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Research Article

Some Identities of Fully Degenerate Dowling Polynomials and Numbers

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Recently, Kim-Kim introduced the degenerate Whitney numbers of the first and second kind involving the degenerate Dowling polynomials and numbers. In this paper, we introduce the fully degenerate Dowling polynomials and the fully degenerate Bell polynomials and derive some identities involving these polynomials. We also obtain the generating functions, expressions, and recurrence relations for the fully degenerate Dowling polynomials and the fully degenerate Bell polynomials and other special polynomials and numbers.

1. Introduction

The definition and properties of some special polynomials are important, and many academics have studied the definitions and properties of some polynomials or degenerate polynomials [1–4]. Kim and Kim [5] introduced the degenerate Dowling polynomials and degenerate Whitney numbers of the first and second kind and obtain some explicit identities. Kim and Kim [6] introduced degenerate r-Dowling polynomials related to the degenerate r-Whitney numbers of the second kind.

Let $(L, \leq)$ be a finite lattice [5, 6], which means it is a finite poset such that every pair x, y of elements in L has a supremum $x \vee y$ and an infimum $x \wedge y$. A finite lattice L is geometric if it is a finite semimodular lattice which is also atomic. For a finite geometric lattice L of rank n, Dowling [7] defined the Whitney numbers $V_L(n,k)$ of the first kind and the Whitney numbers $W_L(n,k)$ of the second kind. In particular, if L is the Dowling lattices [5–7] $Q_n(G)$ of rank n over a finite group G of order m, then the Whitney numbers of the first kind $V_{Q_n(G)}(n,k)$ and the Whitney numbers of the second kind $W_{Q_n(G)}(n,k)$ are, respectively, denoted by $V_m(n,k)$ and $W_m(n,k)$.

At the first, we give some definitions and identities needed in this paper. For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponential functions are defined by

$$e_\lambda^x(t) = (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{1}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = x(x-\lambda)\ldots(x-(n-1)\lambda)$, $(n \geq 1)$ (see [5, 6, 8–11]).

The degenerate Whitney numbers $V_{m,\lambda}(n,k)$ and $W_{m,\lambda}(n,k)$ are defined by the following generating functions with parallel structures,

$$m^n\left(\frac{x-1}{m}\right)_n = \sum_{k=0}^{n} V_{m,\lambda}(n,k)(x)_{k,\lambda}, \tag{2}$$

$$(x)_{n,\lambda} = \sum_{k=0}^{n} W_{m,\lambda}(n,k) m^k \left(\frac{x-1}{m}\right)_k, \tag{3}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\ldots(x-n+1)$, $(n \geq 1)$ (see [5, 6]).

Equivalently, relations (2) and (3) are given by

$$m^n (x)_n = \sum_{k=0}^{n} V_{m,\lambda}(n,k)(mx+1)_{k,\lambda}, \tag{4}$$

$$(mx+1)_{n,\lambda} = \sum_{k=0}^{n} W_{m,\lambda}(n,k) m^k (x)_k. \tag{5}$$

For $n \geq 0$, the degenerate Dowling polynomials (see [5, 11]) are given by

$$e_\lambda(t) e^{x\left(e_\lambda^m(t) - 1/m\right)} = \sum_{n=0}^{\infty} D_{m,\lambda}(n,x) \frac{t^n}{n!}, \tag{6}$$

when $x = 1$, $D_{m,\lambda}(n) = D_{m,\lambda}(n,1)$ are the degenerate Dowling numbers. Note that

$$\lim_{\lambda \longrightarrow 0} D_{m,\lambda}(n,x) = D_m(n,x), \tag{7}$$

where $D_m(n,x)$ are the Dowling polynomials.

From (6), we have

$$D_{m,\lambda}(n,x) = \sum_{k=0}^{n} W_{m,\lambda}(n,k) x^k. \tag{8}$$

(see [6, 11]).

