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Research Article

Conditional Matching Preclusion Number of Graphs

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The conditional matching preclusion number of a graph G, denoted by $mp_1(G)$, is the minimum number of edges whose deletion results in the graph with no isolated vertices that has neither perfect matching nor almost-perfect matching. In this paper, we first give some sharp upper and lower bounds of conditional matching preclusion number. Next, the graphs with large and small conditional matching preclusion numbers are characterized, respectively. In the end, we investigate some extremal problems on conditional matching preclusion number.

1. Introduction

All graphs are undirected, finite, and simple in this paper, refer to the book [1] for notation and terminology not described here. If a cycle contains every vertex of G exactly once, then we called it is a Hamiltonian cycle of G. A connected graph G is Hamiltonian if it exists a Hamiltonian cycle in G. Furthermore, if there exists a Hamiltonian path between any two vertices of G, then G is said Hamiltonian connected. Denoted $E(V_1, V_2)$ to be a edges set has one endpoint in V_1 and another in V_2. If an edge subset F satisfies $G - F$ has neither perfect matching nor almost-perfect matching, then F is a matching preclusion set (MP for short) of G. Denoted $mp(G)$ is the minimum number of edges of all MP set in G. The concept of matching preclusion was introduced in [2] and further studied in [3–18]. Some distributed algorithms require each vertex of the system to be matched by a neighbour vertex, and the matching preclusion number measures the robustness of a graph as a communications network topology. Meanwhile, matching preclusion number has a theoretical connection with conditional connectivity and "changing and unchanging of invariants." In a network, a vertex with a special matching vertex after edge failure any time implies that tasks running on a fault vertex can be change into its matching vertex. Therefore, under this fault assumption, larger mp (G) signifies higher fault tolerance. However, in the network, the probability that the adjacent vertices of the same vertex fail at the same time is very small. So the question is if we delete edges, what are the basic obstructions to a perfect matching or an almost-perfect matching in the resulting graph if no isolated vertices are created. This motivates our next definition. If an edge subset F satisfies $G - F$ has no isolated vertices and has neither perfect matching nor almost-perfect matching, then F is a conditional matching preclusion set (CMP for short). Denoted $mp_1(G)$ is the minimum number of all CMP set. At present, there have been some discussions about the conditional matching preclusion number of special graphs. We mainly want to discuss the conditional matching preclusion number of general graphs. We consider the following three problems in this paper.

Problem 1. Compute the minimum integer $s(n,k) = \min\{|E(G)|: G \in \psi(n,k)\}$, where $\psi(n,k)$ be the set of all graphs of order n with conditional matching preclusion number k

Problem 2. Compute the minimum integer $f(n,k)$ such that for every connected graph G of order n, if $|E(G)| \geq f(n,k)$, then $mp_1(G) \geq k$

Problem 3. Compute the maximum integer $g(n,k)$ such that for every connected graph G of order n, if $|E(G)| \leq g(n,k)$, then $mp_1(G) \leq k$

A basic obstruction to a perfect matching of even graph with no isolated vertex will be the existence of a path $u - w - v$ where the degrees of u and v are 1. Define $v_e(G) = \min\{d_G(u) + d_G(v) - 2 - y_G(u,v)$: u and v are distance 2 apart$\}$, if u and v are adjacent, then $y_G(u,v) = 1$, and 0 otherwise.

Proposition 1 (see [7]). *An even graph G with $\delta(G) \geq 3$, then $mp_1(G) \leq v_e(G)$.*

The basic obstruction to an almost-perfect matching of odd graph with no isolated vertex will be the existence of a three vertices of degree 1 with a common neighbour. Define $v_0(G) = \min\{|F|$: $G - F$ has no isolated vertices and has 3 leaves adjacent to the same vertex$\}$.

Proposition 2 (see [7]). *An odd graph G has $mp_1(G) \leq v_0(G)$.*

For odd vertices bipartite graph $G = (U \cup V, E)$, we have $v_0'(G) = \min\{|F|$: $G - F$ has no isolated vertices and has 2 leaves adjacent to the same vertex in $V\}$.

Proposition 3 (see [7]). *Let G be a bipartite graph with an odd number of vertices. Then, $mp_1(G) \leq v_0'(G)$.*

If H is a spanning subgraph of G, we want to know whether we have $mp_1(H) \leq mp_1(G)$ holds.

Remark 1. For an even number $n (n \geq 12)$, let G be a graph obtained from a $K_3 = u_1v_1w_1u_1$ and a clique K_{n-3} by adding three edges u_1u_2, v_1v_2, w_1w_2, where $u_2, v_2, w_2 \in V(K_{n-3})$. Let H be a graph obtained from G by deleting three edges of K_3. It is clear that H is a spanning subgraph of G. By deleting three edges u_1u_2, v_1v_2, w_1w_2 in G, we can see that there is no perfect matching in the resulting graph; hence, $mp_1(G) \leq 3$. From the definition of $mp_1(H)$, for any $X \subseteq E(H)$, to avoid isolated vertices in $H - X$, we have $X \cap \{u_1u_2, v_1v_2, w_1w_2\} = \varnothing$ and $X \subseteq E(K_{n-3})$, so $mp_1(H) \geq 4 > 3 \geq mp_1(G)$.

Remark 2. For an odd number $n (n \geq 15)$, let G be a graph obtained from a clique K_4 with vertex set $\{u_1, v_1, w_1, t_1\}$ and a clique K_{n-4} by adding four edges $u_1u_2, v_1v_2, w_1w_2, t_1t_2$ between them, where $u_2, v_2, w_2, t_2 \in V(K_{n-4})$. Let H be a graph obtained from G by deleting six edges of the K_4. By deleting $\{u_1u_2, v_1v_2, w_1w_2, t_1t_2\}$ and three edges in $E(K_4)$, we can find there no perfect matching in the resulting graph; hence, $mp_1(G) \leq 7$. For any $X \subseteq E(H)$, to avoid isolated vertices in $H - X$, we have $X \cap \{u_1u_2, v_1v_2, w_1w_2, t_1t_2\} = \varnothing$ and $X \subseteq E(K_{n-4})$, so $mp_1(H) \geq 8 > 7 \geq mp_1(G)$.

2. Sharp Bounds for CMP Number

Lemma 1 (see [2]). *Let $n \geq 2$. Then, $mp(K_{n,n}) = n$, $K_{n,n}$ is super matched.*

Lemma 2 (see [19]). *Let $n \geq 3$. Then, $mp_1(K_{n,n}) = 2n - 2$, $K_{n,n}$ is conditionally super matched.*

Theorem 1 (see [19])

(1) Let $n \geq 4$ be even, then

$$mp_1(K_n) = \begin{cases} \dfrac{(n^2 + 2n)}{8}, & \text{if } n \in \{4, 6, 8\}, \\ 2n - 5, & \text{if } n \geq 10. \end{cases} \quad (1)$$

(2) Let $n \geq 5$ be odd, then

$$mp_1(K_n) = \begin{cases} \dfrac{(n^2 + 4n + 3)}{8}, & \text{if } n \in \{5, 7, 9, 11, 13\}, \\ 3n - 9, & \text{if } n \geq 15. \end{cases} \quad (2)$$

Proposition 4. *Let G be a graph with an even number of vertices and $\delta \geq 3$. Then, $mp_1(G) \leq \delta + \Delta - 2$.*

Proof. Let u be the minimum-degree vertex in G. Since $\delta(G) \geq 3$, it exists a path $u - v - w$ in G. Let $X = E[\{u, w\}, N_G(\{u, w\}) \setminus v]$, then $G - X$ exists a path $u - v - w$ and $d_G(u) = d_G(v) = 1$. By Proposition 1, $mp_1(G) \leq d(u) + d(w) - 2 \leq \delta + \Delta - 2$.

