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Research Article

On Generalized Proportional Fractional Order Derivatives and Darboux Problem for Partial Differential Equations

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The study of the existence and uniqueness of solutions to 2D systems utilizing the generalized proportional fractional derivative operator is the focus of this work. We also derive a finite difference scheme in order to numerically approximate such an operator, and we prove that the method we propose is convergent. Several tests are performed at the end to illustrate the robustness of our algorithm.

1. Introduction

Fractional calculus is a branch of mathematics that allows for the differentiation and integration of noninteger orders, extending the scope of traditional calculus. It involves the study of derivatives and integrals of arbitrary order. This allows for a more general approach to calculus, offering greater flexibility and accuracy. Fractional calculus can be used to model various physical phenomena, including turbulent fluid flow and the diffusion of gases. It is also used in signal processing applications, such as in the design of filters, and in the analysis of fractal images. The goal of fractional calculus is to provide a more comprehensive understanding of phenomena by allowing for the study of derivatives and integrals of arbitrary order. By allowing for the differentiation and integration of noninteger orders, fractional calculus is able to offer greater accuracy and flexibility in the study of various phenomena and has a wide range of applications (see [1–4]). Ordinary and partial fractional differential equations have developed significantly in recent years (see [5–9]).

In [10–12], authors have generalized the Caputo proportional fractional derivatives with respect to another function involving exponential functions. The Caputo derivatives are used to represent the fractional order of derivatives with respect to initial conditions. The exponential functions in the kernels of the derivatives provide a description of the time dynamics of the system. This type of fractional derivatives allows for a more precise description of complex nonlinear systems and provides a better representation of fractional order dynamics (see [13–17]).

The Darboux problem for partial hyperbolic differential equations is an important topic of research in mathematics. It concerns the existence and uniqueness of solutions to certain partial differential equations (PDEs). In recent years, there have been numerous articles discussing the Darboux problem. For instance, readers can refer to [5, 6, 18–22] for further information.

This paper extends [18] to the case of the GPF-Caputo derivative of order ϖ with respect to σ_1, σ_2. The main contribution of this article is to investigate the existence, uniqueness, and numerical solutions of the following fractional partial differential systems:

$$ {}^{C-(\sigma_1,\sigma_2)}D_{a^+}^{\varpi,\nu}\mathcal{U}(\chi_1,\chi_2) = \mathcal{F}(\chi_1,\chi_2,\mathcal{U}(\chi_1,\chi_2)), (\chi_1,\chi_2) \in J = [a_1,b_1]\times[a_2,b_2], \tag{1} $$

$$ \begin{aligned} \mathcal{U}(\chi_1,a_2) &= \varphi(\chi_1), \chi_1 \in [a_1,b_1],\\ \mathcal{U}(a_1,\chi_2) &= \psi(\chi_2), \chi_2 \in [a_2,b_2],\\ \varphi(a_1) &= \psi(a_2), \end{aligned} \tag{2} $$

where $a = (a_1,a_2) \in \mathbb{R}^2$, $\varpi = (\varpi_1,\varpi_2) \in (0,1)^2$, and $\nu = (\nu_1,\nu_2) \in (0,1)^2$; ${}^{C-(\sigma_1,\sigma_2)}D_{a^+}^{\varpi,\nu}$ is the GPF-Caputo derivative of order ϖ with respect to σ_1 and σ_2; and $\mathcal{F}\colon J\times\mathbb{R} \longrightarrow \mathbb{R}$, $\varphi\colon [a_1,b_1] \longrightarrow \mathbb{R}$, and $\psi\colon [a_2,b_2] \longrightarrow \mathbb{R}$ are given continuous functions.

2. Preliminarily

This section presents the definitions, lemmas, and propositions necessary for our findings. In the rest of this paper, we take $J = [a_1,b_1]\times[a_2,b_2]$.

Definition 1. Let $\sigma_1 \in C^1([a_1,b_1],\mathbb{R})$ and $\sigma_2 \in C^1([a_2,b_2],\mathbb{R})$ be positive strictly increasing functions such that $\sigma_1'(\chi_1), \sigma_2'(\chi_2) \neq 0$ for all $\chi_1 \in [a_1,b_1]$ and $\chi_2 \in [a_2,b_2]$. The GPF-integral of order ϖ of $\mathcal{U}(\chi_1,\chi_2) \in L^1(J)$ with respect to σ_1,σ_2 is defined as

$$ \begin{aligned} \left({}^{(\sigma_1,\sigma_2)}I_a^{\varpi,\nu}\mathcal{U}\right)(\chi_1,\chi_2) &= \frac{1}{\nu_1^{\varpi_1}\nu_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1}^{\chi_1}\int_{a_2}^{\chi_2}\sigma_1'(s)\sigma_2'(t)\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(s))\right),\\ &\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(t))\right)\frac{\mathcal{U}(s,t)}{(\sigma_1(\chi_1)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(\chi_2)-\sigma_2(t))^{1-\varpi_2}}dt ds. \end{aligned} \tag{3} $$

where $= (a_1,a_2)$, $\varpi = (\varpi_1,\varpi_2)$, $\nu = (\nu_1,\nu_2) \in (0,1)^2$, and ϖ_1 and ϖ_2 are positive real numbers.

Definition 2. Let $\sigma_1 \in C^1([a_1,b_1],\mathbb{R})$ and $\sigma_2 \in C^1([a_2,b_2],\mathbb{R})$ be positive strictly increasing functions such that $\sigma_1'(\chi_1), \sigma_2'(\chi_2) \neq 0$ for all $\chi_1 \in [a_1,b_1]$ and $\chi_2 \in [a_2,b_2]$. The GPF-Riemann–Liouville derivative of order ϖ of $\mathcal{U}(\chi_1,\chi_2) \in L^1(J)$ with respect to σ_1,σ_2 is defined as

$$ \begin{aligned} \left({}^{(\sigma_1,\sigma_2)}D_a^{\varpi,\nu}\mathcal{U}\right)(\chi_1,\chi_2) &= \frac{{}^{(\sigma_1,\sigma_2)}D^{1,\nu}}{\nu_1^{1-\varpi_1}\nu_2^{1-\varpi_2}\Gamma(1-\varpi_1)\Gamma(1-\varpi_2)}\int_{a_1}^{\chi_1}\int_{a_2}^{\chi_2}\sigma_1'(s)\sigma_2'(t)\\ &\times\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(s))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(t))\right)\\ &\times\frac{\mathcal{U}(s,t)}{(\sigma_1(\chi_1)-\sigma_1(s))^{\varpi_1}(\sigma_2(\chi_2)-\sigma_2(t))^{\varpi_2}}dt ds, \end{aligned} \tag{4} $$

where $= (a_1, a_2), \varpi = (\varpi_1, \varpi_2) \in (0,1)^2, \nu = (\nu_1, \nu_2) \in (0,1)^2, {}^{(\sigma_1,\sigma_2)}D^{1,\nu} = ({}^{\sigma_1}D^{1,\nu_1})({}^{\sigma_2}D^{1,\nu_2})$, $({}^{\sigma_1}D^{1,\nu_1}\mathscr{U})(\chi_1,\chi_2) = (1-\nu_1)\mathscr{U}(\chi_1,\chi_2) + \nu_1\partial_{\chi_1}\mathscr{U}(\chi_1,\chi_2)/\sigma_1(\chi_1)$, and $({}^{\sigma_2}D^{1,\nu_2}\mathscr{U})(\chi_1,\chi_2) = (1-\nu_2)\mathscr{U}(\chi_1,\chi_2) + \nu_2\partial_{\chi_2}\mathscr{U}(\chi_1,\chi_2)/\sigma_2(\chi_2)$.