The degenerate Bell polynomials (see [5, 8–12]) are defined by

$$e^{x\left(e_\lambda(t) - 1\right)} = \sum_{n=0}^{\infty} \mathrm{Bel}_{n,\lambda}(x) \frac{t^n}{n!}, \tag{9}$$

when $x = 1$, $\mathrm{Bel}_{n,\lambda} = \mathrm{Bel}_{n,\lambda}(1)$ are the degenerate Bell numbers.

From (9), we note that

$$\mathrm{Bel}_{n,\lambda}(x) = \sum_{k=0}^{\infty} e^{-x} \frac{(k)_{n,\lambda}}{k!} x^k, \tag{10}$$

where $(k)_{0,\lambda} = 1$, $(k)_{n,\lambda} = k(k-\lambda)(k-2\lambda)\ldots(k-(n-1)\lambda)$, $(n \geq 1)$ (see [9]).

The fully degenerate Bell polynomials (see [8, 9, 13]) are defined by

$$e_\lambda\left(x\left(e_\lambda(t) - 1\right)\right) = \sum_{n=0}^{\infty} \mathrm{Bel}^*_{n,\lambda}(x) \frac{t^n}{n!}. \tag{11}$$

From [13], we have

$$\mathrm{Bel}^*_{n,\lambda}(x) = \sum_{k=0}^{n} e_\lambda(-x) \frac{(1)_{k,\lambda}(k)_{n,\lambda}}{k!} x^k. \tag{12}$$

In [9], we note that

$$\mathrm{Bel}^*_{n,\lambda}(x) = \sum_{k=0}^{n} (1)_{k,\lambda} x^k S_{2,\lambda}(n,k). \tag{13}$$

The degenerate Stirling numbers of the first kind and the second kind (see [6, 8, 9, 11, 14]) are defined by

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n,k)(x)_{k,\lambda}, \ (n \geq 0), \tag{14}$$

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_k, \ (n \geq 0). \tag{15}$$

By the inversion of (14) and (15), we have

$$\frac{1}{k!}\left(\log_\lambda(1+t)\right)^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \tag{16}$$

$$\frac{1}{k!}\left(e_\lambda(t) - 1\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}. \tag{17}$$

Recently, we are interested in the degenerate Dowling polynomials and numbers and the degenerate Bell polynomials and numbers. In this paper, we study the fully degenerate Dowling polynomials and numbers and the fully degenerate Bell polynomials and numbers. Meanwhile, we derive some identities and expressions of them.

2. Fully Degenerate Dowling Polynomials

In this section, we introduce the fully degenerate Dowling polynomials. We also show several identities and properties related to the fully degenerate Dowling polynomials and numbers.

The fully degenerate Dowling polynomials are defined by

$$e_\lambda(t) e_\lambda\left(x \frac{e_\lambda^m(t) - 1}{m}\right) = \sum_{n=0}^{\infty} D^*_{m,\lambda}(n,x) \frac{t^n}{n!}, \tag{18}$$

when $x = 1$, $D^*_{m,\lambda}(n) = D^*_{m,\lambda}(n,1)$ are called the fully degenerate Dowling numbers.

The four identities in the following lemma can be shown just as in Theorem 1, Corollary 1, Theorem 10, and Theorem 17 of [5].

Lemma 1. *For $k \geq 0$,*

$$e_\lambda(t) \frac{1}{k!}\left(\frac{e_\lambda^m(t) - 1}{m}\right)^k = \sum_{n=k}^{\infty} W_{m,\lambda}(n,k) \frac{t^n}{n!}. \tag{19}$$

Lemma 2. *For $n, k \geq 0$ with $n \geq k$,*

$$W_{1,\lambda}(n,k) = S_{2,\lambda}(n+1,k+1) + \lambda n S_{2,\lambda}(n,k+1). \tag{20}$$

Lemma 3. *For $0 \leq k \leq n$,*

$$W_{m+1,\lambda}(n,k) = \frac{1}{(m+1)^k m^{n-k}} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (m+1)^j \langle 1 \rangle_{n-j,m\lambda} W_{m,(m/m+1)\lambda}(j,k). \tag{21}$$