By Lemma 2, $mp_1(K_{n,n}) = 2n - 2, \Delta(K_{n,n}) = \delta(K_{n,n}) = n$, we get $\delta(K_{n,n}) + \Delta(K_{n,n}) - 2 = 2n - 2$ for $n \geq 3$. □

Proposition 5. *Let G be a graph with an odd number of vertices and $\delta \geq 4$. Then, $mp_1(G) \leq \delta + 2\Delta - 3$.*

Proof. Let u be the minimum-degree vertex of G and $ux \in E(G)$. Since $d(x) \geq 4$, there exist two vertices v, w and two edges $xv, xw \in E(G)$. Let $X = E[\{u, v, w\}, N_G(\{u, v, w\}) \setminus x]$, then $G - X$ has no isolated vertices and has 3 leaves adjacent to the same vertex x. By Proposition 2, $mp_1(G) \leq d(u) + d(v) + d(w) - 3 \leq \delta + 2\Delta - 3$. □

3. Graphs with Given CMP Numbers

Theorem 2 (see [20]). *For a graph with order n, if $\delta \geq (n/2)$, then G is Hamiltonian.*

Lemma 3 (see [19]). *A graph G has a perfect matching if and only if G satisfies Tutte's condition, that is, $o(G - S) \leq |S|$.*

Lemma 4 (see [19]). *A graph G has an almost-perfect matching if and only if every subset vertices satisfies Berge's condition, that is, $o(G - S) \leq |S| + 1$.*

The graphs with $mp_1(G) = 2n - k (k \geq 5)$ can be characterized completely.

Proposition 6. *The even graph G has order of $n \geq 10$, and $\delta(G) \geq 3$. Then, $mp_1(G) = 2n - 5$ if and only if $G \cong K_n$.*

Proof. From Theorem 1, we know $mp_1(G) = 2n - 5$ when G is a complete graph. Conversely, if $mp_1(G) = 2n - 5$ and

$G \not\cong K_n$, then there exist $uv \notin E(G)$. If w is a common neighbour vertex of u, v, let $X = E_G[\{u, v\}, N_G(\{u, v\}) \setminus \{w\}]$. So there exists a path $u - w - v$ in $G - X$ satisfies $d_G(u) = d_G(v) = 1$. Since $\delta(G) \geq 3$, $G - X$ has no isolated vertices and no perfect matching exist. Since $|X| = d_G(u) + d_G(v) - 2 \leq 2n - 6$, it means $mp_1(G) \leq |X| \leq 2n - 6$, which contradicts to $mp_1(G) = 2n - 5$. If no exist a common neighbour vertex for u and v, we choose a shortest path between u and v, say $P{:} = ua_1a_2 \ldots a_s v (ua_s \notin E(G))$. Let $X = E_G[\{u, a_2\}, N_G(\{u, a_2\}) \setminus \{a_1\}]$, then there exists a path $u - a_1 - a_2$ in $G - X$ and $d_G(u) = d_G(a_2) = 1$. Since $|X| = d(u) + d(a_2) - 2 \leq n - 3 + n - 2 - 2 = 2n - 7$, then $mp_1(G) \leq |X| \leq 2n - 7$. So $G \cong K_n$. □

Theorem 3. *An even graph G has order $n \geq 8k - 12$ and $\delta(G) \geq 3$. Then, $mp_1(G) = 2n - k$ if and only if $\delta(G) = n + 4 - k$ and $5 \leq k \leq n + 1$.*

Proof. From Proposition 6, we only need to consider $6 \leq k \leq n + 1$. When $mp_1(G) = 2n - k$, we first suppose $\delta(G) \leq n + 3 - k$ and choose a minimum-degree vertex u. Since $\delta(G) \geq 3$, there exists a path $u - v - w$ in G. Let $X = E[\{u, w\}, N_G(\{u, w\}) \setminus \{v\}]$, then there exists a path $u - v - w$ in $E(G) - X$ and $d(u) = d(v) = 1$. If $uw \in E(G)$, then $|X| \leq n + 3 - k - 2 + n - 1 - 2 + 1 = 2n - k - 1$, and we have $mp_1(G) \leq |X| \leq 2n - k - 1$. If $uw \notin E(G)$, then $|X| \leq n + 3 - k - 1 + n - 1 - 2 = 2n - k - 1$, and we have $mp_1(G) \leq |X| \leq 2n - k - 1$. Thus, $\delta(G) \geq n + 4 - k$. Let $\delta(G) \geq n + 4 - k$, and let $\delta(G) = n + 4 - k + t$ and $t \geq 1$. By induction on k, we prove that $mp_1(G) = 2n - k$ if and only if $\delta(G) = n + 4 - k$. From Proposition 6, the result holds for $k = 5$. Suppose that the argument is true for every integer $k'(k' < k)$, that is, $mp_1(G) = 2n - k'$ if and only if $\delta(G) = n + 4 - k'$. Since $\delta(G) = n + 4 - k + t = n + 4 - (k - t)$ and $k - t < k$, then $mp_1(G) = 2n - (k - t)$, it contradicts to $mp_1(G) = 2n - k$. So $\delta(G) = n + 4 - k$.

Conversely, suppose $\delta(G) = n + 4 - k$. We prove that $mp_1(G) = 2n - k$. Let $u \in V(G)$. Since $\delta(G) \geq 3$, it follows that there exists a path $u - v - w$ in G. Let $X = E[\{u, w\}, N_G(\{u, w\}) \setminus v]$, then $E(G) - X$ exists a path $u - v - w$ and $d(u) = d(w) = 1$. If $uw \in E(G)$, then $|X| \leq n + 4 - k - 2 + n - 1 - 2 + 1 = 2n - k$, and we have $mp_1(G) \leq |X| \leq 2n - k$. If $uw \notin E(G)$, then $|X| \leq n + 4 - k - 1 + n - 1 - 2 = 2n - k$, and we have $mp_1(G) \leq |X| \leq 2n - k$. Thus, $mp_1(G) \leq 2n - k$.

Now, we prove that $mp_1(G) \geq 2n - k$. Suppose $\deg_{G[X]}(v) \leq (n/2) + 4 - k$ for every $v \in V(G)$, $\deg_{G-X}(v) = \deg_G(v) - \deg_{G[X]}(v) \geq n + 4 - k - (n/2) - 4 + k = (n/2)$. By Theorem 2, $G - X$ has a Hamiltonian cycle, so there is a perfect matching in $G - X$. It exists a vertex $v \in V(G)$ and $\deg_{G[X]}(v) \geq (n/2) + 5 - k$. Since $G - X$ has no isolated vertices, there has a vertex $u \in V(G)$ and one edge $vu \in E(G \setminus X)$. Let $G_1 = G - \{v, u\}$, then $|X \cap E(G_1)| \leq 2n - k - 1 - ((n/2) + 5 - k) = (3n/2) - 6$.