Definition 3. Let $\sigma_1 \in C^1([a_1,b_1],\mathbb{R})$ and $\sigma_2 \in C^1([a_2,b_2],\mathbb{R})$ be positive strictly increasing functions such that $\sigma_1'(\chi_1), \sigma_2'(\chi_2) \neq 0$ for all $\chi_1 \in [a_1,b_1]$ and $\chi_2 \in [a_2,b_2]$. The GPF-Caputo derivative of order ϖ of $\mathscr{U}(\chi_1,\chi_2) \in L^1(J)$ with respect to σ_1,σ_2 is defined as

$$\begin{aligned}\left({}^{C-(\sigma_1,\sigma_2)}D_{a^+}^{\varpi,\nu}\mathscr{U}\right)(\chi_1,\chi_2) = {}&(\sigma_1,\sigma_2)D^{1,\nu}(\sigma_1,\sigma_2)I_{a^+}^{1-\varpi,\nu}\Bigg[\mathscr{U}(\chi_1,\chi_2) - \exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathscr{U}(\chi_1,a_2)\\ &- \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\mathscr{U}(a_1,\chi_2)\\ &+\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathscr{U}(a_1,a_2)\Bigg],\end{aligned} \tag{5}$$

where $a = (a_1,a_2), \varpi = (\varpi_1,\varpi_2), \nu = (\nu_1,\nu_2) \in (0,1)^2$, and $\varpi_1, \varpi_2, \beta_1, \beta_2$ are positive real numbers.

where $a = (a_1,a_2), \varpi = (\varpi_1,\varpi_2) \in (0,1)^2$, and $\nu = (\nu_1,\nu_2) \in (0,1)^2$.

Lemma 4. *Let $\mathscr{U} \in C(J)$, $\sigma_1 \in C^1([a_1,b_1],\mathbb{R})$, and $\sigma_2 \in C^1([a_2,b_2],\mathbb{R})$ be positive strictly increasing functions such that $\sigma_1'(\chi_1), \sigma_2'(\chi_2) \neq 0$ for all $\chi_1 \in [a_1,b_1]$ and $\chi_2 \in [a_2,b_2]$. Then, we obtain the following relation:*

$$\begin{aligned}&{}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\beta,\nu}\mathscr{U}\right)(\chi_1,\chi_2),\\ &= {}^{(\sigma_1,\sigma_2)}I_{a^+}^{\beta,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\mathscr{U}\right)(\chi_1,\chi_2),\\ &= \left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi+\beta,\nu}\mathscr{U}\right)(\chi_1,\chi_2),\end{aligned} \tag{6}$$

Proof. We have

$$\begin{aligned}&{}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\beta,\nu}\mathscr{U}\right)(\chi_1,\chi_2)\\ &= \frac{1}{\nu_1^{\varpi_1+\beta_1}\nu_2^{\varpi_2+\beta_2}\Gamma(\varpi_1)\Gamma(\varpi_2)\Gamma(\beta_1)\Gamma(\beta_2)}\int_{a_1^+}^{\chi_1}\int_{a_2^+}^{\chi_2}\int_{a_1^+}^{s}\int_{a_2^+}^{t}\sigma_1'(s)\sigma_2'(t)\sigma_1'(\tau)\sigma_2'(u)\\ &\times\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(s))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(t))\right)\\ &\times\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(s)-\sigma_1(\tau))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(t)-\sigma_2(u))\right)\\ &\times\frac{\mathscr{U}(\tau,u)}{(\sigma_1(\chi_1)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(\chi_2)-\sigma_2(t))^{1-\varpi_2}(\sigma_1(s)-\sigma_1(\tau))^{1-\beta_1}(\sigma_2(t)-\sigma_2(u))^{1-\beta_2}}\mathrm{d}u\mathrm{d}\tau\mathrm{d}t\mathrm{d}s.\end{aligned} \tag{7}$$

By using Fubini's theorem, we obtain

$$
\begin{aligned}
&{}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,v}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\beta,v}\mathcal{U}\right)(\chi_1,\chi_2)\\
&=\frac{1}{v_1^{\varpi_1+\beta_1}v_2^{\varpi_2+\beta_2}\Gamma(\varpi_1)\Gamma(\varpi_2)\Gamma(\beta_1)\Gamma(\beta_2)}\int_{a_1^+}^{\chi_1}\int_{a_2^+}^{\chi_2}\sigma_1'(\tau)\sigma_2'(u)\\
&\quad\times\exp\left(\frac{v_1-1}{v_1}(\sigma_1(\chi_1)-\sigma_1(\tau))\right)\exp\left(\frac{v_2-1}{v_2}(\sigma_2(\chi_2)-\sigma_2(u))\right)\mathcal{U}(\tau,u)\\
&\quad\times\int_{\tau}^{\chi_1}\int_{u}^{\chi_2}\frac{\sigma_1'(s)\sigma_2'(t)dtdsdud\tau}{(\sigma_1(\chi_1)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(\chi_2)-\sigma_2(t))^{1-\varpi_2}(\sigma_1(s)-\sigma_1(\tau))^{1-\beta_1}(\sigma_2(t)-\sigma_2(u))^{1-\beta_2}}\\
&=\frac{1}{v_1^{\varpi_1+\beta_1}v_2^{\varpi_2+\beta_2}\Gamma(\varpi_1)\Gamma(\varpi_2)\Gamma(\beta_1)\Gamma(\beta_2)}\int_{a_1^+}^{\chi_1}\int_{a_2^+}^{\chi_2}\sigma_1'(\tau)\sigma_2'(u)\\
&\quad\times\exp\left(\frac{v_1-1}{v_1}(\sigma_1(\chi_1)-\sigma_1(\tau))\right)\exp\left(\frac{v_2-1}{v_2}(\sigma_2(\chi_2)-\sigma_2(u))\right)\mathcal{U}(\tau,u)\\
&\quad\times\left[\int_{\tau}^{\chi_1}\frac{\sigma_1'(s)ds}{(\sigma_1(\chi_1)-\sigma_1(s))^{1-\varpi_1}(\sigma_1(s)-\sigma_1(\tau))^{1-\beta_1}}\int_{u}^{\chi_2}\frac{\sigma_2'(t)dt}{(\sigma_2(\chi_2)-\sigma_2(t))^{1-\varpi_2}(\sigma_2(t)-\sigma_2(u))^{1-\beta_2}}\right]dud\tau.
\end{aligned}
\tag{8}
$$

On the other hand, we have

$$
\begin{aligned}
\int_{\tau}^{\chi_1}\frac{\sigma_1'(s)ds}{(\sigma_1(\chi_1)-\sigma_1(s))^{1-\varpi_1}(\sigma_1(s)-\sigma_1(\tau))^{1-\beta_1}}&=\int_{\sigma_1(\tau)}^{\sigma_1(\chi_1)}\frac{ds}{(\sigma_1(\chi_1)-s)^{1-\varpi_1}(s-\sigma_1(\tau))^{1-\beta_1}},\\
\int_{u}^{\chi_2}\frac{\sigma_2'(t)dt}{(\sigma_2(\chi_2)-\sigma_2(t))^{1-\varpi_2}(\sigma_2(t)-\sigma_2(u))^{1-\beta_2}}&=\int_{\sigma_2(u)}^{\sigma_2(\chi_2)}\frac{dt}{(\sigma_2(\chi_2)-t)^{1-\varpi_2}(t-\sigma_2(u))^{1-\beta_2}}.
\end{aligned}
\tag{9}
$$