Lemma 4. *For $n, k \geq 0$ with $n \geq k$,*

$$W_{m,\lambda}(n,k) = \sum_{i=k}^{n} \binom{n}{i} m^{i-k} (1)_{n-i,\lambda} S_{2,\lambda/m}(i,k). \quad (22)$$

Theorem 5. *For $n \geq 0$, we have*

$$D^{*}_{m,\lambda}(n,x) = \sum_{k=0}^{n} W_{m,\lambda}(n,k) x^{k} (1)_{k,\lambda}. \quad (23)$$

Proof. By Lemma 1 and (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D^{*}_{m,\lambda}(n,x)\frac{t^n}{n!} &= e_\lambda(t) e_\lambda\left(x\frac{e_\lambda^m(t)-1}{m}\right)\\ &= e_\lambda(t)\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{k!}\left(x\frac{e_\lambda^m(t)-1}{m}\right)^k\\ &= \sum_{k=0}^{\infty} x^k (1)_{k,\lambda} e_\lambda(t)\frac{1}{k!}\left(\frac{e_\lambda^m(t)-1}{m}\right)^k\\ &= \sum_{k=0}^{\infty} x^k (1)_{k,\lambda}\sum_{n=k}^{\infty} W_{m,\lambda}(n,k)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} W_{m,\lambda}(n,k) x^k (1)_{k,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \quad (24)$$

So, from (24), we obtain Theorem 5. □

Theorem 6. *For $n \geq 0$, we have*

$$D^{*}_{m,\lambda}(n,x) = \sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}\frac{\langle -1\rangle_{l,\lambda}(-1)^k x^l}{m^l l!}(mk+1)_{n,\lambda}. \quad (25)$$

Proof. From (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D^{*}_{m,\lambda}(n,x)\frac{t^n}{n!} &= e_\lambda(t) e_\lambda\left(x\frac{e_\lambda^m(t)-1}{m}\right)\\ &= e_\lambda(t)\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}}{l!}\left(x\frac{e_\lambda^m(t)-1}{m}\right)^l\\ &= e_\lambda(t)\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}x^l}{m^l l!}\left(e_\lambda^m(t)-1\right)^l\\ &= e_\lambda(t)\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}x^l}{m^l l!}\sum_{k=0}^{l}\binom{l}{k}e_\lambda^{mk}(t)(-1)^{l-k}\\ &= \sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}x^l}{m^l l!}\sum_{k=0}^{l}\binom{l}{k}(-1)^{l-k}e_\lambda^{mk+1}(t)\end{aligned}$$

$$= \sum_{l=0}^{\infty} \frac{(1)_{l,\lambda} x^l}{m^l l!} \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} \sum_{n=0}^{\infty} (mk+1)_{n,\lambda} \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{k=0}^{l} \binom{l}{k} \frac{(1)_{l,\lambda} x^l (-1)^{l-k}}{m^l l!} (mk+1)_{n,\lambda} \right) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{k=0}^{l} \binom{l}{k} \frac{\langle -1 \rangle_{l,\lambda} (-1)^k x^l}{m^l l!} (mk+1)_{n,\lambda} \right) \frac{t^n}{n!}. \tag{26}$$

From (26), we obtain Theorem 6. □

In particular, when $x = 1$, we note that

$$D^*_{m,\lambda}(n) = \sum_{l=0}^{n} \sum_{k=0}^{l} \binom{l}{k} \frac{\langle -1 \rangle_{l,\lambda} (-1)^k}{m^l l!} (mk+1)_{n,\lambda}. \tag{27}$$

Theorem 7. *For $n \geq 0$, we have*

$$D^*_{m,\lambda}(n,x) = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} (1)_{n-k,\lambda} (1)_{l,\lambda} x^l m^{k-l} S_{2,\lambda/m}(k,l). \tag{28}$$