If $G_1 - X$ has an isolated vertex, say a, then $d_{G_1[X]}(a) \geq n + 2 - k$. Since $G - X$ has no isolated vertices, it means a must adjacent to at least one of u and v. If $av \in E(G - X)$, let $G_2 = G - \{v, a\}$, then $|X \cap E(G_2)| \leq |X| - \deg_{G[X]}(v) - \deg_{G[X]}(a) \leq 2n - k - 1 - ((n/2) + 5 - k) - (n + 2 - k) = (n/2) + k - 8$. Clearly, $\deg_{G_2-X}(x) + \deg_{G_2-X}(y) \geq 2(n + 2 - k) - (n/2) - k + 8 = (3n/2) - 3k + 12 \geq n (n \geq 6k - 14)$ for any vertex pairs $x, y \in V(G_2)$, and then, $G_2 - X$ contains a Hamiltonian cycle and a perfect matching M. Furthermore, $M \cup \{av\}$ is a perfect matching of $G - X$. If $av \notin E(G - X)$, then $au \in E(G - X)$. If there is a vertex b and $b \neq u$ such that $vb \in E(G - X)$, we let $G_3 = G - \{v, b, u, a\}$. Clearly, $\deg_{G_3-X}(s) + \deg_{G_3-X}(t) \geq 2(n - k) - (n/2) - k + 8 - 1 = n + (n/2) - 3k + 7 \geq n + (6k - 14/2) - 3k + 7 = n$ $(n \geq 6k - 14)$ for any vertex pairs $s, t \in V(G_3)$, and then, $G_3 - X$ has a Hamiltonian cycle and a perfect matching M in $G_3 - X$. Furthermore, $M \cup \{vb, ua\}$ is a perfect matching of $G - X$. If there exists no vertex b (except u) such that $vb \in E(G - X)$, let $X_1 = E\ [v, N_G(v) \setminus \{u\}] \cup E[a, N_G(a) \setminus \{u\}]$, we have $X_1 \subseteq X$ and $2n + 6 - 2k \leq |X_1| \leq 2n - 6$. So $|X| - |X_1| \geq 2n - k - 1 - 2n + 6 = 5 - k \geq 0$, it means $k \leq 5$, which contradicts to $k \geq 6$.

If $G_1 - X$ has no isolated vertices, suppose $g_{G_1[X]}(x) \leq (n/2) + 2 - k$ for every vertex $x \in V(G_1)$, then $\deg_{G_1-X}(x) = \deg_{G_1}(x) - \deg_{G_1[X]}(x) \geq n + 2 - k - (n/2) - 2 + k = (n/2)$. Then, $G_1 - X$ contains a Hamiltonian cycle, so a perfect matching M in $G_1 - X$, and then, $M \cup \{uv\}$ is a perfect matching in $G - X$. Suppose that there exist two vertices $x, y \in V(G_1)$ with $\deg_{G_1[X]}(x) \geq (n/2) + 3 - k$ such that $xy \notin X$. Let $G_3 = G - \{v, u, x, y\}$, then $|E(G_3) \cap X| \leq (3n/2) - 6 - (n/2) - 3 + k = n + k - 9$. Since $d_{G_3-X}(a) \geq n - k - n - k + 9 = 9 > 1$ for each vertex $a \in V(G_3)$, $G_3 - X$ has no isolated vertices. Suppose $\deg_{G_3[X]}(x') \leq (n/2) - k$ for every vertex $x' \in V(G_3)$, then $\deg_{G_3-X}(x') = \deg_{G_3}(x') - \deg_{G_3[X]}(x') \geq n - k - (n/2) + k = (n/2)$. By Theorem 2, $G_3 - X$ contains a Hamiltonian cycle and a matching M in $G_3 - X$, so $M \cup \{uv, xy\}$ is a perfect matching in $G - X$. Suppose that there exists a vertex $x' \in V(G_3)$ and $\deg_{G_3[X]}(x') \geq (n/2) + 1 - k$ and $x'y' \notin X$. Let $G_4 = G - \left\{v, u, x, y, x', y'\right\}$. $|E(G_4) \cap X| \leq n + k - 9 - (n/2) - 1 + k = (n/2) + 2k - 10$. It follows that for any vertex pair $s, t \in V(G_4)$, $\deg_{G_4-X}(s) + \deg_{G_4-X}(t) \geq 2(n - 2 - k) - (n/2) + 10 - 2k = (3n/2) - 4k + 6 \geq n$. $(n \geq 8k - 12)$. So $G_4 - X$ contains a Hamiltonian cycle, and hence, there is a perfect matching M in $G_4 - X$. Clearly, $M \cup \left\{uv, xy, x'y'\right\}$ is a perfect matching of $G - X$. From the above argument, we get $mp_1(G) = 2n - k$.

Next, we characterize the odd graphs with $mp_1(G) = 3n - 9, 3n - 10, 3n - 11$, respectively. □

Proposition 7. *Let G be an order $n \geq 15$ and $\delta(G) \geq 4$. Then, $mp_1(G) = 3n - 9$ if and only if G is K_n.*

Proof. Suppose $mp_1(G) = 3n - 9$, but $G \neq K_n$. It exists two vertices u and v such that $uv \notin E(G)$. If they have a common neighbourhood w, then there is a vertex a such that $wa \in E(G)$ by $\delta(G) \geq 4$. Let $X = E[\{u, v, a\}, N_G(\{u, v, a\}) \setminus \{w\}]$, then $|X| \leq n - 2 - 2 + n - 2 - 2 + n - 1 - 3 + 2 = 3n - 10$ and u, v, a are three vertices of degree 1 with common neighbour w in $E(G) - X$. From $\delta(G) \geq 4$, that $G - X$ has no isolated vertices, and X is a basic obstruction set to an almost-perfect matching of G. Thus, $mp_1(G) \leq |X| \leq 3n - 10$, and it contradicts to $mp_1(G) = 3n - 9$. If u and v no common neighbourhood, choose a shortest path $p = ua_1a_2 \cdots a_sv$ ($ua_s \notin E(G)$) between u and v. By $\delta(G) \geq 4$, there exists another vertex b_1 such that $a_1b_1 \in E(G)$. Let $X = E[\{u, a_2, b_1\}, N_G(\{u, a_2, b_1\}) \setminus \{a_1\}]$, then $|X| \leq n - 2 - 2 + n - 2 - 2 + n - 1 - 3 + 2 = 3n - 10$ and u, a_2, b_1 are three vertices of degree 1 with the common neighbour a_1 in $E(G) - X$. Thus, $mp_1(G) \leq |X| \leq 3n - 10$, which contradicts to $mp_1(G) = 3n - 9$. □

Proposition 8. *Let G be an odd graph of order $n \geq 15$ and $\delta(G) \geq 4$. Then, $mp_1(G) = 3n - 10$ if and only if $G = K_n - e$, where $e \in E(K_n)$.*

Proof. If $G = K_n - e$, $e = uv \notin E(G)$. Choosing another two vertices x, y from $V(G)$, we know $ux, vx, yx \in E(G)$. Let $X = E[\{u, v, y\}, N_G(\{u, v, y\}) \setminus \{x\}]$, then $|X| \leq 2(n - 2) - 4 + n - 1 - 3 + 2 = 3n - 10$ and u, v, y are three vertices of degree 1 with a common neighbour x in $E(G) - X$. It means that $mp_1(G) \leq 3n - 10$. For every $X \subseteq E(G)$ with $|X| = 3n - 11$, $G - X = K_n - e - X = K_n - (X \cup \{e\})$ is the graph which is obtained from K_n by deleting at most $3n - 10$ edges. By Proposition 7, $G - X$ has an almost-perfect matching, so $mp_1(G) \geq 3n - 10$. By above argument, $mp_1(G) = 3n - 10$.

