Research Article

On Generalized Proportional Fractional Order Derivatives and Darboux Problem for Partial Differential Equations

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The study of the existence and uniqueness of solutions to 2D systems utilizing the generalized proportional fractional derivative operator is the focus of this work. We also derive a finite difference scheme in order to numerically approximate such an operator, and we prove that the method we propose is convergent. Several tests are performed at the end to illustrate the robustness of our algorithm.

1. Introduction

Fractional calculus is a branch of mathematics that allows for the differentiation and integration of noninteger orders, extending the scope of traditional calculus. It involves the study of derivatives and integrals of arbitrary order. This allows for a more general approach to calculus, offering greater flexibility and accuracy. Fractional calculus can be used to model various physical phenomena, including turbulent fluid flow and the diffusion of gases. It is also used in signal processing applications, such as in the design of filters, and in the analysis of fractal images. The goal of fractional calculus is to provide a more comprehensive understanding of phenomena by allowing for the study of derivatives and integrals of arbitrary order. By allowing for the differentiation and integration of noninteger orders, fractional calculus is able to offer greater accuracy and flexibility in the study of various phenomena and has a wide range of applications (see [1–4]). Ordinary and partial fractional differential equations have developed significantly in recent years (see [5–9]).

In [10–12], authors have generalized the Caputo proportional fractional derivatives with respect to another function involving exponential functions. The Caputo derivatives are used to represent the fractional order of derivatives with respect to initial conditions. The exponential functions in the kernels of the derivatives provide a description of the time dynamics of the system. This type of fractional derivatives allows for a more precise description of complex nonlinear systems and provides a better representation of fractional order dynamics (see [13–17]).

The Darboux problem for partial hyperbolic differential equations is an important topic of research in mathematics. It concerns the existence and uniqueness of solutions to certain partial differential equations (PDEs). In recent years, there have been numerous articles discussing the Darboux problem. For instance, readers can refer to [5, 6, 18–22] for further information.
This paper extends [18] to the case of the GPF-Caputo derivative of order $\omega$ with respect to $\sigma_1, \sigma_2$. The main contribution of this article is to investigate the existence, uniqueness, and numerical solutions of the following fractional partial differential systems:

$$C^-(\sigma_1,\sigma_2)^{\omega,\nu}_a\mathcal{U}(x_1,x_2) = \mathcal{F}(x_1,x_2,\mathcal{U}(x_1,x_2)), (x_1,x_2) \in J = [a_1,b_1] \times [a_2,b_2],$$

(1)

where $a = (a_1,a_2) \in \mathbb{R}^2, \omega = (\omega_1,\omega_2) \in (0,1)^2$, and $\nu = (v_1,v_2) \in (0,1)^2$; $C^-(\sigma_1,\sigma_2)^{\omega,\nu}_a\mathcal{U}$ is the GPF-Caputo derivative of order $\omega$ with respect to $\sigma_1$ and $\sigma_2$; and $\mathcal{F}: J \times \mathbb{R} \rightarrow \mathbb{R}$, $\varphi: [a_1,b_1] \rightarrow \mathbb{R}$, and $\psi: [a_2,b_2] \rightarrow \mathbb{R}$ are given continuous functions.

2. Preliminarily

This section presents the definitions, lemmas, and propositions necessary for our findings. In the rest of this paper, we take $J = [a_1,b_1] \times [a_2,b_2]$.

Definition 1. Let $\sigma_1 \in C^1([a_1,b_1], \mathbb{R})$ and $\sigma_2 \in C^1([a_2,b_2], \mathbb{R})$ be positive strictly increasing functions such that $\sigma_1'(x_1),\sigma_2'(x_2) \neq 0$ for all $x_1 \in [a_1,b_1]$ and $x_2 \in [a_2,b_2]$. The GPF-integral of order $\omega$ of $\mathcal{U}(x_1,x_2) \in L^1(J)$ with respect to $\sigma_1, \sigma_2$ is defined as

$$\langle (\sigma_1,\sigma_2)^{\omega,\nu}_a\mathcal{U} \rangle(x_1,x_2) = \frac{1}{v_1^1 \cdot v_2^2 \cdot \Gamma(\omega_1) \Gamma(\omega_2)} \int_{a_1}^{x_1} \int_{a_2}^{x_2} \sigma_1'(s)\sigma_2'(t) \exp\left(\frac{v_1-1}{v_1} (\sigma_1(x_1) - \sigma_1(s))\right) \exp\left(\frac{v_2-1}{v_2} (\sigma_2(x_2) - \sigma_2(t))\right) \frac{U(s,t)}{(\sigma_1(x_1) - \sigma_1(s))^{\omega_1}(\sigma_2(x_2) - \sigma_2(t))^{\omega_2}} \, ds \, dt,$$

(3)

where $a = (a_1,a_2), \omega = (\omega_1,\omega_2), \nu = (v_1,v_2) \in (0,1)^2$, and $\omega_1$ and $\omega_2$ are positive real numbers.

Definition 2. Let $\sigma_1 \in C^1([a_1,b_1], \mathbb{R})$ and $\sigma_2 \in C^1([a_2,b_2], \mathbb{R})$ be positive strictly increasing functions such that $\sigma_1'(x_1),\sigma_2'(x_2) \neq 0$ for all $x_1 \in [a_1,b_1]$ and $x_2 \in [a_2,b_2]$. The GPF-Riemann–Liouville derivative of order $\omega$ of $\mathcal{U}(x_1,x_2) \in L^1(J)$ with respect to $\sigma_1, \sigma_2$ is defined as

$$\langle (\sigma_1,\sigma_2)^{\omega,\nu}_a\mathcal{U} \rangle(x_1,x_2) = \frac{1}{v_1^1 \cdot v_2^2 \cdot \Gamma(1-\omega_1) \Gamma(1-\omega_2)} \int_{a_1}^{x_1} \int_{a_2}^{x_2} \sigma_1'(s)\sigma_2'(t) \times \exp\left(\frac{v_1-1}{v_1} (\sigma_1(x_1) - \sigma_1(s))\right) \exp\left(\frac{v_2-1}{v_2} (\sigma_2(x_2) - \sigma_2(t))\right) \times \frac{U(s,t)}{(\sigma_1(x_1) - \sigma_1(s))^{\omega_1}(\sigma_2(x_2) - \sigma_2(t))^{\omega_2}} \, ds \, dt,$$

(4)
Lemma 4. Let \( \chi, \sigma \in (0, 1)^2 \), \( \nu = (\nu_1, \nu_2) \in (0, 1)^2 \), \( \omega = (\omega_1, \omega_2) \in (0, 1)^2 \), and \( \nu = (\nu_1, \nu_2) \in (0, 1)^2 \). Then, we obtain the following relation:

\[
\left( \mathcal{I}^{\omega_1, \nu_1}_a \mathcal{I}^{\omega_2, \nu_2}_b \mathcal{U} \right)(\chi_1, \chi_2) = \left( \mathcal{I}^{\omega_1, \nu_1}_a \mathcal{I}^{\omega_2, \nu_2}_b \mathcal{U} \right)(\chi_1, \chi_2)
\]

\[
= \frac{1}{\nu_1 \omega_1 + 2 \nu_1 \omega_1 + \nu_1 \omega_1 + 2 \nu_1 \omega_1} \int_{\chi_1}^{\chi_2} \int_{\chi_1}^{\chi_2} \int_{s}^{t} \sigma_1(s) \sigma_2(t) \sigma_1(\tau) \sigma_2(u)
\]

\[
\times \exp \left( \frac{\nu_1 - 1}{\nu_1} (\sigma_1(\chi_1) - \sigma_1(s)) \right) \exp \left( \frac{\nu_2 - 1}{\nu_2} (\sigma_2(\chi_2) - \sigma_2(t)) \right)
\]

\[
\times \exp \left( \frac{\nu_1 - 1}{\nu_1} (\sigma_1(s) - \sigma_1(\tau)) \right) \exp \left( \frac{\nu_2 - 1}{\nu_2} (\sigma_2(t) - \sigma_2(\omega)) \right)
\]

\[
\times \frac{\mathcal{U}(\tau, \omega)}{(\sigma_1(\chi_1) - \sigma_1(\chi_2))^{1-\omega_1}(\sigma_2(\chi_2) - \sigma_2(t))^{1-\omega_2}(\sigma_1(s) - \sigma_1(\tau))^{1-\beta_1}(\sigma_2(t) - \sigma_2(\omega))^{1-\beta_2}}
\]