Proof. From (18), we get

$$\sum_{n=0}^{\infty} D^*_{m,\lambda}(n,x) \frac{t^n}{n!} = e_\lambda(t) e_\lambda\left(x \frac{e^m_\lambda(t) - 1}{m} \right)$$
$$= e_\lambda(t) \sum_{l=0}^{\infty} \frac{(1)_{l,\lambda}}{l!} \left(x \frac{e^m_\lambda(t) - 1}{m} \right)^l$$
$$= e_\lambda(t) \sum_{l=0}^{\infty} \frac{(1)_{l,\lambda} x^l}{m^l l!} (e_{\lambda/m}(mt) - 1)^l$$
$$= e_\lambda(t) \sum_{l=0}^{\infty} \frac{(1)_{l,\lambda} x^l}{m^l} \sum_{k=l}^{\infty} S_{2,\lambda/m}(k,l) \frac{(mt)^k}{k!} \tag{29}$$
$$= \sum_{j=0}^{\infty} (1)_{j,\lambda} \frac{t^j}{j!} \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} (1)_{l,\lambda} x^l m^{k-l} S_{2,\lambda/m}(k,l) \right) \frac{t^k}{k!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} (1)_{n-k,\lambda} (1)_{l,\lambda} x^l m^{k-l} S_{2,\lambda/m}(k,l) \right) \frac{t^n}{n!}.$$

So, by (29), we obtain Theorem 7. □

Theorem 8. *For $n \geq 0$, we have*

$$D^*_{m,\lambda}(n,x) = \sum_{k=0}^{n} \sum_{l=0}^{k} \sum_{j=0}^{l} \binom{n}{k} (1)_{n-k,\lambda} (1)_{l,\lambda} x^l m^{j-l} \lambda^{k-j} S_1(k,j) S_2(j,l). \tag{30}$$

Proof. By (18), we get

$$\begin{aligned}\sum_{n=0}^{\infty} D_{m,\lambda}^{*}(n,x)\frac{t^n}{n!} &= e_\lambda(t)e_\lambda\left(x\frac{e_\lambda^m(t)-1}{m}\right)\\
&= e_\lambda(t)\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}x^l}{m^l l!}\left(e_\lambda^m(t)-1\right)^l\\
&= e_\lambda(t)\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}x^l}{m^l l!}\left(e^{(m\log(1+\lambda t))/\lambda}-1\right)^l\\
&= e_\lambda(t)\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}x^l}{m^l}\sum_{j=l}^{\infty}S_2(j,l)\frac{(m\log(1+\lambda t))^j}{\lambda^j j!}\\
&= e_\lambda(t)\sum_{l=0}^{\infty}\frac{(1)_{l,\lambda}x^l}{m^l}\sum_{j=l}^{\infty}S_2(j,l)\frac{m^j}{\lambda^j}\sum_{k=j}^{\infty}S_1(k,j)\frac{(\lambda t)^k}{k!}\\
&= e_\lambda(t)\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\sum_{j=0}^{l}(1)_{l,\lambda}x^l m^{j-l}\lambda^{k-j}S_1(k,j)S_2(j,l)\right)\frac{t^k}{k!}\\
&= \sum_{i=0}^{\infty}(1)_{i,\lambda}\frac{t^i}{i!}\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\sum_{j=0}^{l}(1)_{l,\lambda}x^l m^{j-l}\lambda^{k-j}S_1(k,j)S_2(j,l)\right)\frac{t^k}{k!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j=0}^{l}\binom{n}{k}(1)_{n-k,\lambda}(1)_{l,\lambda}x^l m^{j-l}\lambda^{k-j}S_1(k,j)S_2(j,l)\right)\frac{t^n}{n!}.\end{aligned} \tag{31}$$

From (31), we get Theorem 8. □

In particular, when $x = 1$, we have

$$D_{m,\lambda}^{*}(n) = \sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{j=0}^{l}\binom{n}{k}(1)_{n-k,\lambda}(1)_{l,\lambda}m^{j-l}\lambda^{k-j}S_1(k,j)S_2(j,l). \tag{32}$$

Theorem 9. *For $n \geq 0$,*

$$D_{m+1,\lambda}^{*}(n,x) = \frac{1}{m^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}(m+1)^j\langle 1\rangle_{n-j,m\lambda}D_{m,(m/m+1)\lambda}^{*}\left(j,\frac{m}{m+1}x\right). \tag{33}$$