By using the change of variables $\zeta = s - \sigma_1(\tau)/\sigma_1(\chi_1) - \sigma_1(\tau)$ and $\eta = t - \sigma_2(u)/\sigma_2(\chi_2) - \sigma_2(u)$ and by using the fact that $\int_0^1 (1-r)^{\alpha-1}r^{\lambda-1}dr = B(\alpha,\lambda)$ and $B(\alpha,\lambda) = \Gamma(\alpha)\Gamma(\lambda)/\Gamma(\alpha+\lambda)$, we get

$$\begin{aligned}
&{}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\beta,\nu}\mathcal{U}\right)(\chi_1,\chi_2)\\
&= \frac{1}{\nu_1^{\varpi_1+\beta_1}\nu_2^{\varpi_2+\beta_2}\Gamma(\varpi_1)\Gamma(\varpi_2)\Gamma(\beta_1)\Gamma(\beta_2)}\int_{a_1^+}^{\chi_1}\int_{a_2^+}^{\chi_2}\sigma_1'(\tau)\sigma_2'(u)\\
&\quad\times\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(\tau))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(u))\right)\\
&\quad\times(\sigma_1(\chi_1)-\sigma_1(\tau))^{\varpi_1+\beta_1-1}(\sigma_2(\chi_2)-\sigma_2(u))^{\varpi_2+\beta_2-1}\mathcal{U}(\tau,u)du d\tau\\
&\quad\times\int_0^1(1-\zeta)^{\varpi_1-1}\zeta^{\beta_1-1}d\zeta\int_0^1(1-\eta)^{\varpi_2-1}\eta^{\beta_2-1}d\eta\\
&= \frac{1}{\nu_1^{\varpi_1+\beta_1}\nu_2^{\varpi_2+\beta_2}\Gamma(\varpi_1+\beta_1)\Gamma(\varpi_2+\beta_2)}\int_{a_1^+}^{\chi_1}\int_{a_2^+}^{\chi_2}\sigma_1'(\tau)\sigma_2'(u)\\
&\quad\times\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(\tau))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(u))\right)\\
&\quad\times(\sigma_1(\chi_1)-\sigma_1(\tau))^{\varpi_1+\beta_1-1}(\sigma_2(\chi_2)-\sigma_2(u))^{\varpi_2+\beta_2-1}\mathcal{U}(\tau,u)du d\tau\\
&= \left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi+\beta,\nu}\mathcal{U}\right)(\chi_1,\chi_2).
\end{aligned}\tag{10}$$

Lemma 5. *Let $\mathcal{U}(\chi_1,\chi_2)\in C(J)$, $\sigma_1\in C^1([a_1,b_1],\mathbb{R})$, and $\sigma_2\in C^1([a_2,b_2],\mathbb{R})$ be positive strictly increasing functions such that $\sigma_1'(\chi_1),\sigma_2'(\chi_2)\neq 0$ for all $\chi_1\in[a_1,b_1]$ and $\chi_2\in[a_2,b_2]$. Then, we have*

$$ {}^{(\sigma_1,\sigma_2)}D_{a^+}^{\varpi,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\mathcal{U}\right)(\chi_1,\chi_2)=\mathcal{U}(\chi_1,\chi_2), \tag{11}$$

where $a=(a_1,a_2)$, $\varpi=(\varpi_1,\varpi_2)\in(0,1)^2$, and $\nu=(\nu_1,\nu_2)\in(0,1)^2$.

Proof. From Definition 2 and Lemma 4, we get

$$\begin{aligned}
&{}^{(\sigma_1,\sigma_2)}D_{a^+}^{\varpi,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\mathcal{U}\right)(\chi_1,\chi_2)\\
&={}^{(\sigma_1,\sigma_2)}D_{a^+}^{1,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{1-\varpi,\nu}\,{}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\mathcal{U}\right)(\chi_1,\chi_2)\\
&={}^{(\sigma_1,\sigma_2)}D^{1,\nu}\left({}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}\mathcal{U}\right)(\chi_1,\chi_2)\\
&=\mathcal{U}(\chi_1,\chi_2).
\end{aligned}\tag{12}$$

□

Using integration by parts, we obtain the following. □

Lemma 6. *Let $\mathcal{U}\in AC^1(J)$, $\sigma_1\in C^1([a_1,b_1],\mathbb{R})$, and $\sigma_2\in C^1([a_2,b_2],\mathbb{R})$ be positive strictly increasing functions* *such that* $\sigma_1'(\chi_1), \sigma_2'(\chi_2) \neq 0$ *for all* $\chi_1 \in [a_1, b_1]$ *and* $\chi_2 \in [a_2, b_2]$. *Then, we have*

$$
\begin{aligned}
{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}\left[\frac{\partial_{\chi_1}\mathcal{U}(\chi_1,\chi_2)}{\sigma_1(\chi_1)}\right] &= \frac{1}{\nu_1\nu_2}\int_{a_2^+}^{\chi_2}\sigma_2'(t)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(t))\right)\mathcal{U}(\chi_1,t)dt \\
&\quad -\frac{1}{\nu_1\nu_2}\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right) \\
&\quad \times\int_{a_2^+}^{\chi_2}\sigma_2'(t)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(t))\right)\mathcal{U}(a_1,t)dt \\
&\quad + (\nu_1-1/)\,{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}\mathcal{U}(\chi_1,\chi_2), \\
{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}\left[\frac{\partial_{\chi_2}\mathcal{U}(\chi_1,\chi_2)}{\sigma_2(\chi_2)}\right] &= \frac{1}{\nu_1\nu_2}\int_{a_1^+}^{\chi_1}\sigma_1'(s)\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(s))\right)\mathcal{U}(s,\chi_2)ds \\
&\quad -\frac{1}{\nu_1\nu_2}\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right) \\
&\quad \times\int_{a_1^+}^{\chi_1}\sigma_1'(s)\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(s))\right)\mathcal{U}(s,a_2)ds \\
&\quad + (\nu_2-1/)\,{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}\mathcal{U}(\chi_1,\chi_2).
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}\left[\frac{\partial^2_{\chi_1\chi_2}\mathcal{U}(\chi_1,\chi_2)}{\sigma_1(\chi_1)\sigma_2(\chi_2)}\right] &= \frac{1}{\nu_1\nu_2}\mathcal{U}(\chi_1,\chi_2)-\frac{1}{\nu_1\nu_2}\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\mathcal{U}(a_1,\chi_2) \\
&\quad -\frac{1}{\nu_1\nu_2}\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(\chi_1,a_2) \\
&\quad +\frac{1}{\nu_1\nu_2}\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right) \\
&\quad \times\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(a_1,a_2) \\
&\quad +\frac{1}{\nu_1\nu_2}\frac{\nu_1-1}{\nu_1}\int_{a_1^+}^{\chi_1}\sigma_1'(s)\exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(s))\right)\mathcal{U}(s,\chi_2)\mathrm{d}s \\
&\quad +\frac{1}{\nu_1\nu_2}\frac{\nu_2-1}{\nu_2}\int_{a_2^+}^{\chi_2}\sigma_2'(t)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_1(\chi_2)-\sigma_2(t))\right)\mathcal{U}(\chi_1,t)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\nu_1\nu_2}\frac{\nu_1-1}{\nu_1}\exp\left(\frac{\nu_2-1}{\nu_2}\left(\sigma_2(\chi_2)-\sigma_2(a_2)\right)\right)\\
&\times\int_{a_1^+}^{\chi_1}\sigma_1'(s)\exp\left(\frac{\nu_1-1}{\nu_1}\left(\sigma_1(\chi_1)-\sigma_1(s)\right)\right)\mathcal{U}(s,a_2)\mathrm{d}s\\
&-\frac{1}{\nu_1\nu_2}\frac{\nu_2-1}{\nu_2}\exp\left(\frac{\nu_1-1}{\nu_1}\left(\sigma_1(\chi_1)-\sigma_1(a_1)\right)\right)\\
&\times\int_{a_2^+}^{\chi_2}\sigma_2'(t)\exp\left(\frac{\nu_2-1}{\nu_2}\left(\sigma_2(\chi_2)-\sigma_2(t)\right)\right)\mathcal{U}(a_1,t)\mathrm{d}t\\
&+\frac{\nu_1-1}{\nu_1}(\nu_2-1/)^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}\mathcal{U}(\chi_1,\chi_2),
\end{aligned}
\tag{14}
$$

where $a=(a_1,a_2),\varpi=(\varpi_1,\varpi_2)\in(0,1)^2$, *and* $\nu=(\nu_1,\nu_2)\in(0,1)^2$.

As a consequence, we obtain the following.