Conversely, it shows that $G = K_n - e$ when $mp_1(G) = 3n - 10$. If $G \neq K_n - e$, then $\overline{G}$ contains a path P_3 = uvw or two independent edges xy, uv as its subgraph. For the former case, if u, v, w have a common neighbour a, let $X = E[\{u, v, w\}, N_G(\{u, v, w\}) \setminus \{a\}]$, we have $|X| \leq 2(n - 2 - 2) + n - 1 - 3 + 1 = 3n - 11$ and u, v, w are three vertices of degree 1 in $E(G) - X$, and $G - X$ has no isolated vertices since $\delta(G) \geq 4$. It follows that $mp_1(G) \leq |X| \leq 3n - 11$. If u, v, w have no common neighbour, it means that $\deg_G(x) \leq n - 2$ for any vertex $x \in V(G) \setminus \{u, v, w\}$. Since $\delta(G) \geq 4$, there exists a vertex $a \notin \{u, w\}$ such that $va \in E(G)$; furthermore, there are two vertices b_1, b_2 such that $ab_1, ab_2 \in E(G)$. Let $X = E[\{v, b_1, b_2\}, N_G(\{v, b_1, b_2\}) \setminus \{a\}]$, then $|X| \leq n - 3 - 3 + 2(n - 2 - 3) + 3 = 3n - 13$ and v, b_1, b_2 are three vertices of degree 1 with common neighbour a in $G - X$. From $\delta(G) \geq 4$, we know $G - X$ has no isolated vertices. Thus, $mp_1(G) \leq |X| \leq 3n - 13$, a contradiction.

For the case of two independent edges $xy, uv \in E(\overline{G})$, there are three situations to be considered. If there are three vertices in $\{x, y, u, v\}$ with a common neighbour a, without loss of generality, they are x, y, u have a common neighbour. Let $X = E[\{x, y, u\}, N_G(\{x, y, u\}) \setminus \{a\}]$, then $|X| \leq n - 2 - 3 + 2(n - 2 - 2) + 2 = 3n - 11$ and v, y, u are three vertices of degree 1 in $G - X$. If there are two vertices in $\{x, y, u, v\}$ with a common neighbour a, then there is a vertex a' ($a' \neq x, y$) such that $aa' \in E(G)$. Clearly, a' is connected to at most two of vertices in x, y, u, v; thus, $\deg_G(a') \leq n - 3$. Let $X = E[\{x, y, a'\}, N_G(\{x, y, a'\}) \setminus \{a\}]$, so x, y, a' are three vertices of degree 1 with common neighbour a in $G - X$ and $|X| \leq n - 3 - 3 + 2(n - 2 - 2) + 2 = 3n - 12$. If any two vertices of x, y, u, v have no common neighbour, it means for every vertex $a \in V(G) \setminus \{x, y, u, v\}$ is connected to at most one of vertices in x, y, u, v, so $\deg_G(a) \leq n - 4$. Choose a vertex a ($a \neq y$) and $xa \in E(G)$, then a must has two other adjacent vertices a_1, a_2. Let $X = E[\{x, a_1, a_2\}, N_G(\{x, a_1, a_2\}) \setminus \{a\}]$, then $|X| \leq n - 2 - 3 + 2(n - 4 - 3) + 3 = 3n - 16$ and x, a_1, a_2 are three vertices of degree 1 in $G - X$. It is a contradiction. Together with the above argument, we can find $G = K_n - e$. □

Proposition 9. *Let G has $n \geq 25$ and $\delta(G) \geq 4$. Then, $mp_1(G) = 3n - 11$ if and only if one of the following conditions holds.*

(1) $\delta(G) = n - 2$ and $G \neq K_n - e$

(2) $G = K_n - E(P_3)$

Proof. Suppose $mp_1(G) = 3n - 11$. Then, we claim $\delta(G) \geq n - 3$. Assume, on the contrary, that $\delta(G) \leq n - 4$. Then, there exists a vertex u in G such that $d_G(u) \leq n - 4$. Choose $v \in N_G(u)$ and $\{a, b\} \in N_G(v)$, since $\delta(G) \geq 4$. Let $X = E[\{u, a, b\}, N_G(\{u, a, b\}) \setminus \{v\}]$, so u, a, b are three vertices of degree 1 with common neighbour v in $G - X$ and $|X| \leq n - 4 - 3 + 2(n - 1 - 3) + 3 = 3n - 12$. Thus, $mp_1(G) \leq |X| \leq 3n - 12$, which contradicts $mp_1(G) = 3n - 11$. By Propositions 7 and 8 and $\delta(G) \geq n - 3$, we have $\delta(G) = n - 2$ and $G \neq K_n - e$ or $\delta(G) = n - 3$.

When $\delta(G) = n - 3$, we need to show $G = K_n - E(P_3)$. Assume $\deg_G(u) = n - 3$. It means there exist two vertices v, w such that $uv, uw \notin E(G)$ and $ux \in E(G)$ for every vertex $x \in V(G) \setminus \{w, v\}$. Assume, on the contrary, that $G \neq K_n - E(P_3)$. Except $uv, uw \notin E(G)$, there exists two vertices x, y in $V(G)$ and $xy \notin E(G)$. If $x = v$, then $\deg_G(v) = n - 3$. Choose $va \in E(G)$ and $ab \in E(G)$; clearly, $ua \in E(G)$. Let $X = E[\{u, v, b\}, N_G(\{u, v, b\}) \setminus \{a\}]$, so u, v, b are three vertices of degree 1 with common neighbour a in $G - X$ and $|X| \leq n - 1 - 3 + 2(n - 3 - 2) + 2 = 3n - 12$. $mp_1(G) \leq |X| \leq 3n - 12$, So $x, y \notin \{v, w\}$. Since $\delta(G) \geq 4$, choose $xa, ab \in E(G)$ and $\{ux, ua, ub\} \subseteq E(G)$. Let $X = E[\{u, x, b\}, N_G(\{u, x, b\}) \setminus \{a\}]$, so u, x, b are three vertices of degree 1 with common neighbour a in $G - X$ and $|X| \leq n - 3 - 3 + n - 2 - 3 + n - 1 - 3 + 3 = 3n - 12$. $mp_1(G) \leq |X| \leq 3n - 12$, which is a contradiction. In summary, we have $G = K_n - E(P_3)$, or $\delta(G) = n - 2$ and $G \neq K_n - e$, as required.

Conversely, by Propositions 7 and 8, we have $mp_1(G) \leq 3n - 11$. We show $mp_1(G) \geq 3n - 11$. If $G = K_n - E(P_3)$, for every $X \subseteq E(G)$ with $|X| \leq 3n - 12$, it follows that $G - X = K_n - (X \cup E(P_3))$ is a graph obtained from K_n by deleting at most $3n - 10$ edges. By Proposition 7, $G - X$ has an almost-perfect matching, as desired. If $\delta(G) = n - 2$ and $G \neq K_n - e$, then $G = K_n - L$, where L is a matching of K_n of size at least 2. It suffices to prove that $G - X$ has an almost-perfect matching for every $X \subseteq E(G)$ with $|X| \leq 3n - 12$ such that $G - X$ has no isolated vertices. Suppose any vertex $v \in V(G)$ has $\deg_{G[X]}(v) \leq (n - 5/2)$, then $\deg_{G-X}(v) = \deg_G(v) - \deg_{G[X]}(v) \geq n - 2 - (n - 5/2) = (n + 1/2) > (n/2)$. By Theorem 2, $G - X$ contains a Hamiltonian cycle. So an almost-perfect matching in $G - X$, a contradiction. Now, we suppose that there exists a vertex $v \in V(G)$ and $\deg_{G[X]}(v) \geq (n - 3/2)$. Since $G - X$ has no isolated vertices, it follows that there exists a vertex $u \in V(G)$ and $vu \in E(G - X)$. Let $G_1 = G - \{v, u\}$, then $|X \cap E(G_1)| \leq 3n - 12 - (n - 3/2) = (5n - 21/2)$.