Proof. From Theorem 5 and Lemma 3, we note that

$$\begin{aligned}D_{m+1,\lambda}^{*}(n,x) &= \sum_{k=0}^{n}W_{m+1,\lambda}(n,k)x^k(1)_{k,\lambda}\\
&= \sum_{k=0}^{n}\left\{\frac{1}{(m+1)^k m^{n-k}}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}(m+1)^j\langle 1\rangle_{n-j,m\lambda}W_{m,(m/m+1)\lambda}(j,k)\right\}x^k(1)_{k,\lambda}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}(m+1)^j\langle 1\rangle_{n-j,m\lambda}\sum_{k=0}^{n}W_{m,(m/m+1)\lambda}(j,k)\left(\frac{m}{m+1}x\right)^k(1)_{k,\lambda}\\
&= \frac{1}{m^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}(m+1)^j\langle 1\rangle_{n-j,m\lambda}D^*_{m,(m/m+1)\lambda}\left(j,\frac{m}{m+1}x\right).
\end{aligned}
\tag{34}
$$

From (34), we obtain Theorem 9. □

Theorem 10. *For* $n \geq 0$,

$$
D^*_{1,\lambda}(n) = \sum_{k=0}^{n}(1)_{k,\lambda}S_{2,\lambda}(n+1,k+1) + \lambda n\sum_{k=0}^{n}(1)_{k,\lambda}S_{2,\lambda}(n,k+1). \tag{35}
$$

Proof. By Theorem 5 and Lemma 2, let $x = 1, m = 1$, we have

$$
\begin{aligned}
D^*_{1,\lambda}(n) &= \sum_{k=0}^{n}W_{1,\lambda}(n,k)(1)_{k,\lambda}\\
&= \sum_{k=0}^{n}(1)_{k,\lambda}\left\{S_{2,\lambda}(n+1,k+1) + \lambda n S_{2,\lambda}(n,k+1)\right\}\\
&= \sum_{k=0}^{n}(1)_{k,\lambda}S_{2,\lambda}(n+1,k+1) + \lambda n\sum_{k=0}^{n}(1)_{k,\lambda}S_{2,\lambda}(n,k+1).
\end{aligned}
\tag{36}
$$

By (36), we obtain Theorem 10. □

Theorem 11. *For* $n \geq 0$,

$$
D^*_{m,\lambda}(n,x) = \sum_{i=0}^{n}\binom{n}{i}m^i(1)_{n-i,\lambda}\mathrm{Bel}^*_{i,\lambda/m}\left(\frac{x}{m}\right). \tag{37}
$$

Proof. By (13), Theorem 5 and Lemma 4, we note that

$$
\begin{aligned}
D^*_{m,\lambda}(n,x) &= \sum_{k=0}^{n}W_{m,\lambda}(n,k)x^k(1)_{k,\lambda}\\
&= \sum_{k=0}^{n}\sum_{i=k}^{n}\binom{n}{i}m^{i-k}(1)_{n-i,\lambda}S_{2,\lambda/m}(i,k)x^k(1)_{k,\lambda}\\
&= \sum_{i=0}^{n}\binom{n}{i}m^i(1)_{n-i,\lambda}\sum_{k=0}^{i}S_{2,\lambda/m}(i,k)\left(\frac{x}{m}\right)^k(1)_{k,\lambda}\\
&= \sum_{i=0}^{n}\binom{n}{i}m^i(1)_{n-i,\lambda}\mathrm{Bel}^*_{i,\lambda/m}\left(\frac{x}{m}\right).
\end{aligned}
\tag{38}
$$