By using Fubini's theorem, we obtain

**Definition 3.** Let \( \sigma_1 \in C^1([a_1, b_1], \mathbb{R}) \) and \( \sigma_2 \in C^1([a_2, b_2], \mathbb{R}) \) be positive strictly increasing functions such that \( \sigma_1'(\chi_1), \sigma_2'(\chi_2) \neq 0 \) for all \( \chi_1 \in [a_1, b_1] \) and \( \chi_2 \in [a_2, b_2] \). The GPF-Caputo derivative of order \( \omega \) of \( \mathcal{U}(\chi_1, \chi_2) \in L^1(I) \) with respect to \( \sigma_1, \sigma_2 \) is defined as

\[
\left( C^{\omega}(\sigma_1, \sigma_2)\mathcal{D}^{\nu}_a \mathcal{U} \right)(\chi_1, \chi_2) = \left( \mathcal{I}^{\nu_1}_a \mathcal{I}^{\nu_2}_b \mathcal{U} \right)(\chi_1, \chi_2),
\]

where \( a = (a_1, a_2), \omega = (\omega_1, \omega_2) \in (0, 1)^2 \), and \( \nu = (\nu_1, \nu_2) \in (0, 1)^2 \).
\[
(\sigma_1, \sigma_2) \mathcal{P}_{\alpha, \nu} \left( (\sigma_1, \sigma_2) \mathcal{P}_{\beta, \gamma} \mathcal{U} \right) (X_1, X_2)
\]

\[
= \frac{1}{\nu_1^{\beta_1} \nu_2^{\beta_2} \Gamma \left( \frac{\alpha_1}{\alpha} \right) \Gamma \left( \frac{\beta_1}{\beta} \right) \Gamma \left( \frac{\gamma_1}{\gamma} \right)} \int_{\alpha_1}^{\alpha_2} \frac{\sigma_1'(r) \sigma_2'(u)}{v_1 - 1} \exp \left( \frac{v_1 - 1}{v_1} (\sigma_1(X_1) - \sigma_1(r)) \right) \exp \left( \frac{v_2 - 1}{v_2} (\sigma_2(X_2) - \sigma_2(u)) \right) \mathcal{U}(r, u) \times \exp \left( \frac{v_2 - 1}{v_2} (\sigma_2(X_2) - \sigma_2(u)) \right) \mathcal{U}(r, u) \times \int_{\gamma_1}^{\gamma_2} \frac{\sigma_1'(s) \sigma_2'(t) ds}{(\sigma_1(X_1) - \sigma_1(s))^{1-\alpha_1} (\sigma_2(X_2) - \sigma_2(t))^{1-\beta_1} (\sigma_1(s) - \sigma_1(t))^{1-\gamma_1} (\sigma_2(t) - \sigma_2(u))^{1-\gamma_2}} \right] \right]
\]

On the other hand, we have

\[
\int_{\gamma_1}^{\gamma_2} \frac{\sigma_1'(s) ds}{(\sigma_1(X_1) - \sigma_1(s))^{1-\alpha_1} (\sigma_2(X_2) - \sigma_2(t))^{1-\beta_1} (\sigma_1(s) - \sigma_1(t))^{1-\gamma_1} (\sigma_2(t) - \sigma_2(u))^{1-\gamma_2}} = \int_{\sigma_1(X_1)}^{\sigma_1(t)} \frac{ds}{(\sigma_1(X_1) - s)^{1-\alpha_1} (s - \sigma_1(t))^{1-\gamma_1}}.
\]

By using the change of variables \( \zeta = s - \sigma_1(t)/\sigma_1(X_1) - \sigma_1(r) \) and \( \eta = t - \sigma_2(u)/\sigma_2(X_2) - \sigma_2(u) \) and by using the fact that \[ \int_0^1 (1-r)^{a-1} r^{1-1} dr = B(a, \lambda) \] and \[ B(a, \lambda) = \Gamma(a) \Gamma(\lambda) / \Gamma(a + \lambda), \] we get

\[
\int_{\gamma_1}^{\gamma_2} \frac{\sigma_1'(t) dt}{(\sigma_2(X_2) - \sigma_2(t))^{1-\beta_1} (\sigma_2(t) - \sigma_2(u))^{1-\beta_2}} = \int_{\sigma_2(X_2)}^{\sigma_2(t)} \frac{dt}{(\sigma_2(t) - \sigma_2(u))^{1-\beta_2}}.
\]
Lemma 5. Let \( \mathcal{U}(x_1,x_2) \in C(J) \), \( \sigma_1 \in C^1([a_1,b_1],[\mathbb{R}]) \), and \( \sigma_2 \in C^1([a_2,b_2],[\mathbb{R}]) \) be positive strictly increasing functions such that \( \sigma_1'(x_1), \sigma_2'(x_2) \neq 0 \) for all \( x_1 \in [a_1,b_1] \) and \( x_2 \in [a_2,b_2] \). Then, we have

\[
\begin{aligned}
\left( \sigma_1, \sigma_2 \right) \int_{\omega}^{\beta} \left( (\sigma_1, \sigma_2) \int_{\omega}^{\beta} \mathcal{U} \right) (x_1, x_2) \\
&= \frac{1}{\Gamma(\omega_1 + \beta_1) \Gamma(\omega_2 + \beta_2)} \left[ \int_{\omega_1}^{\beta_1} \left( \sigma_1(\tau) \sigma_2'(\tau) \right) d\tau \right] \\
&\quad \times \exp \left( \frac{\omega_1}{\omega_1} \left( \sigma_1(x_1) - \sigma_1(x_2) \right) \right) \exp \left( \frac{\omega_2}{\omega_2} \left( \sigma_2(x_2) - \sigma_2(x_2) \right) \right) \\
&\quad \times \left( \sigma_1(x_1) - \sigma_1(x_2) \right)^{\omega_1 + \beta_1 - 1} \left( \sigma_2(x_2) - \sigma_2(x_2) \right)^{\omega_2 + \beta_2 - 1} \mathcal{U}(\tau, u) d\tau d\tau \\
&= \left( \sigma_1, \sigma_2 \right) \int_{\omega}^{\beta} I_{a_1}^{\omega, \beta, v} \mathcal{U} (x_1, x_2).
\end{aligned}
\]

\[\square\]

Lemma 6. Let \( \mathcal{U} \in AC^1(J) \), \( \sigma_1 \in C^1([a_1,b_1],[\mathbb{R}]) \), and \( \sigma_2 \in C^1([a_2,b_2],[\mathbb{R}]) \) be positive strictly increasing functions

\[
\begin{aligned}
\left( \sigma_1, \sigma_2 \right) \int_{\omega}^{\beta} \left( (\sigma_1, \sigma_2) \int_{\omega}^{\beta} \mathcal{U} \right) (x_1, x_2) \\
&= \left( \sigma_1, \sigma_2 \right) \int_{\omega}^{\beta} \left( (\sigma_1, \sigma_2) \int_{\omega}^{\beta} \mathcal{U} \right) (x_1, x_2) \\
&= \left( \sigma_1, \sigma_2 \right) \int_{\omega}^{\beta} \left( (\sigma_1, \sigma_2) \int_{\omega}^{\beta} \mathcal{U} \right) (x_1, x_2) \\
&= \mathcal{U}(x_1, x_2).\]

