^s	0.8377	0.9291	0.9973	0.9992	0.9993	0.9997
q_3^s	0	0.0001	0.0005	0.2401	0.4659	0.5456
$S(q_1^s, q_2^s, q_3^s)$	19.7389	20.2720	20.8122	22.1048	22.5958	23.0088

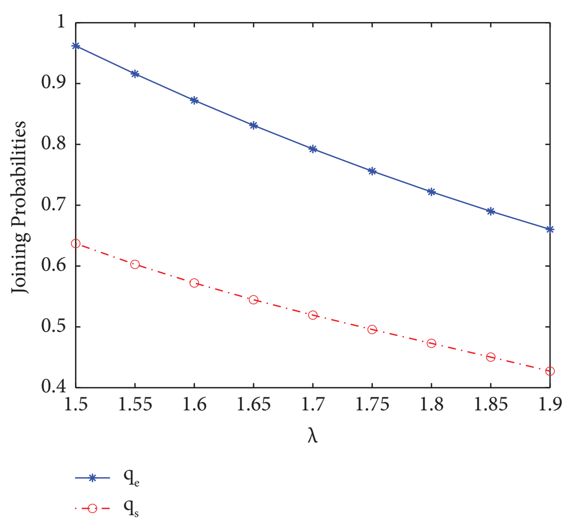

Figure 13: The joining probabilities vs. λ for $\alpha = 4, \mu = 5, \theta = 2, \xi = 0.8, \eta = 2, R = 20, C = 4$.

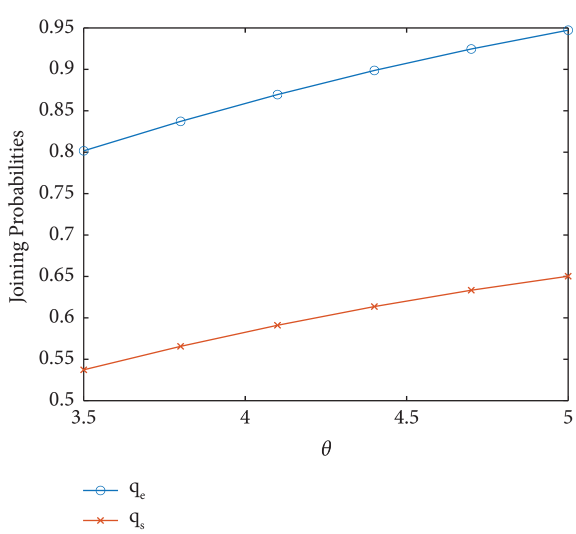

Figure 14: The joining probabilities vs. θ for $\lambda = 2, \alpha = 3, \mu = 4, \xi = 0.5, \eta = 2, R = 20, C = 4$.

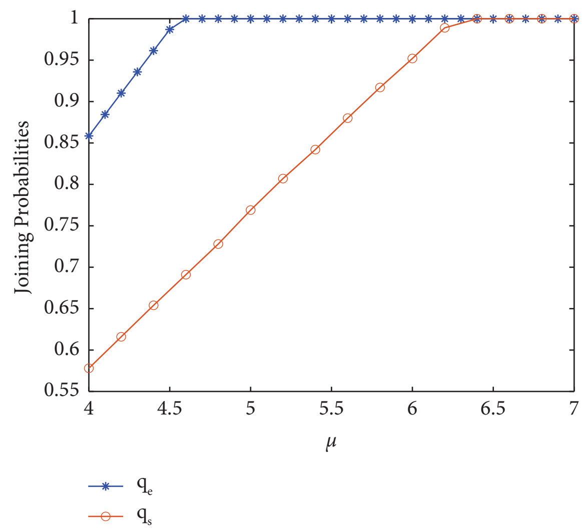

FIGURE 15: The joining probabilities vs. μ for $\lambda = 1.5, \alpha = 3, \theta = 2, \xi = 0.5, \eta = 2, R = 20, C = 4$.

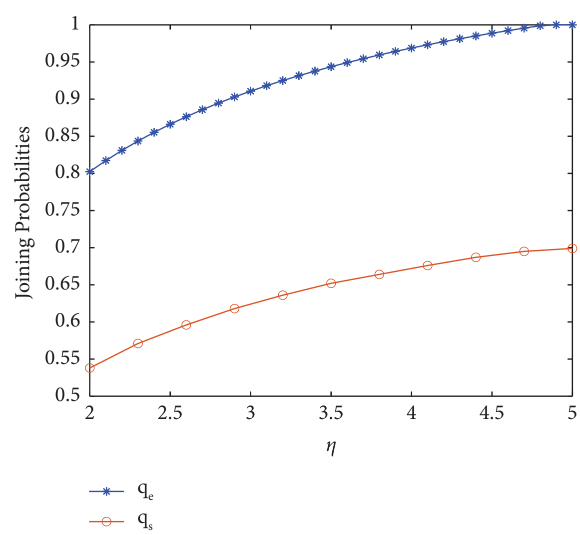

FIGURE 16: The joining probabilities vs. η for $\lambda = 1.8, \alpha = 4, \mu = 5, \theta = 2, \xi = 0.6, R = 20, C = 4$.