By (38), we obtain Theorem 11. □

From [13], we have

$$
\frac{d}{dx}e_\lambda(-x) = -\frac{1}{1-\lambda x}e_\lambda(-x). \tag{39}
$$

We note that

$$
\begin{aligned}
\frac{d}{dt}e^m_\lambda(t) &= \frac{d}{dt}\sum_{k=0}^{\infty}(m)_{k,\lambda}\frac{t^k}{k!} = \sum_{k=0}^{\infty}(m)_{k,\lambda}\frac{t^{k-1}}{(k-1)!}\\
&= \sum_{k=0}^{\infty}(m)_{k,\lambda}\frac{t^k}{k!}(m-k\lambda)\\
&= m\sum_{k=0}^{\infty}(m)_{k,\lambda}\frac{t^k}{k!} - k\lambda\sum_{k=0}^{\infty}(m)_{k,\lambda}\frac{t^k}{k!}\\
&= me^m_\lambda(t) - t\lambda\sum_{k=1}^{\infty}(m)_{k,\lambda}\frac{t^{k-1}}{(k-1)!}\\
&= me^m_\lambda(t) - t\lambda\frac{d}{dt}e^m_\lambda(t).
\end{aligned}
\tag{40}
$$

From (40), we note that

$$
\frac{d}{dt}e^m_\lambda(t) = \frac{m}{1+\lambda t}e^m_\lambda(t) = me^{m-\lambda}_\lambda(t). \tag{41}
$$

So, we have

$$\begin{aligned}
\frac{d}{dt}e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{m^k k!}\left(e_\lambda^m(t)-1\right)^k \\
&= \sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{m^k (k-1)!}\left(e_\lambda^m(t)-1\right)^{k-1}\frac{d}{dt}e_\lambda^m(t) \\
&= \sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{m^{k-1}(k-1)!}\left(e_\lambda^m(t)-1\right)^{k-1}e_\lambda^{m-\lambda}(t) \\
&= \sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{m^k k!}\left(e_\lambda^m(t)-1\right)^k e_\lambda^{m-\lambda}(t)(1-k\lambda) \\
&= \sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{m^k k!}\left(e_\lambda^m(t)-1\right)^k e_\lambda^{m-\lambda}(t) - \lambda\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{m^k(k-1)!}\left(e_\lambda^m(t)-1\right)^k e_\lambda^{m-\lambda}(t) \\
&= e_\lambda^{m-\lambda}(t)e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) - \frac{\lambda}{m}\sum_{k=0}^{\infty}\frac{(1)_{k,\lambda}}{m^{k-1}(k-1)!}\left(e_\lambda^m(t)-1\right)^{k-1}e_\lambda^{m-\lambda}(t)\left(e_\lambda^m(t)-1\right) \\
&= e_\lambda^{m-\lambda}(t)e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) - \frac{\lambda}{m}\left(e_\lambda^m(t)-1\right)\frac{d}{dt}e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right).
\end{aligned} \tag{42}$$

From (42), we have

$$\frac{d}{dt}e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) = \frac{e_\lambda^{m-\lambda}(t)}{1+\lambda\left(e_\lambda^m(t)-1\right)/m}e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) = e_\lambda^{m-\lambda}(t)e_\lambda^{1-\lambda}\left(\frac{e_\lambda^m(t)-1}{m}\right). \tag{43}$$

Theorem 12. *For $n\geq 0$,*

$$D_{m,\lambda}^*(n+1) = \sum_{l=0}^{n}\binom{n}{l}(n-l)!(-\lambda)^{n-l}\left(D_{m,\lambda}^*(l) + \sum_{i=0}^{l}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{l}{i}\binom{k}{p}\frac{\lambda^k(-1)^p}{m^{k-i}}D_{m,\lambda}^*(l-i)(p+1)_{i,\lambda/m}\right). \tag{44}$$

Proof. By (18), (41), and (43), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}D_{m,\lambda}^*(n+1)\frac{t^n}{n!} &= \frac{d}{dt}e_\lambda(t)e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) \\
&= \frac{e_\lambda(t)}{1+\lambda t}e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) + e_\lambda(t)\frac{e_\lambda^{m-\lambda}(t)}{1+\lambda/m\left(e_\lambda^m(t)-1\right)}e_\lambda\left(\frac{e_\lambda^m(t)-1}{m}\right) \\
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty}D_{m,\lambda}^*(l)\frac{t^l}{l!} + e_\lambda^{m-\lambda}(t)e_\lambda^{-\lambda}\left(\frac{e_\lambda^m(t)-1}{m}\right)\sum_{j=0}^{\infty}D_{m,\lambda}^*(j)\frac{t^j}{j!}
\end{aligned}$$