Using integration by parts, we obtain the following. \(\square\)
such that \( \sigma_1'(x_1), \sigma_2'(x_2) \neq 0 \) for all \( x_1 \in [a_1, b_1] \) and \( x_2 \in [a_2, b_2] \). Then, we have

\[
\begin{align*}
\langle \sigma_1, \sigma_2 \rangle \int_{a_i}^{b_i} \left[ \frac{\partial x_i}{\partial t} \frac{\partial \mathcal{U}(x_1, x_2)}{\sigma_1(x_1)} \right] dt &= \frac{1}{v_1 v_2} \int_{a_i}^{b_i} \sigma_2'(t) \exp \left( \frac{v_2 - 1}{v_2} \left( \sigma_2(x_2) - \sigma_2(t) \right) \right) \mathcal{U}(x_1, t) dt \\
&\quad - \frac{1}{v_1 v_2} \exp \left( \frac{v_1 - 1}{v_1} \left( \sigma_1(x_1) - \sigma_1(t) \right) \right) \mathcal{U}(a_i, t) dt \\
&\quad \times \int_{a_i}^{b_i} \sigma_1'(t) \exp \left( \frac{v_2 - 1}{v_2} \left( \sigma_2(x_2) - \sigma_2(t) \right) \right) \mathcal{U}(a_i, t) dt \\
&\quad + \left( v_1 - 1 \right) \langle \sigma_1, \sigma_2 \rangle \int_{a_i}^{b_i} \mathcal{U}(x_1, x_2) dt,
\end{align*}
\]

\[
\begin{align*}
\langle \sigma_1, \sigma_2 \rangle \int_{a_i}^{b_i} \left[ \frac{\partial^2 x_i}{\partial t^2} \frac{\partial \mathcal{U}(x_1, x_2)}{\sigma_1(x_1) \sigma_2(x_2)} \right] dt &= \frac{1}{v_1 v_2} \mathcal{U}(x_1, x_2) - \frac{1}{v_1 v_2} \exp \left( \frac{v_1 - 1}{v_1} \left( \sigma_1(x_1) - \sigma_1(t) \right) \right) \mathcal{U}(a_1, x_2) \\
&\quad - \frac{1}{v_1 v_2} \exp \left( \frac{v_2 - 1}{v_2} \left( \sigma_2(x_2) - \sigma_2(a_2) \right) \right) \mathcal{U}(x_1, a_2) \\
&\quad + \frac{1}{v_1 v_2} \exp \left( \frac{v_1 - 1}{v_1} \left( \sigma_1(x_1) - \sigma_1(a_1) \right) \right) \\
&\quad \times \exp \left( \frac{v_2 - 1}{v_2} \left( \sigma_2(x_2) - \sigma_2(a_2) \right) \right) \mathcal{U}(a_1, a_2) \\
&\quad + \frac{1}{v_1 v_2} \frac{v_1 - 1}{v_1} \int_{a_i}^{b_i} \sigma_1'(t) \exp \left( \frac{v_1 - 1}{v_1} \left( \sigma_1(x_1) - \sigma_1(t) \right) \right) \mathcal{U}(s, x_2) ds \\
&\quad + \frac{1}{v_1 v_2} \frac{v_2 - 1}{v_2} \int_{a_i}^{b_i} \sigma_2'(t) \exp \left( \frac{v_2 - 1}{v_2} \left( \sigma_2(x_2) - \sigma_2(t) \right) \right) \mathcal{U}(x_1, t) dt
\end{align*}
\]
where $a = (a_1, a_2), \omega = (\omega_1, \omega_2) \in (0, 1)^2$, and $v = (v_1, v_2) \in (0, 1)^2$.

As a consequence, we obtain the following:

\[
\begin{align*}
&\left(\int_{a_i} s \, \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_1(x_1) - \sigma_1(a_1))\right)\right) (x_1, x_2) = \mathcal{U}(x_1, x_2) - \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_1(x_1) - \sigma_1(a_1))\right) \mathcal{U}(x_1, x_2) \\
&\quad \quad - \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_1(x_1) - \sigma_1(a_1))\right) \mathcal{U}(x_1, x_2) \\
&\quad \quad + \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_1(x_1) - \sigma_1(a_1))\right) \\
&\quad \quad \times \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_1(x_1) - \sigma_1(a_1))\right) \mathcal{U}(x_1, x_2),
\end{align*}
\]

where $a = (a_1, a_2), \omega = (\omega_1, \omega_2) \in (0, 1)^2$, and $v = (v_1, v_2) \in (0, 1)^2$.

**Lemma 7.** Let $\mathcal{U} \in AC^1(J)$, $\sigma_1 \in C^1([a_1, b_1], \mathbb{R})$, and $\sigma_2 \in C^1([a_2, b_2], \mathbb{R})$ be positive strictly increasing functions such that $\sigma_1(x_1), \sigma_2(x_2) \neq 0$ for all $x_1 \in [a_1, b_1]$ and $x_2 \in [a_2, b_2]$. Then, we have

\[
\left(\int_{a_i} s \, \mathcal{U}(x_1, x_2)\right) (x_1, x_2) = \mathcal{U}(x_1, x_2) - \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_2(x_1) - \sigma_2(a_2))\right) \mathcal{U}(x_1, x_2) \\
&\quad \quad - \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_2(x_1) - \sigma_2(a_2))\right) \mathcal{U}(x_1, x_2) \\
&\quad \quad + \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_2(x_1) - \sigma_2(a_2))\right) \\
&\quad \quad \times \exp\left(-\frac{v_1 - 1}{v_1} (\sigma_2(x_1) - \sigma_2(a_2))\right) \mathcal{U}(x_1, x_2),
\]

where $a = (a_1, a_2), \omega = (\omega_1, \omega_2) \in (0, 1)^2$, and $v = (v_1, v_2) \in (0, 1)^2$.

**Proposition 8.** Let $\sigma_1 \in C^1([a_1, b_1], \mathbb{R})$ and $\sigma_2 \in C^1([a_2, b_2], \mathbb{R})$ be positive strictly increasing functions.
where \( a = (a_1, a_2), \omega = (\omega_1, \omega_2) \in (0,1)^2 \), and \( v = (v_1, v_2) \in (0,1)^2 \).

**Definition 9.** (see [23]). Let \( m \in \mathbb{N}^* \), \( \omega, \beta, z, \rho \in \mathbb{C} \), such that \( R(\omega_j), R(\beta_j) > 0 \) for \( j = 1, 2, \ldots, m \). The generalized Mittag–Leffler function is defined by

\[
E_{\rho}((\omega_j, \beta_j)_{j=1}^{m}; (z)) = \sum_{k=0}^{\infty} \frac{\rho_k}{\Gamma(\omega_j + \beta_j)} \frac{z^k}{k!},
\]

where

\[
(\rho)_k = \rho(\rho + 1), \ldots, (\rho + k - 1) = \frac{(\rho + k)}{\Gamma(\rho)}.
\]

We have the following particular case \( (m = 2 \text{ and } \rho = 1) \):

\[
E_{1}((\omega_j, \beta_j)_{j=1}^{2}; (z)) = \frac{z}{\Gamma(\omega_j + \beta_j)}
\]

**3. Existence Results**

We first require the following lemma in order to demonstrate the existence and uniqueness results.