Lemma 7. *Let* $\mathcal{U}\in AC^1(J)$, $\sigma_1\in C^1([a_1,b_1],\mathbb{R})$, *and* $\sigma_2\in C^1([a_2,b_2],\mathbb{R})$ *be positive strictly increasing functions such that* $\sigma_1'(\chi_1),\sigma_2'(\chi_2)\neq 0$ *for all* $\chi_1\in[a_1,b_1]$ *and* $\chi_2\in[a_2,b_2]$. *Then, we have*

$$
\begin{aligned}
{}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\left({}^{C-(\sigma_1,\sigma_2)}D^{\varpi,\nu}\mathcal{U}\right)(\chi_1,\chi_2)&=\mathcal{U}(\chi_1,\chi_2)-\exp\left(\frac{\nu_2-1}{\nu_2}\left(\sigma_2(\chi_2)-\sigma_2(a_2)\right)\right)\mathcal{U}(\chi_1,a_2)\\
&-\exp\left(\frac{\nu_1-1}{\nu_1}\left(\sigma_1(\chi_1)-\sigma_1(a_1)\right)\right)\mathcal{U}(a_1,\chi_2)\\
&+\exp\left(\frac{\nu_1-1}{\nu_1}\left(\sigma_1(\chi_1)-\sigma_1(a_1)\right)\right)\\
&\times\exp\left(\frac{\nu_2-1}{\nu_2}\left(\sigma_2(\chi_2)-\sigma_2(a_2)\right)\right)\mathcal{U}(a_1,a_2),
\end{aligned}
\tag{15}
$$

where $a=(a_1,a_2),\varpi=(\varpi_1,\varpi_2)\in(0,1)^2$, *and* $\nu=(\nu_1,\nu_2)\in(0,1)^2$.

Proposition 8. *Let* $\sigma_1\in C^1([a_1,b_1],\mathbb{R})$ *and* $\sigma_2\in C^1([a_2,b_2],\mathbb{R})$ *be positive strictly increasing functions such that* $\sigma_1'(\chi_1),\sigma_2'(\chi_2)\neq 0$ *for all* $\chi_1\in[a_1,b_1]$ *and* $\chi_2\in[a_2,b_2]$. *The GPF-Caputo derivative of order* ϖ *of* $\mathcal{U}(\chi_1,\chi_2)\in AC^1(J)$ *with respect to* σ_1,σ_2 *is given by*

$$
\begin{aligned}
\left({}^{C-(\sigma_1,\sigma_2)}D_{a^+}^{\varpi,\nu}\mathcal{U}\right)(\chi_1,\chi_2)&={}^{(\sigma_1,\sigma_2)}I_{a^+}^{1-\varpi,\nu}\left({}^{(\sigma_1,\sigma_2)}D^{1,\nu}\mathcal{U}(\chi_1,\chi_2)\right)\\
&=\frac{1}{\nu_1^{1-\varpi_1}\nu_2^{1-\varpi_2}\Gamma(1-\varpi_1)\Gamma(1-\varpi_2)}\int_{a_1^+}^{\chi_1}\int_{a_2^+}^{\chi_2}\sigma_1'(s)\sigma_2'(t)\\
&\times\exp\left(\frac{\nu_1-1}{\nu_1}\left(\sigma_1(\chi_1)-\sigma_1(s)\right)\right)\exp\left(\frac{\nu_2-1}{\nu_2}\left(\sigma_2(\chi_2)-\sigma_2(t)\right)\right)\\
&\times\frac{{}^{(\sigma_1,\sigma_2)}D^{1,\nu}\mathcal{U}(s,t)}{\left(\sigma_1(\chi_1)-\sigma_1(s)\right)^{\varpi_1}\left(\sigma_2(\chi_2)-\sigma_2(t)\right)^{\varpi_2}}dtds,
\end{aligned}
\tag{16}
$$

where $a = (a_1, a_2), \varpi = (\varpi_1, \varpi_2) \in (0,1)^2$, *and* $\nu = (\nu_1, \nu_2) \in (0,1)^2$.

Definition 9. (see [23]). Let $m \in \mathbb{N}^*$, $\varpi_j, \beta_j, z, \rho \in \mathbb{C}$, such that $\Re(\varpi_j), \Re(\beta_j) > 0$ for $j = 1, 2, \ldots, m$. The generalized Mittag–Leffler function is defined by

$$\mathbb{E}_\rho\left(\left(\varpi_j, \beta_j\right)_{j=1,m}; (z)\right) = \sum_{k=0}^{+\infty} \frac{(\rho)_k}{\prod_{j=1}^{m} \Gamma\left(k\varpi_j + \beta_j\right)} \frac{z^k}{k!}, \quad (17)$$

where

$$\begin{aligned}(\rho)_k &= \rho(\rho+1), \ldots, (\rho+k-1)\\ &= \frac{\Gamma(\rho+k)}{\Gamma(\rho)}.\end{aligned} \quad (18)$$

We have the following particular case ($m = 2$ and $\rho = 1$):

$$\begin{aligned}\mathbb{E}_\rho\left(\left(\varpi_j, \beta_j\right)_{j=1,2}; (z)\right) &= \mathbb{E}\left(\left(\varpi_j, \beta_j\right)_{j=1,2}; (z)\right)\\ &= \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\varpi_1 + \beta_1)\Gamma(k\varpi_2 + \beta_2)}.\end{aligned} \quad (19)$$

3. Existence Results

We first require the following lemma in order to demonstrate the existence and uniqueness results.

Lemma 10. *$\mathcal{U} \in C(J)$ satisfies (1)-(2) if and only if*

$$\begin{aligned}\mathcal{U}(\chi_1,\chi_2) &= \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\psi(\chi_2) + \exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\varphi(\chi_1)\\ &\quad - \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\varphi(a_1)\\ &\quad + {}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\mathcal{F}(\chi_1,\chi_2,\mathcal{U}(\chi_1,\chi_2)).\end{aligned} \quad (20)$$

Proof. Assume that $\mathcal{U}$ satisfies (20). By applying ${}^{C-(\sigma_1,\sigma_2)}D_{a^+}^{\varpi,\nu}$ and using Lemma 5, we get $\mathcal{U}$ which satisfies (1). We have the integral as zero when $\chi_1 = a_1$ or $\chi_2 = a_2$; therefore, the initial conditions in (2) are satisfied. Then, $\mathcal{U}$ satisfies (1)-(2). Conversely, suppose $\mathcal{U}$ satisfies (1)-(2) and let

$$\begin{aligned}h(\chi_1,\chi_2) &= \mathcal{F}(\chi_1,\chi_2,\mathcal{U}(\chi_1,\chi_2))\\ &\quad + \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(a_1,a_2)\Big]\\ &= {}^{(\sigma_1,\sigma_2)}D_{a^+}^{1,\nu}\,{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1-\varpi,\nu}\Big[\mathcal{U}(\chi_1,\chi_2) - \exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(\chi_1,a_2)\\ &\quad - \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\mathcal{U}(a_1,\chi_2)\\ &\quad + \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(a_1,a_2)\Big].\end{aligned} \quad (21)$$

By applying ${}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}$ to (21), we find

$$
\begin{aligned}
{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}h(\chi_1,\chi_2) = {}^{(\sigma_1,\sigma_2)}I_{a^+}^{1-\varpi,\nu}\Big[&\mathcal{U}(\chi_1,\chi_2) - \exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(\chi_1,a_2) \\
&- \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\mathcal{U}(a_1,\chi_2) \\
&+ \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(a_1,a_2)\Big].
\end{aligned}
\tag{22}
$$

Applying the operator ${}^{(\sigma_1,\sigma_2)}D_{a^+}^{1-\varpi,\nu}$ to this last equation, we obtain

$$
\begin{aligned}
&\mathcal{U}(\chi_1,\chi_2) - \exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(\chi_1,a_2) - \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\mathcal{U}(a_1,\chi_2) \\
&\quad + \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\mathcal{U}(a_1,a_2) \\
&= {}^{(\sigma_1,\sigma_2)}D_{a^+}^{1-\varpi,\nu}\,{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}h(\chi_1,\chi_2) \\
&= {}^{(\sigma_1,\sigma_2)}D^{1,\nu}\,{}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}\,{}^{(\sigma_1,\sigma_2)}I_{a^+}^{1,\nu}h(\chi_1,\chi_2) \\
&= {}^{(\sigma_1,\sigma_2)}I_{a^+}^{\varpi,\nu}h(\chi_1,\chi_2).
\end{aligned}
\tag{23}
$$