If $G_1 - X$ has no isolated vertices, suppose every vertex $x \in V(G_1)$ has $\deg_{G[X]}(x) \le (n - 9/2)$. Then, $\deg_{G_1 - X}(x) = \deg_{G_1}(x) - \deg_{G_1[X]}(x) \ge n - 4 - (n - 9/2) = (n + 1/2)$. By Theorem 2, $G_1 - X$ contains a Hamiltonian cycle. So an almost-perfect matching M' in $G_1 - X$ and $M' \cup \{uv\}$ is an almost-perfect matching in $G - X$, a contradiction.

We suppose that there exists a vertex $x \in V(G_1)$ and $\deg_{G_1[X]}(x) \ge (n - 7/2)$ and $xy \notin X$. Let $G_2 = G - \{v, u, x, y\}$, then $|E(G_2) \cap X| \le (5n - 21/2) - (n - 7/2) = (4n - 14/2)$. Recall that $G = K_n - L$, where L is a matching of K_n of size at least 2; thus, $G_2 = K_n - L - \{u, v, x, y\} = K_{n-4} - L$. It means removing up to $(n - 5/2)$ edges from K_{n-4}. Since $n \ge 25$, $(n - 5/2) + (4n - 14/2) = (5n - 19/2) < 3(n - 4) - 9$. By Proposition 7, $G_2 - X$ has an almost-perfect matching M', and $M' \cup \{uv, xy\}$ is an almost-perfect matching of $G - X$.

If $G_1 - X$ has two isolated vertices a and b, then $3n - 12 - \deg_{[X]}(v) - \deg_{[X]}(a) - \deg_{[X]}(b) \le 3n - 12 - (n - 3/2) - 2(n - 4) = (n - 5/2)$. Since $G - X$ has no isolated vertices, the vertices a and b must be at least adjacent to one of $\{u, v\}$, respectively. Since $\delta = n - 2$, we can assume $au \in E(G)$ and $bv \in E(G)$. Let $G_2 = G - \{u, v, a, b\} = K_n - L - \{u, v, a, b\}$. Since $(n - 5/2) + (n - 5/2) = n - 5$, $mp_1(K_{n-4}) = 3(n - 4) - 9 = 3n - 21$ and $n - 5 < 3n - 21$. So $G_2 - X$ has an almost-perfect matching M', and $M' \cup \{au, bv\}$ is an almost-perfect matching of $G - X$.

If $G_1 - X$ has one isolated vertex a, then $\deg_{G[X]}(a) \ge n - 4$. Let $G_2 = G - \{u, v, a\}$, clearly, $|E(G_2) \cap X| \le 3n - 12 - (n - 3/2) - n + 4 = (3n - 13/2)$. Suppose that every vertices $b \in V(G_2)$ have $\deg_{G_2[X]}(b) \le (n - 9/2)$, then $\deg_{G_2 - X}(b) = \deg_{G_2}(b) - \deg_{G_2[X]}(b) \ge n - 5 - (n - 9/2) > (n - 3/2)$. By Theorem 2, $G_2 - X$ contains a Hamiltonian cycle. So there is a perfect matching M in $G_2 - X$, and $M \cup \{uv\}$ is an almost-perfect matching in $G - X$ missing a. We suppose that there exists a vertex $b \in V(G_2)$ and $\deg_{G_2[X]}(b) \ge (n - 7/2)$. Since $G_1 - X$ has only one isolated vertex a, it follows that there exists one vertex b' and $bb' \notin X$. Let $G_3 = G - \{v, u, a, b, b'\}$, then $|E(G_3) \cap X| \le (3n - 13/2) - (n - 7/2) = n - 3$. Clearly, $G_3 = K_n - L - \{v, u, a, b, b'\}$ and $|V(G_3)| = n - 5$ is even, and we have $mp_1(K_{n-5}) = 2n - 15$. Since $K_{n-5} - L$ means from K_{n-5} remove at most $(n - 5/2)$ edges and $(n - 5/2) + n - 3 = (3n - 11/2) < 2n - 15$ $(n \ge 25)$, $G_3 - X$ has a perfect matching M, and $M \cup \{uv, bb'\}$ is an almost-perfect matching of $G - X$ missing a, a contradiction. Thus, $mp_1(G) = 3n - 11$. □

4. Extremal Problems on CMP Number

Consider the three extremal problems in the introduction.

Lemma 5. *Let n, k be two positive integers and $n \ge 3$ be odd. Then,*

(1) $s(n, 0) = 0$

(2) $s(n, 1) = (n + 5/2)$

(3) $s(n, 2) = (n + 7/2)$

Proof

(1) Let G be a no edges with order n graph. Clearly, $mp_1(G) = 0$. So $s(n, 0) = 0$.

(2) Let $H = (n - 5/2)P_2 \cup S_5^+$, where S_5^+ is a graph obtained from a star $K_{1,4}$ by adding one edge between two leaves. Clearly, $mp_1(H) = 1$ and H has $(n + 5/2)$ edges. Then, $s(n, 1) \le (n + 5/2)$. Conversely, assume that $s(n, 1) \le (n + 3/2)$, it means exists an odd graph G of order n with $s(n, 1) \le (n + 3/2)$ thus $mp_1(G) = 1$. Assume that $C_1, C_2, \cdots, C_t$ are the connected components in G. If two of $C_1, C_2, \cdots, C_t$ are odd components in G, then $mp_1(G) = 0$. So there is exactly one odd component in G, suppose it is C_t. We may assume $|V(C_1)| \ge |V(C_2)| \ge \cdots \ge |V(C_{t-1})|$. If $|V(C_2)| \ge 4$, then $|E(G)| \ge 2(4 - 1) + (n - 8 - 3/2) + 3 - 1 = (n + 5/2)$, it contradicts with $|E(G)| = s(n, 1) \le (n + 3/2)$. Then, $|V(C_1)| \ge 4$ and $|V(C_2)| = |V(C_3)| = \cdots = |V(C_{t-1})| = 2$. If $|V(C_1)| \ge 6$, then $|E(G)| \ge 6 - 1 + (n - 6 - 3/2) + 3 - 1 = (n + 5/2)$, a contradiction. So $|V(C_1)| = 4$ or $|V(C_1)| = 2$. If $|V(C_1)| = 4$, we claim $|V(C_t)| = 3$. Otherwise, $|V(C_t)| \ge 5$ and $|E(G)| \ge 4 - 1 + (n - 4 - 5/2) + 5 - 1 = (n + 5/2)$. We have $(n - 7/2)P_2$ in G, and $|E(C_1)| + |E(C_t)| \le (n + 3/2) - (n - 7/2) = 5$. So $G = (n - 7/2)P_2 \cup P_3 \cup P_4$ or $G = (n - 7/2)P_2 \cup P_3 \cup S_4$. For two cases, we know $mp_1(G) = 0$. If $|V(C_1)| = 2$, we claim $|V(C_t)| = 5$. Otherwise, $|V(C_t)| = 3$ or $|V(C_t)| \ge 7$. If $|V(C_t)| = 3$, then $|E(C_t)| \le (n - 3/2) - (n - 3/2) = 3$ and $|E(C_t)| \ge 2$. So $G = (n - 3/2)P_2 \cup C_3$ or $G = (n - 3/2)P_2 \cup P_3$. For two cases, we know $mp_1(G) = 0$. If $|V(C_t)| \ge 7$, then $|E(G)| \ge 6 + (n - 7/2) = (n + 5/2)$, which is a contradiction. So $|V(C_t)| = 5$, and $|E(C_t)| \le (n + 3/2) - (n - 5/2) = 4$. Thus, $G = (n - 5/2)P_2 \cup P_5$ or $G = (n - 5/2)P_2 \cup S_5$. For two cases, we know $mp_1(G) = 0$. So $s(n, 1) = (n + 5/2)$.