**Lemma 10.** \( \mathcal{U} \in C(\mathbb{J}) \) satisfies (1)-(2) if and only if

\[
\mathcal{U}(\chi_1, \chi_2) = \exp\left(\frac{v_1 - 1}{v_1} (\sigma_1(\chi_1) - \sigma_1(a_1))\right) \psi(\chi_2) + \exp\left(\frac{v_2 - 1}{v_2} (\sigma_2(\chi_2) - \sigma_2(a_2))\right) \psi(\chi_1)
\]

\[
- \exp\left(\frac{v_1 - 1}{v_1} (\sigma_1(\chi_1) - \sigma_1(a_1))\right) \exp\left(\frac{v_2 - 1}{v_2} (\sigma_2(\chi_2) - \sigma_2(a_2))\right) \psi(a_1)
\]

\[
+ (\sigma, \sigma)_{\omega}^{\rho} \mathcal{F}(\chi_1, \chi_2, \mathcal{U}(\chi_1, \chi_2)).
\]

**Proof.** Assume that \( \mathcal{U} \) satisfies (20). By applying \( C^{-(\sigma, \sigma)} D_{a^*}^{\rho, \nu} \) and using Lemma 5, we get \( \mathcal{U} \) which satisfies (1). We have the integral as zero when \( \chi_1 = a_1 \) or \( \chi_2 = a_2 \); therefore, the initial conditions in (2) are satisfied. Then, \( \mathcal{U} \) satisfies (1)-(2). Conversely, suppose \( \mathcal{U} \) satisfies (1)-(2) and let

\[
h(\chi_1, \chi_2) = \mathcal{F}(\chi_1, \chi_2, \mathcal{U}(\chi_1, \chi_2))
\]

\[
+ \exp\left(\frac{v_1 - 1}{v_1} (\sigma_1(\chi_1) - \sigma_1(a_1))\right) \exp\left(\frac{v_2 - 1}{v_2} (\sigma_2(\chi_2) - \sigma_2(a_2))\right) \mathcal{U}(a_1, a_2)
\]

\[
- \exp\left(\frac{v_1 - 1}{v_1} (\sigma_1(\chi_1) - \sigma_1(a_1))\right) \mathcal{U}(a_1, \chi_2)
\]

\[
+ \exp\left(\frac{v_1 - 1}{v_1} (\sigma_1(\chi_1) - \sigma_1(a_1))\right) \exp\left(\frac{v_2 - 1}{v_2} (\sigma_2(\chi_2) - \sigma_2(a_2))\right) \mathcal{U}(a_1, a_2).
\]

By applying \( (\sigma, \sigma)_{\omega}^{\rho, \nu} \) to (21), we find
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Let

\[
\begin{align*}
\Upsilon(x, y) &= \exp\left(\frac{v_1 - 1}{v_1} (\sigma_1 (y) - \sigma_1 (a_1))\right) \psi(y) + \exp\left(\frac{v_2 - 1}{v_2} (\sigma_2 (y) - \sigma_2 (a_2))\right) \psi(x) \\
&\quad - \exp\left(\frac{v_1 - 1}{v_1} (\sigma_1 (x) - \sigma_1 (a_1))\right) \exp\left(\frac{v_2 - 1}{v_2} (\sigma_2 (x) - \sigma_2 (a_2))\right) \psi(a_1),
\end{align*}
\]

where

\[
G = \{(x, y, \Upsilon) \text{ such that } (x, y) \in [a_1, h_1^*] \times [a_2, h_2^*] \text{ and } |\Upsilon - \Upsilon(x, y)| \leq k\}.
\]

\[
M = \sup_{(x, y, \Upsilon) \in G} |\Upsilon(x, y, \Upsilon)|,
\]

and

\[
(h_1, h_2) = \begin{cases} (h_1^*, h_2^*), & \text{if } M = 0, \\
\min\left(h_1^*, \sigma_1^{-1} \left(\frac{k^2}{M^{1/2}} \left(\frac{1}{\omega_1} + 1\right)^{1/\omega_1}\right)\right), \min\left(h_2^*, \sigma_2^{-1} \left(\frac{k^2}{M^{1/2}} \left(\frac{1}{\omega_2} + 1\right)^{1/\omega_2}\right)\right), & \text{otherwise}.
\end{cases}
\]
Then, problem (1)-(2) has at least one solution \( U \in C([a_1, h_1] \times [a_2, h_2], \mathbb{R}) \).

**Proof.** If \( M = 0 \), then \( \mathcal{F}(x_1, x_2, U) = 0 \) for all \((x_1, x_2, U) \in \mathcal{G} \). Then, the function \( U: [a_1, h_1] \times [a_2, h_2] \rightarrow \mathbb{R} \) with \( U(x_1, x_2) = \mathcal{F}(x_1, x_2) \) satisfies (1)-(2). For \( M \neq 0 \), define the set by

\[
\mathcal{Y} = \{ U \in C([a_1, h_1] \times [a_2, h_2], \mathbb{R}) \mid \| U - \mathcal{T} \|_{\infty} \leq k \},
\]

(27)

Clearly that \( \mathcal{Y} \) is nonempty, closed, and convex subset of \( C([a_1, h_1] \times [a_2, h_2], \mathbb{R}) \). We define the operator \( \mathcal{A} \) on this set \( \mathcal{Y} \) by

\[
(\mathcal{A}U)(x_1, x_2) = \mathcal{F}(x_1, x_2, U) + \frac{1}{v_1 v_2^\omega \Gamma(\omega_1) \Gamma(\omega_2)} \int_{a_1}^{x_1} \int_{a_2}^{x_2} \sigma_1(s) \sigma_2(t) \exp \left( \frac{v_1 - 1}{v_1} \left( \sigma_1(x_1) - \sigma_1(s) \right) \right) \cdot \exp \left( \frac{v_2 - 1}{v_2} \left( \sigma_2(x_2) - \sigma_2(t) \right) \right) \frac{\mathcal{F}(s, t, U(s, t))}{(\sigma_1(x_1) - \sigma_1(s))^{1 - \omega_1} (\sigma_2(x_2) - \sigma_2(t))^{1 - \omega_2}} dt ds.
\]

(28)

We shall show that \( \mathcal{A} \) satisfies the assumption of Schauder’s fixed point theorem. We have \( \mathcal{A} \) as continuous.

Now, we prove that \( \mathcal{A} \) is defined to \( \mathcal{Y} \) into itself; let \( U \in \mathcal{Y} \) and \((x_1, x_2) \in [a_1, h_1] \times [a_2, h_2] \), then

\[
\| \mathcal{A}(U) \|_{\infty} \leq \| \mathcal{T} \|_{\infty} + k.
\]

(30)

Thus, we have \( \mathcal{A}(\mathcal{U}) \in \mathcal{Y} \) if \( \mathcal{U} \in \mathcal{Y} \). Now, we show that \( \mathcal{A}(\mathcal{Y}) \) is relatively compact. Firstly, we show that \( \mathcal{U} \in \mathcal{Y} \) is uniformly bounded. Indeed, from the previous step, we get

\[
\mathcal{Y} = \{ U \in C([a_1, h_1] \times [a_2, h_2], \mathbb{R}) \}
\]

Secondly, we show that \( \mathcal{A}(\mathcal{U}) \) is equicontinuous. Let \((x_1, y_1), (x_2, y_2) \in [a_1, h_1] \times [a_2, h_2] \) such that \( x_1 < x_2 \) and \( y_1 < y_2 \) and \( \mathcal{U} \in \mathcal{Y} \). Then,
As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the abovementioned inequality tends to zero. Hence, $\mathcal{U}(\mathcal{U})$ is equicontinuous, since $\mathcal{F}$ is uniformly continuous in $[a_1, h_1] \times [a_2, h_2]$. As a consequence of Arzela–Ascoli theorem and Schauder’s fixed point theorem, we deduce that $\mathcal{U}$ has a fixed point $\mathcal{U} \in \mathcal{W}$. This fixed point is a solution of problem (1)-(2).