□

Theorem 11. *Let $h_1^* > 0$, $h_2^* > 0$, and $k > 0$. Define the function as follows:*

$$
\begin{aligned}
\mathcal{T}(\chi_1,\chi_2) &= \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\psi(\chi_2) + \exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\varphi(\chi_1) \\
&\quad - \exp\left(\frac{\nu_1-1}{\nu_1}(\sigma_1(\chi_1)-\sigma_1(a_1))\right)\exp\left(\frac{\nu_2-1}{\nu_2}(\sigma_2(\chi_2)-\sigma_2(a_2))\right)\varphi(a_1), \\
G &= \left\{(\chi_1,\chi_2,\mathcal{U})\,\mathrm{such that}\,(\chi_1,\chi_2)\in[\mathrm{a}_1,\mathrm{h}_1^*]\times[\mathrm{a}_2,\mathrm{h}_2^*]\,\mathrm{and}\,|\mathcal{U}-\mathcal{T}(\chi_1,\chi_2)|\le \mathrm{k}\right\},
\end{aligned}
\tag{24}
$$

where

$$
M = \sup_{(\chi_1,\chi_2,\mathcal{U})\in G}|\mathcal{F}(\chi_1,\chi_2,\mathcal{U})|,
\tag{25}
$$

$$
(h_1,h_2) = \begin{cases} (h_1^*,h_2^*), & \text{if } M = 0, \\ \min\left(h_1^*,\sigma_1^{-1}\left(\nu_1\left(\dfrac{k^{1/2}\Gamma(\varpi_1+1)}{M^{1/2}}\right)^{1/\varpi_1}\right)\right), \min\left(h_2^*,\sigma_2^{-1}\left(\nu_2\left(\dfrac{k^{1/2}\Gamma(\varpi_2+1)}{M^{1/2}}\right)^{1/\varpi_2}\right)\right), & \text{otherwise.}\end{cases}
\tag{26}
$$

Then, problem (1)-(2) has at least one solution $\mathscr{U} \in C([a_1, h_1] \times [a_2, h_2], \mathbb{R})$.

Proof. If $M = 0$, then $\mathscr{F}(\chi_1, \chi_2, \mathscr{U}) = 0$ for all $(\chi_1, \chi_2, \mathscr{U}) \in G$. Then, the function $\mathscr{U}: [a_1, h_1^*] \times [a_2, h_2^*] \longrightarrow \mathbb{R}$ with $\mathscr{U}(\chi_1, \chi_2) = \mathscr{T}(\chi_1, \chi_2)$ satisfies (1)-(2). For $M \neq 0$, define the set by

$$\mathscr{V} = \{\mathscr{U} \in C([a_1, h_1] \times [a_2, h_2], \mathbb{R}) \text{such that} \|\mathscr{U} - \mathscr{T}\|_\infty \leq k\}, \tag{27}$$

clearly that $\mathscr{V}$ is nonempty, closed, and convex subset of $C([a_1, h_1] \times [a_2, h_2], \mathbb{R})$. We define the operator $\mathscr{A}$ on this set $\mathscr{V}$ by

$$\begin{aligned}(\mathscr{A}\mathscr{U})(\chi_1, \chi_2) = \mathscr{T}(\chi_1, \chi_2) + \frac{1}{v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1)\Gamma(\varpi_2)} \int_{a_1}^{\chi_1} \int_{a_2}^{\chi_2} \sigma_1'(s)\sigma_2'(t) \exp\left(\frac{v_1 - 1}{v_1}(\sigma_1(\chi_1) - \sigma_1(s))\right) \\ \cdot \exp\left(\frac{v_2 - 1}{v_2}(\sigma_2(\chi_2) - \sigma_2(t))\right) \frac{\mathscr{F}(s, t, \mathscr{U}(s, t))}{(\sigma_1(\chi_1) - \sigma_1(s))^{1-\varpi_1}(\sigma_2(\chi_2) - \sigma_2(t))^{1-\varpi_2}} dt ds.\end{aligned} \tag{28}$$

We shall show that $\mathscr{A}$ satisfies the assumption of Schauder's fixed point theorem. We have $\mathscr{A}$ as continuous. Now, we prove that $\mathscr{A}$ is defined to $\mathscr{V}$ into itself; let $\mathscr{U} \in \mathscr{V}$ and $(\chi_1, \chi_2) \in [a_1, h_1] \times [a_2, h_2]$, then

$$\begin{aligned}\left|(\mathscr{A}\mathscr{U})(\chi_1, \chi_2) - \mathscr{T}(\chi_1, \chi_2)\right| &= \frac{1}{v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1)\Gamma(\varpi_2)} \int_{a_1}^{\chi_1} \int_{a_2}^{\chi_2} \sigma_1'(s)\sigma_2'(t) \exp\left(\frac{v_1 - 1}{v_1}(\sigma_1(\chi_1) - \sigma_1(s))\right) \\ &\quad \cdot \exp\left(\frac{v_2 - 1}{v_2}(\sigma_2(\chi_2) - \sigma_2(t))\right) \frac{|\mathscr{F}(s, t, \mathscr{U}(s, t))|}{(\sigma_1(\chi_1) - \sigma_1(s))^{1-\varpi_1}(\sigma_2(\chi_2) - \sigma_2(t))^{1-\varpi_2}} dt ds \\ &\leq \frac{M}{v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1)\Gamma(\varpi_2)} \int_{a_1}^{\chi_1} \int_{a_2}^{\chi_2} \sigma_1'(s)\sigma_2'(t) \exp\left(\frac{v_1 - 1}{v_1}(\sigma_1(\chi_1) - \sigma_1(s))\right) \\ &\quad \cdot \exp\left(\frac{v_2 - 1}{v_2}(\sigma_2(\chi_2) - \sigma_2(t))\right) \frac{1}{(\sigma_1(\chi_1) - \sigma_1(s))^{1-\varpi_1}(\sigma_2(\chi_2) - \sigma_2(t))^{1-\varpi_2}} dt ds \\ &\leq \frac{M}{v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1)\Gamma(\varpi_2)} \int_{a_1}^{\chi_1} \int_{a_2}^{\chi_2} \frac{\sigma_1'(s)\sigma_2'(t)}{(\sigma_1(\chi_1) - \sigma_1(s))^{1-\varpi_1}(\sigma_2(\chi_2) - \sigma_2(t))^{1-\varpi_2}} dt ds \\ &\leq \frac{M}{v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1 + 1)\Gamma(\varpi_2 + 1)} (\sigma_1(h_1) - \sigma_1(a_1))^{\varpi_1} (\sigma_2(h_2) - \sigma_2(a_2))^{\varpi_2} \\ &\leq \frac{M}{v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1 + 1)\Gamma(\varpi_2 + 1)} (\sigma_1(h_1))^{\varpi_1} (\sigma_2(h_2))^{\varpi_2} \\ &\leq \frac{M}{v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1 + 1)\Gamma(\varpi_2 + 1)} \frac{k v_1^{\varpi_1} v_2^{\varpi_2} \Gamma(\varpi_1 + 1)\Gamma(\varpi_2 + 1)}{M} \\ &\leq k.\end{aligned} \tag{29}$$

Thus, we have $\mathscr{A}(\mathscr{U}) \in \mathscr{V}$ if $\mathscr{U} \in \mathscr{V}$. Now, we show that $\mathscr{A}(\mathscr{V})$ is relatively compact. Firstly, we show that $\mathscr{U} \in \mathscr{V}$ is uniformly bounded. Indeed, from the previous step, we get

$$\|\mathscr{A}(\mathscr{U})\|_\infty \leq \|\mathscr{T}\|_\infty + k. \tag{30}$$