(3) Let $H = (n - 7/2)P_2 \cup P_3 \cup K_4 - \{e\}$. Clearly, $mp_1(H) = 2$ and H has $(n + 7/2)$ edges. Then, $s(n, 2) \le (n + 7/2)$. Since $s(n, 1) = (n + 5/2)$, it follows that $(n + 5/2) \le s(n, 1) \le s(n, 2) \le (n + 7/2)$. Now, we prove $s(n, 2) \ge (n + 7/2)$. If we assume that $s(n, 2) = (n + 5/2)$, then it exists an odd graph G of order n with $s(n, 2) = (n + 5/2)$ such that $mp_1(G) = 2$. So G is not connected. Let $C_1, C_2, \cdots, C_t$ be the connected components in G. If two of $C_1, C_2, \cdots, C_t$ are odd components, then $mp_1(G) = 0$. So there is exactly one odd component in G, and we can assume that $|V(C_t)|$ is odd and $|V(C_i)|$ is even for $1 \le i \le t - 1$. Assume $|V(C_1)| \ge |V(C_2)| \ge \cdots \ge |V(C_{t-1})|$. If $|V(C_t)| \ge 9$, then $|E(G)| \ge |V(C_t)| - 1 + (n - |V(C_t)|/2) = (|V(C_t)| + n - 2/2) \ge (n + 7/2)$, but $|E(G)| = (n + 5/2)$. So $3 \le |V(C_t)| \le 7$. Now, we consider three cases of the number of $|V(C_t)|$. First assume $|V(C_t)| = 7$, if $|V(C_1)| \ge 4$, then $|E(G)| \ge 6 + 3 + (n - 7 - 4/2) = (n + 7/2)$, a contradiction. All even components are P_2, and $|E(C_t)| = (n - 5/2) - (n - 7/2) = 6$. By $|V(C_t)| = 7$ and $|E(C_t)| = 6$, the structure of C_t must be P_7 or P_6 add one pendant edge or P_5 add two pendant edges. For above three structure of C_t, we have $mp_1(G) = 0$ or $mp_1(G) = 1$. Next assume $|V(C_t)| = 5$, if $|V(C_1)| \ge 4$, then

$|E(G)| \geq 4 + 3 + (n-9/2) = (n+5/2)$. Since $|E(G)| = (n+5/2)$, So $|V(C_1)| = 4$ and $|V(C_2)| = |V(C_3)| = \cdots = |V(C_{t-1})| = 2$. Since C_t has an almost-perfect matching, so C_t must be P_5 or P_4 add one pendant edge. For any structure of C_t, $mp_1(G) = 0$. At last, $|V(C_t)| = 3$ If $|V(C_2)| \geq 4$, then $|E(G)| \geq 2 + 2(4-1) + (n-8-3/2) = (n+5/2)$. Since $|E(G)| = (n+5/2)$. So $G = (n-11/2)P_2 \cup 2P_4$ and $mp_1(G) = 0$. So only have $|V(C_1)| \geq 4$, other even components are P_2. If $|V(C_1)| \geq 6$, then $|E(G)| \geq 5 + 2 + (n-6-3/2) = (n+5/2)$. Since $|E(G)| = (n+5/2)$. It follows that $C_t = P_3$, so C_1 must be P_6 or P_5 add one pendant edge or P_4 add two pendant edges. For above structure of C_t, $mp_1(G) = 0$ or $mp_1(G) = 1$. If $|V(C_1)| = 4$, then $|E(C_1 \cup C_t)| = (n+5/2) - (n+7/2) = 6$. So $|E(C_1)| = 2$ and $|E(C_t)| = 4$, or $|E(C_1)| = 3$ and $|E(C_t)| = 3$. It follows that $G = (n-7/2)P_2 \cup P_3 \cup C_4$ or $G = (n-7/2)P_2 \cup P_3 \cup S_3^+$ or $G = (n-7/2)P_2 \cup C_3 \cup P_4$. For any case, we know $mp_1(G) = 0$ or $mp_1(G) = 1$. So $s(n,2) = (n+7/2)$. □

Theorem 4. *Let $n \geq 5$ be an odd integer, $3 \leq k \leq 3n-9$. Then,*

(1) If $\lceil (k-3/3) \rceil$ is odd, then

$$s(n,k) \leq \frac{\lceil (k-3/3) \rceil^2 + n + 2 + 2k}{2}. \quad (3)$$

(2) If $\lceil (k-3/3) \rceil$ is even, then

$$s(n,k) \leq \frac{\lceil (k-3/3) \rceil^2 + 2\lceil (k-3/3) \rceil + 2k + 3 + n}{2}. \quad (4)$$

Proof

(1) Let H_1 be a graph obtained from u, K_3, $K_{\lceil (k-3/3) \rceil}$, by arbitrarily adding $k-3$ edges between K_3 and $K_{\lceil (k-3/3) \rceil}$, and making the join graphs: $\{u\} \vee K_3$ and $\{u\} \vee K_{\lceil (k-3/3) \rceil}$. Let $H = H_1 \cup (n-4-\lceil (k-3/3) \rceil/2)P_2$. Suppose v_1, v_2, v_3 are three vertices in K_3, the degree of v_1, v_2, v_3 in H_1 is less than or equal to the degree of other vertices in $V(H_1 \backslash \{v_1, v_2, v_3\})$. Clearly, we delete $k-3$ edges between K_3 and $K_{\lceil (k-3/3) \rceil}$ and three edges in K_3 of H, and then, v_1, v_2, v_3 are three vertices of degree 1 with the common neighbour u and the rests of H have no isolated vertices. And $|E(H)| = 3 + \lceil (k-3/3) \rceil + k + (\lceil (k-3/3) \rceil/2) + (n-4-\lceil (k-3/3) \rceil/2) = (\lceil (k-3/3) \rceil^2 + n + 2 + 2k/2)$, so $s(n,k) \leq (\lceil (k-3/3) \rceil^2 + n + 2 + 2k/2)$.