3.1. Uniqueness of Solution. In this subsection, we discuss the uniqueness of solution of problem (1)-(2).

Lemma 12. Assume that there exists a constant $L>0$ such that

$$\|\mathcal{F}(x_1, x_2, \mathcal{U}_1) - \mathcal{F}(x_1, x_2, \mathcal{U}_2)\| \leq L\|\mathcal{U}_1 - \mathcal{U}_2\|,$$  

(32)
for all \((\chi_1, \chi_2) \in [a_1, h_1] \times [a_2, h_2]\) and \(\mathcal{U}_1, \mathcal{U}_2 \in \mathbb{R}_e\), then we have

\[
\|\mathcal{A}(\mathcal{U}_1) - \mathcal{A}(\mathcal{U}_2)\|_{C([-a, \xi] \times [\alpha, \xi])} \leq \frac{L\|\mathcal{U}_1 - \mathcal{U}_2\|_{C([-a, \xi] \times [\alpha, \xi])}}{v_1^\omega U_2^\omega \Gamma(\psi_1 + \omega) \Gamma(\psi_2 + 1)} \left(\sigma_1(\chi_1)^\omega (\sigma_2(\chi_2))^\omega\right). 
\]

(33)

**Proof.** Let \((\chi_1, \chi_2) \in [a_1, h_1] \times [a_2, h_2]\) and \((v, w) \in [a_1, \chi_1] \times [a_2, \chi_2]\), we have

\[
\|\mathcal{A}(\mathcal{U}_1)(v, w) - \mathcal{A}(\mathcal{U}_2)(v, w)\|
\]

\[
= \frac{1}{v_1^\omega U_2^\omega \Gamma(\psi_1) \Gamma(\psi_2)} \int_{a_1}^{\psi_1} \int_{a_2}^{\psi_2} \sigma_1'(s) \sigma_2'(t) \left[\frac{\mathcal{F}(s, t, \mathcal{U}_1(s, t)) - \mathcal{F}(s, t, \mathcal{U}_2(s, t))}{(\sigma_1(v) - \sigma_1(s))^{1+\omega}(\sigma_2(w) - \sigma_2(t))^{1+\omega}}\right] dt dt
\]

\[
\leq \frac{L\|\mathcal{U}_1 - \mathcal{U}_2\|_{C([-a, \xi] \times [\alpha, \xi])}}{v_1^\omega U_2^\omega \Gamma(\psi_1) \Gamma(\psi_2)} \int_{a_1}^{\psi_1} \int_{a_2}^{\psi_2} \sigma_1'(s) \sigma_2'(t) \left[\frac{|\mathcal{U}_1(s, t) - \mathcal{U}_2(s, t)|}{(\sigma_1(v) - \sigma_1(s))^{1+\omega}(\sigma_2(w) - \sigma_2(t))^{1+\omega}}\right] dt dt
\]

(34)

It implies that

\[
\|\mathcal{A}(\mathcal{U}_1) - \mathcal{A}(\mathcal{U}_2)\|_{C([-a, \xi] \times [\alpha, \xi])} \leq \frac{L\|\mathcal{U}_1 - \mathcal{U}_2\|_{C([-a, \xi] \times [\alpha, \xi])}}{v_1^\omega U_2^\omega \Gamma(\psi_1 + \omega) \Gamma(\psi_2 + 1)} \left(\sigma_1(\chi_1)^\omega (\sigma_2(\chi_2))^\omega\right). 
\]

(35)

**Theorem 13.** Suppose that the assumptions of Theorem 11 are satisfied, and suppose that the function \(\mathcal{F}\) satisfies the Lipschitz condition with respect to the third variable with the Lipschitz constant \(L\). Also, let \(j \in \mathbb{N}, (\chi_1, \chi_2) \in [a_1, h_1] \times [a_2, h_2]\) and \(\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{F}\). Then,

\[
\|\mathcal{A}^j(\mathcal{U}_1) - \mathcal{A}^j(\mathcal{U}_2)\|_{C([-a, \xi] \times [\alpha, \xi])} \leq \frac{L\|\mathcal{U}_1 - \mathcal{U}_2\|_{C([-a, \xi] \times [\alpha, \xi])}}{v_1^\omega U_2^\omega \Gamma(\psi_1 + \omega) \Gamma(\psi_2 + 1)} \left(\sigma_1(\chi_1)^\omega (\sigma_2(\chi_2))^\omega\right). 
\]

(36)

**Proof.** For \(j = 0\), the inequality (36) holds. Suppose that (36) is true for \(j \in \mathbb{N}\), then for all \((\chi_1, \chi_2) \in [a_1, h_1] \times [a_2, h_2]\) and \((v, w) \in [a_1, \chi_1] \times [a_2, \chi_2]\), we have
\[ \left\| \mathcal{A}^j \mathcal{U}_1 \right\|_C([a_1, a_1]) - \left\| \mathcal{A}^j \mathcal{U}_2 \right\|_C([a_2, a_2]) \leq \frac{L^j}{v_1^{a_1} v_2^{a_2} \Gamma(\alpha_1 \Gamma(\alpha_2) \Gamma(1 + \alpha_1 (j - 1)) \Gamma(1 + \omega_1 (j - 1)) \Gamma(1 + \alpha_2 (j - 1)) \Gamma(1 + \omega_2 j)} \left( (\sigma_1(t))^{a_1, j} (\sigma_2(t))^{a_2, j} \right) \left( (\sigma_1(x))^{a_1} (\sigma_2(y))^{a_2} \right). \]
3.3, and the set $\mathcal{Y}$ as defined in 3.4. We will use Weissinger’s fixed point theorem to show that $\mathcal{A}$ has a unique fixed point. From 3.6, we get

$$
\left\| (\mathcal{A}^j \mathcal{U}_1) - (\mathcal{A}^j \mathcal{U}_2) \right\|_{C([a,h_1],[a,h_2])} \leq \frac{L}{\Delta \omega_j} \frac{\left\| \mathcal{U}_1 - \mathcal{U}_2 \right\|_{C([a,h_1],[a,h_2])}}{\Gamma(1 + \omega_j) \Gamma(1 + \omega_j)} \left( \sigma_1(h_1) \right)^{\omega_j} \left( \sigma_2(h_2) \right)^{\omega_j}.
$$

(39)

Let

$$
\mathcal{F}_j = \frac{1}{\Delta \omega_j} \left( \frac{L}{\Gamma(1 + \omega_j) \Gamma(1 + \omega_j)} \right) \left( \sigma_1(h_1) \right)^{\omega_j} \left( \sigma_2(h_2) \right)^{\omega_j}.
$$

(40)

so the series converges. This completes the proof. □

4. Numerical Method for the Approximation of the 2D Generalized Proportional Fractional Derivative

We derive and investigate in this section a finite difference method to approximate the solution of system (1)-(2).

$$
\mathcal{F}(x_{i+t+1}, x_{j+1}, u_{i,j}) = \frac{1}{\Delta x \Delta y} \sum_{i=1}^{J} \sum_{j=1}^{J} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \Omega_1^{\omega_1}(x_{i+1} - s) \Omega_2^{\omega_2}(x_{j+1} - t) dt ds,
$$

(41)

$$
\left( \Delta x \Delta y \quad v_i, v_j, u_{i,j} + \Delta x \quad v_i v_j(u_{i,j+1} - u_{i,j}) \right) \quad \nu_i \nu_j(u_{i+1,j+1} - u_{i+1,j} + u_{i,j}),
$$

(42)

with $v_i = 1 - u_i$ and $\Omega_\omega(x) = \exp((\nu - 1/\nu)z^{1-\omega}/(1-\omega))$. In the sequel, we will assume that $\Delta x_1 = \Delta x_2 = \Delta$ for simplicity.

4.1. The Finite Difference Scheme. Let $M$ and $N$ be positive integers and let $\{x_{1i}, x_{2i}, \ldots, x_{Mi}\} \times \{x_{1j}, x_{2j}, \ldots, x_{Nj}\}$ be a partition of $J$ defined by $x_{1i} = \sigma_1(i)$ and $x_{2j} = \sigma_2(j)$, where $\sigma_1(j) = \sigma_1(a_1) + (j - 1) \Delta x$ and $\sigma_2(j) = \sigma_2(a_2) + (j - 1) \Delta y$ with $\Delta x = (\sigma_1(b_1) - \sigma_1(a_1))/N$ and $\Delta y = (\sigma_2(b_2) - \sigma_2(a_2))/M$. We introduce the approximation $u_{i+1,j+1}$ of the solution $\mathcal{U}(x_{i+1}, x_{j+1})$ with $0 \leq i \leq N$ and $0 \leq j \leq M$ as follows: $\forall 1 \leq i \leq N$ and $\forall 1 \leq j \leq M$.