Secondly, we show that $\mathscr{A}(\mathscr{U})$ is equicontinuous. Let $(x_1, y_1), (x_2, y_2) \in [a_1, h_1] \times [a_2, h_2]$ such that $x_1 < x_2$ and $y_1 < y_2$ and $\mathscr{U} \in \mathscr{V}$. Then,

$$\begin{aligned}
&\left|(\mathscr{A}\mathscr{U})(x_2,y_2)-(\mathscr{A}\mathscr{U})(x_1,y_1)\right| \le \left|T(x_2,y_2)-T(x_1,y_1)\right| \\
&\quad+\frac{M}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1^+}^{x_1}\int_{a_2^+}^{y_1}\sigma_1'(s)\sigma_2'(t)\Bigg(\frac{1}{(\sigma_1(x_1)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(y_1)-\sigma_2(t))^{1-\varpi_2}} \\
&\quad-\frac{1}{(\sigma_1(x_2)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(y_2)-\sigma_2(t))^{1-\varpi_2}}\Bigg)dtds \\
&\quad+\frac{M}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1^+}^{x_1}\int_{a_2^+}^{y_1}\sigma_1'(s)\sigma_2'(t) \\
&\quad\times\left[\exp\left(\frac{v_1-1}{v_1}(\sigma_1(x_1)-\sigma_1(s))\right)\exp\left(\frac{v_2-1}{v_2}(\sigma_2(y_1)-\sigma_2(t))\right)\right. \\
&\quad\left.-\exp\left(\frac{v_1-1}{v_1}(\sigma_1(x_2)-\sigma_1(s))\right)\exp\left(\frac{v_2-1}{v_2}(\sigma_2(y_2)-\sigma_2(t))\right)\right] \\
&\quad\times\frac{1}{(\sigma_1(x_1)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(y_1)-\sigma_2(t))^{1-\varpi_2}}dt\,ds \\
&\quad+\frac{M}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1^+}^{x_1}\int_{y_1}^{y_2}\sigma_1'(s)\sigma_2'(t)\frac{1}{(\sigma_1(x_2)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(y_2)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\quad+\frac{M}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{x_1}^{x_2}\int_{a_2^+}^{y_1}\sigma_1'(s)\sigma_2'(t)\frac{1}{(\sigma_1(x_2)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(y_2)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\quad+\frac{M}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\sigma_1'(s)\sigma_2'(t)\frac{1}{(\sigma_2(x_2)-\sigma_2(s))^{1-\varpi_1}(y_2-t)^{1-\varpi_2}}dt\,ds \\
&\le\left|T(x_1,y_1)-T(x_2,y_2)\right| \\
&\quad+\frac{M}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1+1)\Gamma(\varpi_2+1)}\left[\left(\left(\frac{1-v_1}{v_1}\right)\left|\sigma_1(x_1)-\sigma_1(x_2)\right|+\left(\frac{1-v_2}{v_2}\right)\left|\sigma_2(y_1)-\sigma_2(y_2)\right|\right)\right. \\
&\quad\times(\sigma_1(h_1)-\sigma_1(a_1))^{\varpi_1}(\sigma_2(h_2)-\sigma_2(a_2))^{\varpi_2} \\
&\quad\left.+2\left((\sigma_1(x_2)-\sigma_1(a_1))^{\varpi_1}(\sigma_2(y_2)-\sigma_2(y_1))^{\varpi_2}+(\sigma_2(y_2)-\sigma_2(a_2))^{\varpi_2}(\sigma_1(x_2)-\sigma_1(x_1))^{\varpi_1}\right)\right].
\end{aligned} \tag{31}$$

As $x_1 \longrightarrow x_2$ and $y_1 \longrightarrow y_2$, the right-hand side of the abovementioned inequality tends to zero. Hence, $\mathscr{A}(\mathscr{U})$ is equicontinuous, since $\mathscr{T}$ is uniformly continuous in $[a_1,h_1]\times[a_2,h_2]$. As a consequence of Arzela–Ascoli theorem and Schauder's fixed point theorem, we deduce that $\mathscr{A}$ has a fixed point $\mathscr{U}\in\mathscr{V}$. This fixed point is a solution of problem (1)-(2). □

3.1. Uniqueness of Solution. In this subsection, we discuss the uniqueness of solution of problem (1)-(2).

Lemma 12. *Assume that there exists a constant $L>0$ such that*

$$\left|\mathscr{F}(\chi_1,\chi_2,\mathscr{U}_1)-\mathscr{F}(\chi_1,\chi_2,\mathscr{U}_2)\right|\le L\left|\mathscr{U}_1-\mathscr{U}_2\right|, \tag{32}$$

for all $(\chi_1,\chi_2) \in [a_1,h_1] \times [a_2,h_2]$ *and* $\mathscr{U}_1,\mathscr{U}_2 \in \mathbb{R}$, *then we have*

$$\left\|(\mathscr{A}\mathscr{U}_1) - (\mathscr{A}\mathscr{U}_2)\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])} \leq \frac{L\|\mathscr{U}_1 - \mathscr{U}_2\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{\upsilon_1^{\varpi_1}\upsilon_2^{\varpi_2}\Gamma(\varpi_1+1)\Gamma(\varpi_2+1)}(\sigma_1(\chi_1))^{\varpi_1}(\sigma_2(\chi_2))^{\varpi_2}. \tag{33}$$

Proof. Let $(\chi_1,\chi_2) \in [a_1,h_1] \times [a_2,h_2]$ and $(v,w) \in [a_1,\chi_1] \times [a_2,\chi_2]$, we have

$$\begin{aligned}
&\left|(\mathscr{A}\mathscr{U}_1)(v,w) - (\mathscr{A}\mathscr{U}_2)(v,w)\right| \\
&= \frac{1}{\upsilon_1^{\varpi_1}\upsilon_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1}^{v}\int_{a_2}^{w}\sigma_1'(s)\sigma_2'(t)\frac{\left|\mathscr{F}(s,t,\mathscr{U}_1(s,t)) - \mathscr{F}(s,t,\mathscr{U}_2(s,t))\right|}{(\sigma_1(v)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(w)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\leq \frac{L}{\upsilon_1^{\varpi_1}\upsilon_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1}^{v}\int_{a_2}^{w}\sigma_1'(s)\sigma_2'(t)\frac{\left|\mathscr{U}_1(s,t) - \mathscr{U}_2(s,t)\right|}{(\sigma_1(v)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(w)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\leq \frac{L\|\mathscr{U}_1 - \mathscr{U}_2\|_{C([a_1,s]\times[a_2,t])}}{\upsilon_1^{\varpi_1}\upsilon_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1}^{v}\int_{a_2}^{w}\frac{\sigma_1'(s)\sigma_2'(t)}{(\sigma_1(v)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(w)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\leq \frac{L\|\mathscr{U}_1 - \mathscr{U}_2\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{\upsilon_1^{\varpi_1}\upsilon_2^{\varpi_2}\Gamma(\varpi_1+1)\Gamma(\varpi_2+1)}(\sigma_1(\chi_1))^{\varpi_1}(\sigma_2(\chi_2))^{\varpi_2}.
\end{aligned} \tag{34}$$

It implies that

$$\left\|(\mathscr{A}\mathscr{U}_1) - (\mathscr{A}\mathscr{U}_2)\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])} \leq \frac{L\|\mathscr{U}_1 - \mathscr{U}_2\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{\upsilon_1^{\varpi_1}\upsilon_2^{\varpi_2}\Gamma(\varpi_1+1)\Gamma(\varpi_2+1)}(\sigma_1(\chi_1))^{\varpi_1}(\sigma_2(\chi_2))^{\varpi_2}. \tag{35}$$