$$
\begin{aligned}
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty} D_{m,\lambda}^{*}(l)\frac{t^l}{l!} + e_\lambda^{m-\lambda}(t)\sum_{j=0}^{\infty} D_{m,\lambda}^{*}(j)\frac{t^j}{j!}\sum_{k=0}^{\infty}\frac{(-\lambda)_{k,\lambda}}{m^k k!}\left(e_\lambda^{m}(t)-1\right)^k \\
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty} D_{m,\lambda}^{*}(l)\frac{t^l}{l!} + e_\lambda^{m-\lambda}(t)\sum_{j=0}^{\infty} D_{m,\lambda}^{*}(j)\frac{t^j}{j!}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{m^k}\sum_{p=0}^{k}\binom{k}{p}e_\lambda^{mp}(t)(-1)^{k-p} \\
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty} D_{m,\lambda}^{*}(l)\frac{t^l}{l!} + e_\lambda^{-\lambda}(t)\sum_{j=0}^{\infty} D_{m,\lambda}^{*}(j)\frac{t^j}{j!}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\lambda^k(-1)^p}{m^k}\binom{k}{p}e_\lambda^{mp+m}(t) \\
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty} D_{m,\lambda}^{*}(l)\frac{t^l}{l!} + e_\lambda^{-\lambda}(t)\sum_{j=0}^{\infty} D_{m,\lambda}^{*}(j)\frac{t^j}{j!}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\frac{\lambda^k(-1)^p}{m^k}\binom{k}{p}\sum_{i=0}^{\infty}(mp+m)_{i,\lambda}\frac{t^i}{i!} \\
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty} D_{m,\lambda}^{*}(l)\frac{t^l}{l!} + e_\lambda^{-\lambda}(t)\sum_{j=0}^{\infty} D_{m,\lambda}^{*}(j)\frac{t^j}{j!}\sum_{i=0}^{\infty}\left(\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{k}{p}\frac{\lambda^k(-1)^p}{m^k}(mp+m)_{i,\lambda}\right)\frac{t^i}{i!} \\
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty} D_{m,\lambda}^{*}(l)\frac{t^l}{l!} + e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty}\left(\sum_{i=0}^{l}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{l}{i}\binom{k}{p}\frac{\lambda^k(-1)^p}{m^k}D_{m,\lambda}^{*}(l-i)(mp+m)_{i,\lambda}\right)\frac{t^l}{l!} \\
&= e_\lambda^{-\lambda}(t)\sum_{l=0}^{\infty}\left(D_{m,\lambda}^{*}(l) + \sum_{i=0}^{l}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{l}{i}\binom{k}{p}\frac{\lambda^k(-1)^p}{m^k}D_{m,\lambda}^{*}(l-i)(mp+m)_{i,\lambda}\right)\frac{t^l}{l!} \\
&= \sum_{g=0}^{\infty}(-\lambda)^g t^g\sum_{l=0}^{\infty}\left(D_{m,\lambda}^{*}(l) + \sum_{i=0}^{l}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{l}{i}\binom{k}{p}\frac{\lambda^k(-1)^p}{m^k}D_{m,\lambda}^{*}(l-i)(mp+m)_{i,\lambda}\right)\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{n}{l}(n-l)!(-\lambda)^{n-l}\left(D_{m,\lambda}^{*}(l) + \sum_{i=0}^{l}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{l}{i}\binom{k}{p}\frac{\lambda^k(-1)^p}{m^k}D_{m,\lambda}^{*}(l-i)(mp+m)_{i,\lambda}\right)\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{l=0}^{n}\binom{n}{l}(n-l)!(-\lambda)^{n-l}\left(D_{m,\lambda}^{*}(l) + \sum_{i=0}^{l}\sum_{k=0}^{i}\sum_{p=0}^{k}\binom{l}{i}\binom{k}{p}\frac{\lambda^k(-1)^p}{m^{k-i}}D_{m,\lambda}^{*}(l-i)(p+1)_{i,\lambda/m}\right)\right\}\frac{t^n}{n!}.
\end{aligned}
\tag{45}
$$