Proposition 15. If $\mathcal{U} \in C^2(J)$, then scheme (41)-(42) is consistent with order (at least) one.

Proof. Let $1 \leq i \leq N$ and $1 \leq j \leq M$. We define the truncation error at a grid node $(x_{i+1}, x_{j+1})$ by

$$
R_{i+1,j+1} = \frac{1}{\Delta x \Delta y} \sum_{i=1}^{J} \sum_{j=1}^{J} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \Omega_1^{\omega_1}(x_{i+1} - s) \Omega_2^{\omega_2}(x_{j+1} - t) dt ds,
$$

(43)

$$
\left( \Delta x \Delta y \quad v_i v_i(u_{i+1,j+1} - u_{i+1,j} - u_{i,j}) \right) \quad \nu_i \nu_j(u_{i+1,j+1} - u_{i+1,j} + u_{i,j}),
$$

where $\mathcal{F}_{i,j} := \mathcal{F}(x_{i+1}, x_{j+1})$. We obtain by Taylor expansions of $\mathcal{U}$ in $(s,t) \in [x_{i+1}, x_{j+1}] \times [x_{i+1}, x_{j+1}]$. 

Discrete Dynamics in Nature and Society
\[ \mathcal{W}_{i,j} = \mathcal{U}(s, t) + \mathcal{O}(\Delta), \]
\[ \mathcal{W}_{i+1,j} - \mathcal{W}_{i,j} = \Delta \left( \partial_{x_1} \mathcal{U}(s, t) + (x_{1i+1/2} - s) \partial^2_{x_1x_1} \mathcal{U}(s, t) \right) + \mathcal{O}(\Delta^2), \]
\[ \mathcal{W}_{i,j+1} - \mathcal{W}_{i,j} = \Delta \left( \partial_{x_2} \mathcal{U}(s, t) + (x_{2j+1/2} - t) \partial^2_{x_2x_2} \mathcal{U}(s, t) \right) + \mathcal{O}(\Delta^2), \]
\[ \mathcal{W}_{i+1,j+1} - \mathcal{W}_{i+1,j} - \mathcal{W}_{i,j+1} + \mathcal{W}_{i,j} = \Delta^2 \partial^2_{x_1x_2} \mathcal{U}(s, t) + \mathcal{O}(\Delta^3). \] 

with \( \xi_{i+1/2} = \left( \xi_i + \xi_{i+1} \right)/2 \), yielding by (1).

\[ R_{r_1,1+i} = \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{X_{ii}}^{X_{ji}} \int_{X_{ij}}^{X_{ji}} \Omega_{x_1}^{\phi_1}(x_{1i+1} - s) \Omega_{x_2}^{\phi_2}(x_{2j+1} - t) dt ds \]
\[ \left( \partial_{x_1} \mathcal{U}(s, t) + v_1 \partial_{x_2} \mathcal{U}(s, t) + v_2 \partial_{x_1} \mathcal{U}(s, t) + \partial^2_{x_1x_1} \mathcal{U}(s, t) + (x_{1i+1/2} - s) \partial^2_{x_1x_1} \mathcal{U}(s, t) \right) + \mathcal{O}(\Delta(t)) \]
\[ = \int_{X_{ii}}^{X_{ji}} \int_{X_{ij}}^{X_{ji}} \Omega_{x_1}^{\phi_1}(x_{1i+1} - s) \Omega_{x_2}^{\phi_2}(x_{2j+1} - t) dt ds \]
\[ + \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{X_{ii}}^{X_{ji}} \int_{X_{ij}}^{X_{ji}} \Omega_{x_1}^{\phi_1}(x_{1i+1} - s) \Omega_{x_2}^{\phi_2}(x_{2j+1} - t) dt ds \]
\[ \left( (x_{1i+1/2} - s) \partial^2_{x_1x_1} \mathcal{U}(s, t) + (x_{2j+1/2} - t) \partial^2_{x_2x_2} \mathcal{U}(s, t) + \mathcal{O}(\Delta) \right) - \mathcal{F}(x_{1i+1}, x_{2j+1}, \mathcal{W}_{i+1,1+j}) + \mathcal{O}(\Delta) \]
\[ = \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{X_{ii}}^{X_{ji}} \int_{X_{ij}}^{X_{ji}} \Omega_{x_1}^{\phi_1}(x_{1i+1} - s) \Omega_{x_2}^{\phi_2}(x_{2j+1} - t) dt ds \]
\[ \left( (x_{1i+1/2} - s) \partial^2_{x_1x_1} \mathcal{U}(s, t) + (x_{2j+1/2} - t) \partial^2_{x_2x_2} \mathcal{U}(s, t) + \mathcal{O}(\Delta) \right). \] 

Hence, we get for any \( 1 \leq i \leq N \) and \( 1 \leq i + 1 \leq M \).

\[ |R_{r_1,1+i}| \leq \kappa_1 \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{X_{ii}}^{X_{ji}} (x_{1i+1/2} - s) \Omega_{x_1}^{\phi_1}(x_{1i+1} - s) ds \int_{X_{ij}}^{X_{ji}} \Omega_{x_2}^{\phi_2}(x_{2j+1} - t) dt \]
\[ + \kappa_1 \sum_{i=1}^{l} \sum_{j=1}^{l} \int_{X_{ii}}^{X_{ji}} (x_{2j+1/2} - t) \Omega_{x_2}^{\phi_2}(x_{2j+1} - t) dt \int_{X_{ij}}^{X_{ji}} \Omega_{x_1}^{\phi_1}(x_{1i+1} - s) ds + \mathcal{O}(\Delta), \]

with \( \kappa_1 = \max(\|\partial^2_{x_1x_1} \mathcal{U}\|_{L^\infty}(f), \|\partial^2_{x_2x_2} \mathcal{U}\|_{L^\infty}(f)) \). Moreover, we have

\[ \left| \int_{X_{ii}}^{X_{ji}} \Omega_{x_1}^{\phi_1}(x_{1i+1} - s) ds \right| = \frac{\gamma_1^{\phi_1} - 1}{\Gamma(1 - \phi_1)} \int_0^{\gamma_1^{\phi_1} - a_i} x_1^{-\phi_1} \exp \left( - \frac{\gamma_1^{\phi_1} - a_i}{v_1} \right) d\chi_1 \]
\[ = \frac{\gamma_1^{\phi_1} - 1}{\Gamma(1 - \phi_1)} (\gamma_1^{\phi_1} - a_i)^{1 - \phi_1} \eta^{\phi_1 - 1} d\eta \]
\[ \leq \frac{\gamma_1^{-\phi_1}}{v_1^{\phi_1}}, \]
Assume that \( \epsilon_i \) denotes the perturbations. Define the error function \( \mu(\theta_i, \chi_i) \), and then (41) and (52) yield

\[
\theta_{i,l} = \mu(\chi_{i}, a_{i}), \quad 1 \leq l \leq N + 1; \quad \theta_{i,j} = \mu(a_{i}, \chi_{j}), \quad 1 \leq j \leq M + 1,
\]

where \( \epsilon_{i+1,j+1} \geq 0 \) denotes the perturbations.

**Proof.** All we need is to prove that scheme (41)-(42) is stable with respect to perturbations. Let \( (u_{i,j})_{1 \leq i \leq M, 1 \leq j \leq N+1} \) be the solution of (41)-(42) and let \( \left( \theta_{i,j} \right)_{1 \leq i \leq N+1, 1 \leq j \leq M+1} \) be the solution of system (41)-(42) with additional perturbations for all \( 1 \leq i \leq N \) and \( 1 \leq j \leq M \).