□

Theorem 13. *Suppose that the assumptions of Theorem 11 are satisfied, and suppose that the function* $\mathscr{F}$ *satisfies the Lipschitz condition with respect to the third variable with the Lipschitz constant* L. *Also, let* $j \in \mathbb{N}$, $(\chi_1,\chi_2) \in [a_1,h_1] \times [a_2,h_2]$ *and* $\mathscr{U}_1,\mathscr{U}_2 \in \mathscr{V}$. *Then,*

$$\left\|\left(\mathscr{A}^j\mathscr{U}_1\right) - \left(\mathscr{A}^j\mathscr{U}_2\right)\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])} \leq \frac{L^j\|\mathscr{U}_1 - \mathscr{U}_2\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{\upsilon_1^{\varpi_1 j}\upsilon_2^{\varpi_2 j}\Gamma(1+\varpi_1 j)\Gamma(1+\varpi_2 j)}(\sigma_1(\chi_1))^{\varpi_1 j}(\sigma_2(\chi_2))^{\varpi_2 j}. \tag{36}$$

Proof. For $j = 0$, the inequality (36) holds. Suppose that (36) is true for $j \in \mathbb{N}$, then for all $(\chi_1,\chi_2) \in [a_1,h_1] \times [a_2,h_2]$ and $(v,w) \in [a_1,\chi_1] \times [a_2,\chi_2]$, we have

$$\begin{aligned}
&\left|\left(\mathscr{A}^{j}\mathscr{U}_1\right)(v,w)-\left(\mathscr{A}^{j}\mathscr{U}_2\right)(v,w)\right| \\
&=\left|\left(\mathscr{A}\mathscr{A}^{j-1}\mathscr{U}_1\right)(v,w)-\left(\mathscr{A}\mathscr{A}^{j-1}\mathscr{U}_2\right)(v,w)\right| \\
&=\frac{1}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1}^{v}\int_{a_2}^{w}\sigma_1'(s)\sigma_2'(t)\frac{\left|\mathscr{F}\left(s,t,\mathscr{A}^{j-1}\mathscr{U}_1(s,t)\right)-\mathscr{F}\left(s,t,\mathscr{A}^{j-1}\mathscr{U}_2(s,t)\right)\right|}{(\sigma_1(v)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(w)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\leq\frac{L}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1}^{v}\int_{a_2}^{w}\sigma_1'(s)\sigma_2'(t)\frac{\left|\mathscr{A}^{j-1}\mathscr{U}_1(s,t)-\mathscr{A}^{j-1}\mathscr{U}_2(s,t)\right|}{(\sigma_1(v)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(w)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\leq\frac{L}{v_1^{\varpi_1}v_2^{\varpi_2}\Gamma(\varpi_1)\Gamma(\varpi_2)}\int_{a_1}^{v}\int_{a_2}^{w}\sigma_1'(s)\sigma_2'(t)\frac{\left\|\mathscr{A}^{j-1}\mathscr{U}_1-\mathscr{A}^{j-1}\mathscr{U}_2\right\|_{C([a_1,s]\times[a_2,t])}}{(\sigma_1(v)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(w)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\leq\frac{L^{j}\left\|\mathscr{U}_1-\mathscr{U}_2\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{v_1^{\varpi_1 j}v_2^{\varpi_2 j}\Gamma(\varpi_1)\Gamma(\varpi_2)\Gamma(1+\varpi_1(j-1))\Gamma(1+\varpi_2(j-1))} \\
&\quad\times\int_{a_1}^{v}\int_{a_2}^{w}\sigma_1'(s)\sigma_2'(t)\frac{(\sigma_1(s))^{\varpi_1(j-1)}(\sigma_2(t))^{\varpi_2(j-1)}}{(\sigma_1(v)-\sigma_1(s))^{1-\varpi_1}(\sigma_2(w)-\sigma_2(t))^{1-\varpi_2}}dtds \\
&\leq\frac{L^{j}\left\|\mathscr{U}_1-\mathscr{U}_2\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{v_1^{\varpi_1 j}v_2^{\varpi_2 j}\Gamma(\varpi_1)\Gamma(\varpi_2)\Gamma(1+\varpi_1(j-1))\Gamma(1+\varpi_2(j-1))} \\
&\quad\times\frac{\Gamma(\varpi_1)\Gamma(\varpi_2)\Gamma(1+\varpi_1(j-1))\Gamma(1+\varpi_2(j-1))}{\Gamma(1+\varpi_1 j)\Gamma(1+\varpi_2 j)}(\sigma_1(x))^{\varpi_1 j}(\sigma_2(y))^{\varpi_2 j} \\
&\leq\frac{L^{j}\left\|\mathscr{U}_1-\mathscr{U}_2\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{v_1^{\varpi_1 j}v_2^{\varpi_2 j}\Gamma(1+\varpi_1 j)\Gamma(1+\varpi_2 j)}(\sigma_1(\chi_1))^{\varpi_1 j}(\sigma_2(\chi_2))^{\varpi_2 j}.
\end{aligned} \tag{37}$$

It implies that

$$\left\|\left(\mathscr{A}^{j}\mathscr{U}_1\right)-\left(\mathscr{A}^{j}\mathscr{U}_2\right)\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}\leq\frac{L^{j}\left\|\mathscr{U}_1-\mathscr{U}_2\right\|_{C([a_1,\chi_1]\times[a_2,\chi_2])}}{v_1^{\varpi_1 j}v_2^{\varpi_2 j}\Gamma(1+\varpi_1 j)\Gamma(1+\varpi_2 j)}(\sigma_1(\chi_1))^{\varpi_1 j}(\sigma_2(\chi_2))^{\varpi_2 j}. \tag{38}$$

Hence, the proof is complete. □

Theorem 14. *Let h_1 and h_2 be the same in Theorem 11. Suppose that $\mathscr{F}\colon G\longrightarrow\mathbb{R}$ satisfies a Lipschitz condition with respect to the third variable with the Lipschitz constant L, where the set G is defined as in Theorem 11. Then, there exists a unique solution $\mathscr{U}\in C([a_1,h_1]\times[a_2,h_2],\mathbb{R})$ of problem (1)-(2).*

Proof. It follows from Theorem 11 that (1)-(2) has a solution. To show the uniqueness, we adapt Theorem 13. We use the operator $\mathscr{A}$ as defined in 3.5, the function $\mathscr{T}$ as defined in

3.3, and the set $\mathscr{V}$ as defined in 3.4. We will use Weissinger's fixed point theorem to show that $\mathscr{A}$ has a unique fixed point. From 3.6, we get

$$\left\|\left(\mathscr{A}^j\mathscr{U}_1\right)-\left(\mathscr{A}^j\mathscr{U}_2\right)\right\|_{C([a_1,h_1]\times[a_2,h_2])}\leq\frac{L^j\left\|\mathscr{U}_1-\mathscr{U}_2\right\|_{C([a_1,h_1]\times[a_2,h_2])}}{v_1^{\varpi_1 j}v_2^{\varpi_2 j}\Gamma(1+\varpi_1 j)\Gamma(1+\varpi_2 j)}(\sigma_1(h_1))^{\varpi_1 j}(\sigma_2(h_2))^{\varpi_2 j}. \tag{39}$$

Let $\mathscr{Z}_j=L^j(\sigma_1(h_1))^{\varpi_1 j}(\sigma_2(h_2))^{\varpi_2 j}/v_1^{\varpi_1 j}v_2^{\varpi_2 j}\Gamma(1+\varpi_1 j)\Gamma(1+\varpi_2 j)$. Then, we have

$$\begin{aligned}\sum_{j=0}^{\infty}\mathscr{Z}_j&=\sum_{j=0}^{\infty}\frac{\left(L(\sigma_1(h_1))^{\varpi_1}(\sigma_2(h_2))^{\varpi_2}/v_1^{\varpi_1}v_2^{\varpi_2}\right)^j}{\Gamma(1+\varpi_1 j)\Gamma(1+\varpi_2 j)}\\&\leq\mathbb{E}\left((\varpi_i,1)_{i=1,2};\left(\frac{L(\sigma_1(h_1))^{\varpi_1}(\sigma_2(h_2))^{\varpi_2}}{v_1^{\varpi_1}v_2^{\varpi_2}}\right)\right),\end{aligned} \tag{40}$$

so the series converges. This completes the proof. □

4. Numerical Method for the Approximation of the 2D Generalized Proportional Fractional Derivative

We derive and investigate in this section a finite difference method to approximate the solution of system (1)-(2).