(2) Let H_1 be a graph obtained from tree cliques u, K_3, $K_{\lceil (k-3/3) \rceil + 1}$. By arbitrarily adding $k-3$ edges between K_3 and $K_{\lceil (k-3/3) \rceil + 1}$, then make the join graphs: $\{u\} \vee K_3$ and $\{u\} \vee K_{\lceil (k-3/3) \rceil + 1}$. Let $H = H_1 \cup (n-5-\lceil (k-3/3) \rceil/2)P_2$. The degree of three vertices v_1, v_2, v_3 in K_3 is less than or equal to the degree of other vertices in $V(H_1 \backslash K_3)$. Clearly, when we delete $k-3$ edges between K_3 and $K_{\lceil (k-3/3) \rceil}$ and three edges in K_3, then v_1, v_2, v_3 are three vertices of degree 1 with common neighbour u, and the rest of graphs have no isolated vertices. This just deletes the k edges from H and no almost-perfect matching in result graph.$|E(H)| = 3 + \lceil (k-3/3) \rceil + 1 + k + \binom{\lceil (k-3/3) \rceil + 1}{2} + (n-5-\lceil (k-3/3) \rceil/2) = (\lceil (k-3/3) \rceil^2 + 2\lceil (k-3/3) \rceil + 2k + 3 + n/2)$, so $s(n,k) \leq (\lceil (k-3/3) \rceil^2 + 2\lceil (k-3/3) \rceil + 2k + 3 + n/2)$. □

Lemma 6. *Let n, k be two positive integers and $n \geq 4$ be even. Then,*

(1) $s(n,0) = 0$

(2) $s(n,1) = (n/2) + 2$

(3) $s(n,2) = (n/2) + 3$

(4) $s(n,3) = (n/2) + 4$

Proof

(1) Let G be the graph of order n with no edges. Clearly, $mp_1(G) = 0$. So $s(n,0) = 0$.

(2) Let $H = (n-4/2)P_2 \cup S_4^+$. S_4^+ is a star of order 4 and add one edge between two of pendent vertices. Clearly, $mp_1(H) = 1$ and H has $(n/2) + 2$ edges. Then, $s(n,1) \leq (n/2) + 2$. Conversely, assume that $s(n,1) \leq (n/2) + 1$. Then, there exists an even graph G of order n with $s(n,1) \leq (n/2) + 1$ edges such that $mp_1(G) = 1$. Because G exists a perfect matching, so $|E(G)| \geq (n/2)$. If $|E(G)| = (n/2)$, then $G = (n/2)P_2$ and $mp_1(G) = 0$. If $|E(G)| = (n/2) + 1$, then $G = (n-4/2)P_2 \cup P_4$ and $mp_1(G) = 0$. So $s(n,1) = (n/2) + 2$.

(3) Let $H = (K_4 - \{e\}) \cup (n-4/2)P_2$. Clearly, $mp_1(H) = 2$ and H has $(n/2) + 3$ edges. Then, $s(n,2) \leq (n/2) + 3$. Since $s(n,1) = (n/2) + 2$, it follows $(n/2) + 2 = s(n,1) \leq s(n,2) \leq (n/2) + 3$. Assume that $s(n,2) = (n/2) + 2$. Then, there exists an even graph G of order n with $s(n,2) = (n/2) + 2$ edges such that $mp_1(G) = 2$. Because G has a perfect matching, so $|E(G)| \geq (n/2)$, now G is a graph of add two edges to a perfect match of size $(n/2)$. So G has the following situations: $G = C_4 \cup (n-4/2)P_2$, $G = P_6 \cup (n-6/2)P_2$, $G = 2P_4 \cup (n-8/2)P_2$, $G = K_3^+ \cup (n-4/2)P_2$, $G = (n-6/2)P_2 \cup$ (P_5 add one pendant edge on the middle vertex), and for any of the above, we have $mp_1(G) = 0$ or $mp_1(G) = 1$. So $s(n,2) = (n/2) + 3$.

(4) Let $H = K_4 \cup (n-4/2)P_2$. Clearly, $mp_1(H) = 3$ and H has $(n/2) + 4$ edges. Then, $s(n,3) \leq (n/2) + 4$. Since $s(n,2) = (n/2) + 3$, it follows that $(n/2) + 3 = s(n,2) \leq s(n,3) \leq (n/2) + 4$. Assume that $s(n,3) = (n/2) + 3$. Then, there exists an even graph G of order n with $s(n,3) = (n/2) + 3$ edges such that $mp_1(G) = 2$. Since G has a perfect matching, so G is a

graph of add three edges to a perfect match of size $(n/2)$. All possible structures of G are shown as follows (Figure 1). For any structure, we have $mp_1(G) = 0$ or $mp_1(G) = 1$ or $mp_1(G) = 2$. So $s(n,3) = (n/2) + 4$. □

Theorem 5. *Let n, k be two positive integers and $n \geq 6$ be even and $4 \leq k \leq 2n - 5$. Then,*

(1) If $\lceil (k-1/2) \rceil$ is odd, then $s(n,k) \leq (\lceil (k-1/2) \rceil^2 + n + 2k + 1/2)$

(2) If $\lceil (k-1/2) \rceil$ is even, then $s(n,k) \leq (\lceil (k-1/2) \rceil^2 + 2\lceil (k-1/2) \rceil + n + 2k + 2/2)$

Proof

(1) Let H_1 be a graph obtained from $K_3 = u_1u_2u_3$ and $K_{\lceil (k-1/2) \rceil}$. add $k-1$ edges between $\{u_2, u_3\}$ and $K_{\lceil (k-1/2) \rceil}$, then make the join graph: $\{u_1\} \vee K_{\lceil (k-1/2) \rceil}$. Let $H = H_1 \cup (n - 3 - \lceil (k-1/2) \rceil / 2) P_2$. And the degree of three vertices u_2, u_3 in K_3 is less than or equal to the degree of other vertices in $V(H_1 \backslash \{u_2, u_3\})$. Clearly, when we delete edge $\{u_2u_3\}$ and $k-1$ edges between $\{u_2, u_3\}$ and $K_{\lceil (k-1/2) \rceil}$, then u_2, u_3 are two vertices of degree 1 with common neighbour u_1, and the rest of the graphs have no isolated vertices. This just deletes the k edges from H and no perfect matching in result graph. $|E(H)| = 2 + \lceil (k-1/2) \rceil + k + \binom{\lceil (k-1/2) \rceil}{2} + (n - 3 - \lceil (k-1/2) \rceil / 2) = (\lceil (k-1/2) \rceil^2 + n + 1 + 2k/2)$, so $s(n,k) \leq (\lceil (k-1/2) \rceil^2 + n + 1 + 2k/2)$.

(2) Let H_1 be a graph obtained from $K_3 = u_1u_2u_3$ and $K_{\lceil (k-1/2) \rceil + 1}$. Arbitrarily add $k-1$ edges between $\{u_2, u_3\}$ and $K_{\lceil (k-1/2) \rceil + 1}$, then make the join graph: $\{u_1\} \vee K_{\lceil (k-1/2) \rceil + 1}$. Let $H = H_1 \cup (n - 4 - \lceil (k-1/2) \rceil / 2) P_2$. From the construction of graph H_1, the degree of three vertices u_2, u_3 in K_3 is less than or equal to the degree of other vertices in $V(H_1 \backslash \{u_2, u_3\})$. Clearly, when we delete edge $\{u_2u_3\}$ and $k-1$ edges between $\{u_2, u_3\}$ and $K_{\lceil (k-1/2) \rceil + 1}$, then u_2, u_3 are two vertices of degree 1 with common neighbour u_1, and the rest of the graphs have no isolated vertices. This just deletes the k edges from$H$ and no perfect matching in result graph. $|E(H)| = 3 + \lceil (k-1/2) \rceil + k + \binom{\lceil (k-1/2) \rceil + 1}{2} + (n - 4 - \lceil (k-1/2) \rceil / 2) = (\lceil (k-1/2) \rceil^2 + 2\lceil (k-1/2) \rceil + n + 2 + 2k/2)$, so $s(n,k) \leq (\lceil (k-1/2) \rceil^2 + 2\lceil (k-1/2) \rceil + n + 2 + 2k/2)$. □

Observation 1. $g(n,k) = s(n, k+1) - 1$, n, k be two positive integers.