Theorem 16. Assume that \( \mathcal{F} \) is \( \Gamma \)-Lipschitz with respect to its third variable. Then, scheme (41)-(42) is convergent.

\[
\sum_{i=1}^{N} \sum_{j=1}^{M+1} \int_{X_{i}}^{X_{i+1}} \int_{X_{j}}^{X_{j+1}} \Omega^{\phi}_{i}(X_{i+1} - s) \Omega^{\phi}_{j}(X_{j+1} - t) \ dt \ ds(\Delta^2) \ 
\]

\[
+ \nu_{i,j} \left( \theta_{i+1,j+1} - \theta_{i,j+1} + \theta_{i,j+1} - \theta_{i,j} \right) - \Delta^2 \ \mathcal{F}(X_{i+1}, X_{j+1}, \theta_{i,j}) = \Delta^2 \ \epsilon_{i+1,j+1},
\]

where \( \epsilon_{i+1,j+1} \geq 0 \) denotes the perturbations. Define the error function \( \epsilon_{i,j} \).
\[
Y_{i+1,j+1} = \frac{1}{\Delta t} \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} \int_{X_i}^{X_{i+1}} \int_{X_j}^{X_{j+1}} \Omega^0_{\varphi_i} (\chi_{i+1} - s) \Omega^0_{\varphi_j} (\chi_{j+1} - t) \, dt \, ds \\
\cdot (\Delta^2 \varphi_i \varphi_j e_{i,j} + \Delta \varphi_i e_{i+1,j} - e_{i,j}) + \Delta \varphi_j (e_{i,j+1} - e_{i,j}) \\
+ \varphi_i \varphi_j (e_{i+1,j+1} - e_{i+1,j} + e_{i,j+1}) \\
= e_{i+1,j+1} + \mathcal{F}(\chi_{i+1}, \chi_{j+1}, \theta_{i,j}) - \mathcal{F}(\chi_{i+1}, \chi_{j+1}, u_{i,j}) \\
\leq e_{i+1,j+1} + L|e_{i,j}|.
\]

(54)

In addition, we have for all \(1 \leq i \leq I\),
\[
\int_{X_i}^{X_{i+1}} \Omega^0_{\varphi_i} (\chi_{i+1} - s) \, ds = \varphi_i^{\varphi_i^{-1}}(\kappa_{i+1}^1 - \kappa_{i-1}^1),
\]
and for all \(1 \leq j \leq J\),
\[
\int_{X_j}^{X_{j+1}} \Omega^0_{\varphi_j} (\chi_{j+1} - t) \, dt = \varphi_j^{\varphi_j^{-1}}(\kappa_{j+1}^2 - \kappa_{j-1}^2),
\]
(55)

(56)

\[
C_0 Y_{i+1,j+1} = \Delta^2 \varphi_i \varphi_j \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} (\kappa_{i+1}^1 - \kappa_{i-1}^1)(\kappa_{j+1}^2 - \kappa_{j-1}^2)e_{i,j} \\
+ \Delta \varphi_i \varphi_j \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} (\kappa_{j+1}^2 - \kappa_{j-1}^2)(e_{i+1,j} - e_{i,j}) \\
+ \Delta \varphi_j \varphi_i \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} (\kappa_{i+1}^1 - \kappa_{i-1}^1)(e_{i,j+1} - e_{i,j}) \\
+ \varphi_i \varphi_j \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} (\kappa_{i+1}^1 - \kappa_{i-1}^1)(e_{i+1,j+1} - e_{i+1,j} - e_{i,j+1} + e_{i,j}).
\]

(57)

Using some index changes, one may find
\[
C_0 Y_{i+1,j+1} = S_{i+1,j+1} + C_0 (\kappa_{i+1}^1 - \kappa_{i}^1)(\kappa_{j+1}^2 - \kappa_{j}^2)e_{i+1,j+1}
\]
(58)

\[
S_{i+1,j+1} = C_1 \sum_{i=1}^{I-1} \sum_{j=1}^{J-1} (\kappa_{i+1}^1 - \kappa_{i}^1)(\kappa_{j+1}^2 - \kappa_{j}^2)e_{i,j} \\
+ C_2 \sum_{i=2}^{I-1} \sum_{j=1}^{J-1} (\kappa_{i+2}^1 - \kappa_{i+1}^1)(\kappa_{j+2}^2 - \kappa_{j+1}^2)e_{i,j} \\
+ C_3 \sum_{i=1}^{I-1} \sum_{j=2}^{J-1} (\kappa_{i+1}^1 - \kappa_{i}^1)(\kappa_{j+2}^2 - \kappa_{j+1}^2)e_{i,j} \\
+ C_4 \sum_{i=1}^{I-1} \sum_{j=2}^{J-1} (\kappa_{i+2}^1 - \kappa_{i+1}^1)(\kappa_{j+2}^2 - \kappa_{j+1}^2)e_{i,j} \\
+ C_5 \sum_{i=1}^{I-1} \sum_{j=2}^{J-1} (\kappa_{i+1}^1 - \kappa_{i}^1)(\kappa_{j+2}^2 - \kappa_{j+1}^2)e_{i+1,j+1} \\
+ C_6 \sum_{j=2}^{J-1} (\kappa_{i+1}^1 - \kappa_{i}^1)(\kappa_{j+2}^2 - \kappa_{j+1}^2)e_{i+1,j+1}
\]

(59)
\[ C_1 = \Delta^2 - \Delta(1 + \Delta)(v_1 + v_2) + (1 + \Delta)^2 v_1 v_2, \]
\[ C_2 = v_1(1 - (1 + \Delta)v_2), \]
\[ C_3 = v_2(1 - (1 + \Delta)v_1), \]
\[ C_4 = v_1 v_2. \]  

(60)

Now, we need the following results.

Lemma 17. Let \( D_i = v_i^{\omega_i - \varphi}/(2 - \omega_i) \) and \( E_i = v_i^{\varphi - \omega_i}/(2 - \omega_i) \), \( i = 1, 2 \). Then, we have

\[ \kappa_i^j \geq D_i \Delta^{1 - \omega_i}, \quad i = 1, 2, \]  

(61)

and \( \forall n \in \mathbb{N} \),

\[ \kappa_n^i \leq E_i \Delta^{1 - \omega_i}, \quad i = 1, 2. \]  

(62)

Proof. The first inequality is a direct consequence of the estimate \( \varphi(z, \zeta) \geq e^{-\Delta}/(1 + \zeta)z^t \) for any \( z \in [0, \mathcal{Z}] \). Using the identity \( b^t - a^t \leq (b - a)a^t \) for any \( a \leq b \) and \( t \in [0, 1] \), we obtain for \( 1 \leq i \leq 1 + 1 \),

\[ \kappa_{i+1}^1 = \frac{v_i^{\omega_i - 1}}{\Gamma(1 - \omega_i)} \int_{x_i}^{x_{i+1}} (x_{i+1} - s)^{-\omega_i} e^{-\omega_i (x_{i+1} - s)} \, ds \]

\[ \leq \frac{v_i^{\omega_i - 1}}{(2 - \omega_i)} \left( (x_{i+1} - x_i)^{1 - \omega_i} - (x_{i+1} - x_i)^{1 - \omega_i} \right) \]

\[ \leq \frac{v_i^{\omega_i - 1}}{\Gamma(1 - \omega_i) (2 - \omega_i)}. \]  

(63)

We obtain the following from (54) and (58) and Lemma 17:

\[ v_1 v_2 D_1 D_2 \Delta^{(\varphi_i + \varphi_j)} e_{i+1,j+1} \leq C_0(e_{i+1,j+1} + L|e_{i,j}|) - S_{i+1,j+1}, \]  

(64)

yielding

\[ |e_{i+1,j+1}| \leq C_0D_2 \Delta^{(\varphi_i + \varphi_j)} e_{i+1,j+1} + D_3 \Delta^{(\varphi_i + \varphi_j)} S_{i+1,j+1}, \]  