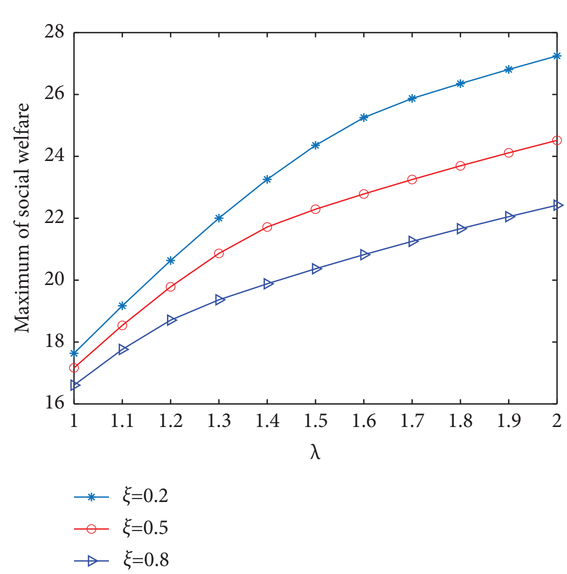

FIGURE 17: The maximum of social welfare vs. λ for $\alpha = 3, \mu = 4, \theta = 5, \eta = 2, R = 20, C = 4$.

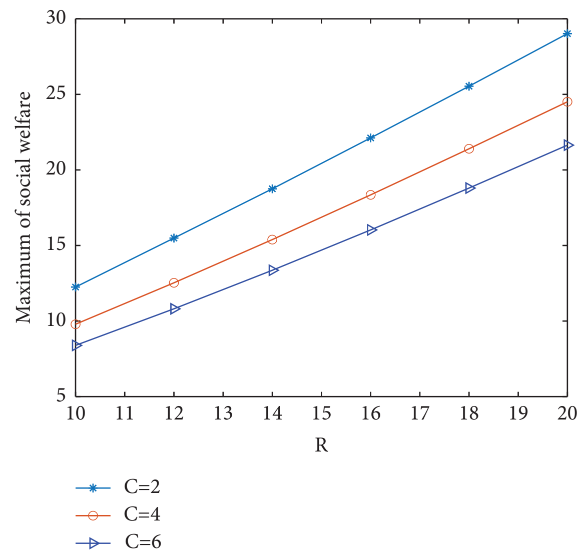

Figure 18: The maximum of social welfare vs. R for $\lambda = 2, \alpha = 3, \mu = 4, \theta = 5, \xi = 0.5, \eta = 2$.

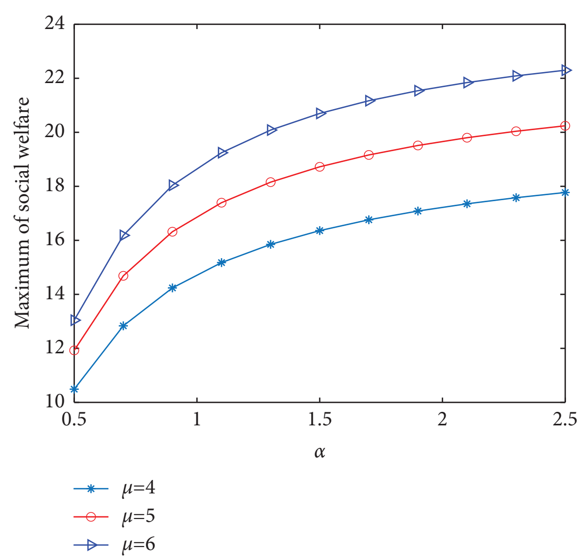

Figure 19: The maximum of social welfare vs. α for $\lambda = 1.5, \theta = 3, \xi = 0.8, \eta = 2, R = 20, C = 4$.

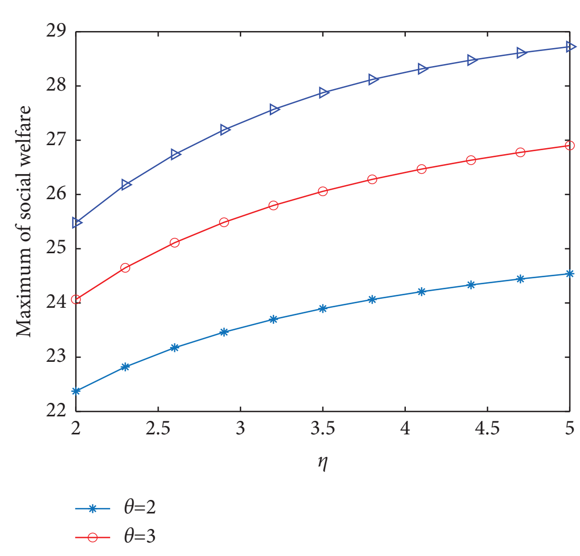

Figure 20: The maximum of social welfare vs. η for $\lambda = 1.8, \alpha = 4, \mu = 5, \xi = 0.6, R = 20, C = 4$.

6. Conclusion

In this paper, we analyze a repairable M/M/1 retrial queue with setup times. Under the stability condition, we construct balance equations to obtain the steady-state probabilities of the server in different states. And we derive the performance measures of the system. Next, the effects of parameters on the performance measures of the system and the cost function are analyzed by numerical examples. Finally, we present an extensive analysis of customers' equilibrium joining behavior and socially optimal strategies in the almost and fully unobservable cases.

Furthermore, there are also significant limitations in this paper. Our discussions are based on the assumption of the exponential distribution, which is convenient to obtain analytical solutions. However, this assumption may not apply to some practical scenarios. It is worth challenging in some directions. One is to consider that the service time obeys the general distribution, which can be studied using the supplementary variable method and differential equation. In addition, we can incorporate the present model in a profit-maximizing framework, where the owner or manager of the system imposes an entrance fee.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there ae no conflicts of interest.

Acknowledgments

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