Therefore, from (45), we obtain Theorem 12. □

3. Remarks

Recently, the degenerate Poisson random variable X_λ with parameter $\alpha>0$, probability mass function is given by (see [15])

$$p_\lambda(i) = e_\lambda^{-1}(\alpha)\frac{\alpha^i}{i!}(1)_{i,\lambda}, \quad (i=1,2,3,\ldots). \tag{46}$$

Now, we would like to give the relation between the fully degenerate Dowling polynomials and degenerate Poisson random variables. So, we slightly modify our fully degenerate Dowling polynomials which are given by

$$e_\lambda(t)e_\lambda^{-1}\left(\frac{x}{m}\right)e_\lambda\left(\frac{x}{m}e_\lambda^{m}(t)\right) = \sum_{n=0}^{\infty} D_{m,\lambda}^{*}(n,x)\frac{t^n}{n!}. \tag{47}$$

Theorem 13

$$E\left[(mX_\lambda+1)_{n,\lambda}\right] = D_{m,\lambda}^{*}(n,\alpha),\ (n\geq 0), \tag{48}$$

where X_λ is the degenerate Poisson random variable with parameters α/m.

Proof. Let X_λ be the degenerate Poisson random variable with parameters α/m, then we have

$$
\begin{aligned}
E\left[e_\lambda^{mX_\lambda+1}(t)\right] &= \sum_{k=0}^{\infty} e_\lambda^{mk+1}(t)p_\lambda(k) \\
&= e_\lambda(t)e_\lambda^{-1}\left(\frac{\alpha}{m}\right)\sum_{k=0}^{\infty} e_\lambda^{mk}(t)\frac{(1)_{k,\lambda}}{k!}\left(\frac{\alpha}{m}\right)^k \\
&= e_\lambda(t)e_\lambda^{-1}\left(\frac{\alpha}{m}\right)e_\lambda\left(\frac{\alpha}{m}e_\lambda^{m}(t)\right) \\
&= \sum_{n=0}^{\infty} D_{m,\lambda}^{*}(n,\alpha)\frac{t^n}{n!}.
\end{aligned}
\tag{49}
$$

On the other hand, we easily get

$$E\left[e_\lambda^{mX_\lambda+1}(t)\right] = \sum_{n=0}^{\infty} E\left[(mX_\lambda+1)_{n,\lambda}\right]\frac{t^n}{n!}. \tag{50}$$

So, by (49) and (50), we obtain Theorem 13. □

4. Conclusion

In this paper, we introduced the fully degenerate Dowling polynomials $D^{*}_{m,\lambda}(n,x)$ and the fully degenerate Bell polynomials $\mathrm{Bel}^{*}_{n,\lambda}(x)$, which are degenerate versions of the Dowling polynomials $D_m(n,x)$ and the Bell polynomials $\mathrm{Bel}_n(x)$.

We showed the fully degenerate Dowling polynomials with Whitney numbers in Theorem 5. We used different methods for the fully degenerate Dowling polynomials to obtain some identities related to the Stirling numbers of the first kind and the second kind, the degenerate Stirling numbers of the second kind and the fully degenerate Bell polynomials in Theorems 6–8 and 11. Meanwhile, we investigated the recurrence relations for the fully degenerate Dowling polynomials in Theorem 9. In particular, we let $x = 1$ and $m = 1$, and we have also obtained the special relation between the fully degenerate Dowling polynomials and the degenerate Stirling numbers of the second kind in Theorem 10. Furthermore, we used the differential equation to obtain the identities associated with the fully degenerate Dowling polynomials in Theorem 12. Assume that X_λ is the degenerate Poisson random variable with parameter α/m, we showed that the Poisson degenerate central moments $E[(mX_\lambda + 1)_{n,\lambda}]$ is equal to $D^{*}_{m,\lambda}(n,\alpha)$ in Theorem 13.

This has profound implications for us to continue to study various degenerate versions of many special polynomials and numbers in the future.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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