(65)

with \( D_3 = (v_1 v_2 D_1 D_2)^{-1} \). Let us notice that \( C_1 \geq 0, \quad C_2 \leq 0, \quad C_3 \leq 0, \) and \( C_4 \geq 0 \) if \( \Delta \) is sufficiently small (a sufficient condition is \( \Delta \leq \min(v_1/1 - v_1, v_2/1 - v_2) \)). Thus, we deduce

\[ |e_{i+1,j+1}| \leq C_0D_2 \Delta^{(\varphi_i + \varphi_j)} e_{i+1,j+1} + L|e_{i,j}| \]

\[ \times \exp \left\{ D_3 \Delta^{(\varphi_i + \varphi_j)} \left[ (C_1 \sum_{i=1}^I \sum_{j=1}^J (\kappa_i^1 - \kappa_i^j) (\kappa_j^1 - \kappa_j^2) 
\right. \right. \]

\[ \left. \left. \quad - C_2 \sum_{i=2}^I \sum_{j=1}^J (\kappa_i^1 - \kappa_i^j) (\kappa_j^1 - \kappa_j^2) \right) \right. \]

\[ \left. \left. \quad - C_3 \sum_{i=1}^I \sum_{j=2}^J (\kappa_i^1 - \kappa_i^j) (\kappa_j^2 - \kappa_j^2) \right) \right. \]

\[ \left. \left. \quad + C_4 \sum_{j=2}^J (\kappa_i^1 - \kappa_i^j) (\kappa_j^2 - \kappa_j^2) \right) \right. \]

\[ \left. \left. \quad + C_5 \sum_{i=1}^I \sum_{j=2}^J (\kappa_i^1 - \kappa_i^j) (\kappa_j^2 - \kappa_j^2) \right) \right. \]

\[ \left. \left. \quad \right) \right. \]

\[ \leq C_0D_2 \Delta^{(\varphi_i + \varphi_j)} e_{i+1,j+1} + L|e_{i,j}| \]

\[ \times \exp \left\{ D_3 \Delta^{(\varphi_i + \varphi_j)} \left[ (C_1 - C_2 - C_3 + C_4) \kappa_i^1 \kappa_i^2 + C_4 \kappa_i^1 \kappa_i^2 \right] \right\}. \]
In addition, there exists a constant $C_5 > 0$ that depends on $v_1$ and $v_2$ only but not on $\Delta$ such that
\[
C_1 - C_2 - C_3 + C_4 = \Delta^2 - \Delta(\Delta + 2)(v_1 + v_2) + (\Delta + 2)^2 v_1 v_2 \leq C_5 \Delta^2. \tag{67}
\]

Using Lemma 17 once more, we deduce
\[
\left| e_{i+1,j+1} \right| \leq \tilde{D}_4 \Delta^{\bar{\omega}_1 + \bar{\omega}_2} \max_{1 \leq i \leq \tilde{i} + 1, 1 \leq j \leq \tilde{j} + 1} \bar{e}_{i,j} \tag{68}
\]
with $\tilde{D}_4 = \tilde{v}_1^{\bar{\omega}_1} \tilde{v}_2^{\bar{\omega}_2} D_3 (1 + L) e^{D_1 E_1 E_2 (C_3 + C_5)}$ is a constant that do not depend on $\Delta$. This achieves the proof. □

4.2. Numerical Tests

4.2.1. Example 1. We consider system (1)-(2) with the data as follows:
\[
\begin{align*}
  \mathcal{F}(\chi_1, \chi_2, \mathcal{U}) &= 0, \quad (\chi_1, \chi_2, \mathcal{U}) \in [0, 1] \times [0, 1] \times \mathbb{R}, \\
  \phi(\chi_1) &= \cos(\pi \chi_1 / 2), \chi_1 \in [0, 1], \\
  \psi(\chi_2) &= e^{\chi_2}, \chi_2 \in [0, 1].
\end{align*}
\tag{69}
\]
Figure 2: The numerical solutions (a) versus the exact solutions (b) relative to example 2. The parameters used are \((\varpi_1, \varpi_2) = (0.5, 0.2)\) and from top to bottom \((v_1; v_2) = (0.5; 0.5), (v_1; v_2) = (0.9; 0.2),\) and \((v_1; v_2) = (0.1; 0.4)\).
Table 2: $L^\infty$-errors and convergence orders for example 2 with different parameters $v_1$ and $v_2$.

<table>
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<th>$\Delta$</th>
<th>$(v_1; v_2) = (0.5; 0.5)$</th>
<th>Order</th>
<th>$(v_1; v_2) = (0.9; 0.2)$</th>
<th>Order</th>
<th>$(v_1; v_2) = (0.1; 0.4)$</th>
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<td>—</td>
<td>5.1722 (−04)</td>
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<td>5.1705 (−04)</td>
<td>2.2091</td>
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</tbody>
</table>

and $(\sigma_1, \sigma_2)(\xi) = (\sqrt{1 + \xi^2}, \ln(1 + \xi))$. According to Lemma 10, the solution of system (1)-(2) is given by

$$
\mathcal{U}(\chi_1, \chi_2) = \exp\left(\frac{v_1 - 1}{v_1} \left(\sqrt{1 + x^2} - 1\right)\right)e^{\phi} + (1 + y)^{\nu_1-1/\nu_2} \cos\left(\pi \chi_1/2\right)
$$

$$
- \exp\left(\frac{v_1 - 1}{v_1} \left(\sqrt{1 + x^2} - 1\right)\right)(1 + y)^{\nu_2-1/\nu_1}.
$$

Figure 1 shows the numerical solution obtained by using the FD scheme (41)-(42) versus the exact solution. The parameters used are $[a_1, b_1] = [a_2, b_2] = [0, 1]$, $(\alpha_1, \alpha_2) = (0.5, 0.5)$, and $(v_1, v_2) = (0.7, 0.4)$. Clearly, the approximated solution fits very well with the theoretical one. Table 1 reports the $L^\infty$ difference of the solutions, as well as the orders of convergence for various values of $(v_1, v_2)$. Notice that these errors go to zero when the path $\Delta$ tends to zero, which is in total agreement with our theoretical study.

4.2.2. Example 2. Consider system (1)-(2) with the data:

$$
f(\chi_1, \chi_2, \mathcal{U}) = C \frac{x_1^2 x_2}{1 + x_2} \mathcal{U}, (\chi_1, \chi_2, \mathcal{U}) \in [0, 1] \times [0, 1] \times \mathbb{R},
$$

$$
\psi(\chi_1) = e^{1/\chi_1}, x_1 \in [0, 1],
$$

$$
\psi(\chi_2) = e^{1/\chi_2}, x_2 \in [0, 1],
$$

with $\lambda_i = v_i - 1/v_i, i = 1, 2$, $C = 2\nu_1\nu_2 + \lambda_1 v_1 \nu_2 + \lambda_2 v_2 \nu_1 + \lambda_1 \lambda_2 v_1 \nu_2 / v_1 - \omega_1$, $v_2^{-1/\omega_2} \Gamma(2 - \omega_2) \Gamma(2 - \omega_2)$, $\sigma_1(\xi) = \xi^{2/1 - \omega_1}$, and $\sigma_2(\xi) = (2\xi/1 + \xi)^{1/1 - \omega_2}$. One can check that the solution of (1)-(2) is given by $\mathcal{U}(\chi_1, \chi_2) = e^{1/\chi_1} v_1^{\nu_1} \chi_2$. We plotted in Figure 2 the exact versus the numerical solutions obtained by using the FD scheme (41)-(42) for various values of $(v_1, v_2)$. The fractional order is set to $(\alpha_1, \alpha_2) = (0.5, 0.2)$. One can notice that all the solutions are in good agreement. We also compute the errors in the $L^\infty$ norm between the solutions in Table 2. The obtained results clearly confirm the robustness of the proposed scheme.

5. Conclusion

The existence and uniqueness of solutions to 2D systems using the proportional fractional derivative operator with respect to other functions are introduced and studied in this work. We also use the finite differences method to study the numerical approximation of such operator, and we establish the convergence of the numerical scheme.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References