Corollary 1. *Let n, k be two positive integers and $n \geq 5$ be odd, $3 \leq k \leq 3n - 9$. Then,*

(1) If $\lceil (k-3/3) \rceil$ is odd, then $g(n,k) \leq (\lceil (k-2/3) \rceil^2 + n + 2k + 2/2)$

(2) If $\lceil (k-3/3) \rceil$ is even, then $g(n,k) \leq (\lceil (k-2/3) \rceil^2 + 2\lceil (k-2/3) \rceil + 2k + 3 + n/2)$

Corollary 2. *Let $n \geq 6$, $4 \leq k \leq 2n - 5$ be two positive integers and be even. Then,*

(1) If $\lceil (k-1/2) \rceil$ is odd, then $g(n,k) \leq (\lceil (k/2) \rceil^2 + n + 2k + 1/2)$

(2) If $\lceil (k-3/3) \rceil$ is even, then $g(n,k) \leq (\lceil (k/2) \rceil^2 + 2\lceil (k/2) \rceil + n + 2k + 2/2)$

Theorem 6. *Let n, k be two positive integers. Then,*

(1) If $n \geq 10$ is even and $1 \leq k \leq 2n - 5$, then $f(n,k) = \binom{n-3}{2} + k + n - 1$

(2) If $n \geq 15$ is odd and $1 \leq k \leq 3n - 9$, then $f(n,k) = \binom{n-4}{2} + k + n - 1$

Proof

(1) First show $f(n,k) \geq \binom{n-3}{2} + k + n - 1$, we construct H as follows, give three components $H_1 = P_3 = v_1v_2v_3$, $H_2 = K_{\lceil (k-1/2) \rceil}$, $H_3 = K_{n-3-\lceil (k-1/2) \rceil}$. $k-1$edges are connected between $\{v_1, v_3\}$ and H_2, $\{v_2\}$ adjacents to all vertices in H_2 and H_3, and then make a join between H_2 and H_3. Clearly, H is a connected graph on n vertices, $E(H) = 2 + k - 1 + n - 3 + \binom{n-3}{2} = \binom{n-3}{2} + k + n - 2$, and $mp_1(H) = k - 1 < k$. So $f(n,k) \geq \binom{n-3}{2} + k + n - 1$. Now to show $f(n,k) \leq \binom{n-3}{2} + k + n - 1$, let G be a graph with n vertices such $|E(G)| \geq \binom{n-3}{2} + n + k - 1$. For any $X \subseteq E(G)$, $|X| = k - 1$, so $|E(G - X)| \geq \binom{n-3}{2} + n$ and $|E(\overline{G - X})| \leq \binom{n}{2} - \binom{n-3}{2} - n = 2n - 6$. Since $mp_1(K_n) = 2n - 5$ by Theorem 1, thus $G - X$ has a perfect matching, and hence, $mp_1(H) \geq k$. So $f(n,k) = \binom{n-3}{2} + k + n - 1$.

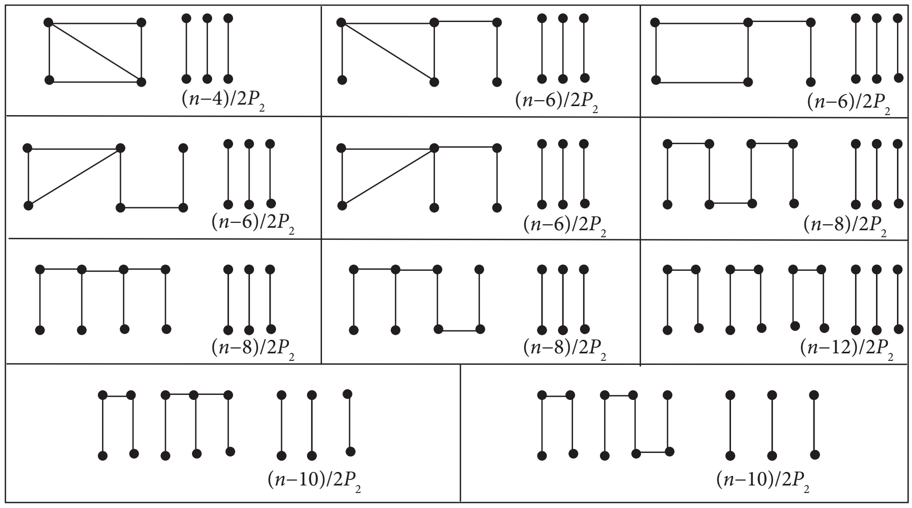

Figure 1: All possible structures of G.

(2) First prove $f(n,k) \geq \binom{n-4}{2} + k + n - 1$, we construct H as follows: give three components H_1, H_2, H_3. H_1 is a star graph with central vertex u and three pendant vertices v_1, v_2, v_3. $H_2 = K_{\lceil (k-1/3) \rceil}$, $H_3 = K_{n-4-\lceil (k-1/3) \rceil}$. $k-1$ edges are connected between $\{v_1, v_2, v_3\}$ and H_2, $\{u\}$ adjacent to all vertices in H_2 and H_3 and then make a join between H_2 and H_3. Clearly, H is a connected graph on n vertices, $E(H) = 3 + k - 1 + n - 4 + \binom{n-4}{2} = \binom{n-4}{2} + k + n - 2$, and $mp_1(H) = k - 1 < k$. So $f(n,k) \geq \binom{n-4}{2} + k + n - 1$. Now to show $f(n,k) \leq \binom{n-4}{2} + k + n - 1$, let G be a graph with n vertices such $|E(G)| \geq \binom{n-4}{2} + n + k - 1$. For any $X \subseteq E(G)$, $|X| = k - 1$, so $|E(G - X)| \geq \binom{n-4}{2} + n$ and $|E(\overline{G - X})| \leq \binom{n}{2} - (n - 42) - n = 3n - 10$. Since $mp_1(K_n) = 3n - 9$, tt follows that $G - X$ has an almost-perfect matching, and hence, $mp_1(H) \geq k$. So $f(n,k) = \binom{n-4}{2} + k + n - 1$. □

5. Conclusion

The concept of matching preclusion was introduced in [2]. The matching preclusion number measures the robustness of a graph as a communications network topology. In a network, a vertex with a special matching vertex after edge failure any time implies that tasks running on a fault vertex can be change into its matching vertex. Larger $mp(G)$ signifies higher fault tolerance. However, the probability that the adjacent vertices of the same vertex fail at the same time is very small. In the paper, we mainly want to discuss the conditional matching preclusion number of general graphs. First of all, we want to discuss the bound of the conditional matching preclusion number of a general graph. It is necessary to discuss the bounds of reaching the number of conditional matches according to the parity of the number of vertices of the graph. Next, we will draw a graph from the conditional matching preclusion number. When the number of conditional matches is a special value, what property does the obtained graph satisfy. From the perspective of graph description, this is very meaningful. Finally, we discussed three extreme value problems. It is actually an extension of the two problems discussed previously, but it is more difficult to solve than the previous problems.

Data Availability

No data are used in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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