4.1. The Finite Difference Scheme. Let M and N be positive integers and let $[\chi_{1i},\chi_{1i+1}]\times[\chi_{2j},\chi_{2j+1}]_{1\leq i\leq N+1,1\leq j\leq M+1}$ be a partition of J defined by $\chi_{1i}=\sigma_1^{-1}(\overline{\chi_{1i}})$ and $\chi_{2j}=\sigma_2^{-1}(\overline{\chi_{2j}})$, where $\overline{\chi_{1i}}=\sigma_1(a_1)+(i-1)\Delta x$ and $\overline{\chi_{2j}}=\sigma_2(a_2)+(j-1)\Delta y$ with $\Delta x=(\sigma_1(b_1)-\sigma_1(a_1))/N$ and $\Delta y=(\sigma_2(b_2)-\sigma_2(a_2))/M$. We introduce the approximation $u_{\imath+1,J+1}$ of the solution $\mathscr{U}(\chi_{1\imath+1},\chi_{2J+1})$ with $0\leq\imath\leq N$ and $0\leq J\leq M$ as follows: $\forall 1\leq\imath\leq N$ and $\forall 1\leq J\leq M$.

$$\begin{aligned}\mathscr{F}\left(\chi_{1\imath+1},\chi_{2J+1},u_{\imath,J}\right)&=\frac{1}{\Delta x\Delta y}\sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{1i}}^{\chi_{1i+1}}\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega_{v_1}^{\varpi_1}(\chi_{1\imath+1}-s)\quad\Omega_{v_2}^{\varpi_2}\left(\chi_{2J+1}-t\right)dtds,\\&\left(\Delta x\Delta y\quad\overline{v}_1\overline{v}_2u_{i,j}+\Delta y\quad v_1\overline{v}_2\left(u_{i+1,j}-u_{i,j}\right)+\Delta x\quad v_2\overline{v}_1\left(u_{i,j+1}-u_{i,j}\right)+v_1v_2\left(u_{i+1,j+1}-u_{i+1,j}-u_{i,j+1}+u_{i,j}\right)\right),\end{aligned} \tag{41}$$

$$u_{\imath,1}=\mathscr{U}(\chi_{1\imath},a_2),1\leq\imath\leq\mathrm{N}+1;\ \mathrm{u}_{1,\mathrm{J}}=\mathscr{U}\left(\mathrm{a}_1,\chi_{2\mathrm{J}}\right),1\leq\mathrm{J}\leq\mathrm{M}+1, \tag{42}$$

with $\overline{v}_i=1-v_i$ and $\Omega_v^{\varpi}(z)=\exp(v-1/vz)/v^{1-\varpi}\Gamma(1-\varpi)z^{\varpi}$. In the sequel, we will assume that $\Delta\chi_1=\Delta\chi_2=\Delta$ for simplicity.

Proposition 15. *If $\mathscr{U}\in C^2(J)$, then scheme (41)-(42) is consistent with order (at least) one.*

Proof. Let $1\leq\imath\leq\mathrm{N}$ and $1\leq J\leq M$. We define the truncation error at a grid node $(\chi_{1\imath+1},\chi_{2J+1})$ by

$$\begin{aligned}R_{\imath+1,\mathrm{J}+1}&=\frac{1}{\Delta^2}\sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{1i}}^{\chi_{1i+1}}\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega_{v_1}^{\varpi_1}(\chi_{1\imath+1}-s)\quad\Omega_{v_2}^{\varpi_2}\left(\chi_{2J+1}-t\right)\quad dtds,\\&\left(\Delta^2\overline{v}_1\overline{v}_2\overline{\mathscr{U}}_{i,j}+\Delta v_1\overline{v}_2\left(\overline{\mathscr{U}}_{i+1,j}-\overline{\mathscr{U}}_{i,j}\right)+\Delta v_2\overline{v}_1\left(\overline{\mathscr{U}}_{i,j+1}-\overline{\mathscr{U}}_{i,j}\right)+v_1v_2\left(\overline{\mathscr{U}}_{i+1,j+1}-\overline{\mathscr{U}}_{i+1,j}-\overline{\mathscr{U}}_{i,j+1}+\overline{\mathscr{U}}_{i,j}\right)\right)-\mathscr{F}\left(\chi_{1\imath+1},\chi_{2J+1},\overline{\mathscr{U}}_{\imath,\mathrm{J}}\right),\end{aligned} \tag{43}$$

where $\overline{\mathscr{U}}_{i,j}:=\mathscr{U}(\chi_{1i},\chi_{2j})$. We obtain by Taylor expansions of $\mathscr{U}$ in $(s,t)\in[\chi_{1i},\chi_{1i+1}]\times[\chi_{2j},\chi_{2j+1}]$.

$$\begin{aligned}\overline{\mathscr{U}}_{i,j} &= \mathscr{U}(s,t) + \mathscr{O}(\Delta),\\ \overline{\mathscr{U}}_{i+1,j} - \overline{\mathscr{U}}_{i,j} &= \Delta\left(\partial_{\chi_1}\mathscr{U}(s,t) + (\chi_{1i+1/2} - s)\partial^2_{\chi_1\chi_1}\mathscr{U}(s,t)\right) + \mathscr{O}\left(\Delta^2\right),\\ \overline{\mathscr{U}}_{i,j+1} - \overline{\mathscr{U}}_{i,j} &= \Delta\left(\partial_{\chi_2}\mathscr{U}(s,t) + \left(\chi_{2j+1/2} - t\right)\partial^2_{\chi_2\chi_2}\mathscr{U}(s,t)\right) + \mathscr{O}\left(\Delta^2\right),\\ \overline{\mathscr{U}}_{i+1,j+1} - \overline{\mathscr{U}}_{i+1,j} - \overline{\mathscr{U}}_{i,j+1} + \overline{\mathscr{U}}_{i,j} &= \Delta^2\partial^2_{\chi_1\chi_2}\mathscr{U}(s,t) + \mathscr{O}\left(\Delta^3\right).\end{aligned} \tag{44}$$

with $\xi_{i+1/2} = (\xi_i + \xi_{i+1})/2$, yielding by (1).

$$\begin{aligned}R_{\imath+1,\mathrm{J}+1} &= \sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{1i}}^{\chi_{1i+1}}\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega^{\varpi_1}_{\nu_1}(\chi_{1\imath+1} - s)\quad\Omega^{\varpi_2}_{\nu_2}\left(\chi_{2J+1} - t\right)dtds\\ &\quad\left(\overline{\nu}_1\overline{\nu}_2\mathscr{U}(s,t) + \nu_1\overline{\nu}_2\partial_{\chi_1}\mathscr{U}(s,t) + \nu_2\overline{\nu}_1\partial_{\chi_2}\mathscr{U}(s,t) + \nu_1\nu_2\partial^2_{\chi_1\chi_2}\mathscr{U}(s,t) + (\chi_{1i+1/2} - s)\partial^2_{\chi_1\chi_1}\mathscr{U}(s,t)\right.\\ &\quad\left.+\left(\chi_{2j+1/2} - t\right)\partial^2_{\chi_2\chi_2}\mathscr{U}(s,t) + \mathscr{O}(\Delta)\right) - \mathscr{F}\left(\chi_{1\imath+1}, \chi_{2J+1}, \overline{\mathscr{U}}_{\imath+1,\mathrm{J}+1} + \mathscr{O}(\Delta)\right)\Big)\\ &= \int_{a_1}^{\chi_{1\imath+1}}\int_{a_2}^{\chi_{2J+1}}\Omega^{\varpi_1}_{\nu_1}(\chi_{1\imath+1} - s)\quad\Omega^{\varpi_2}_{\nu_2}\left(\chi_{2J+1} - t\right)\quad D^{1,\nu}\mathscr{U}(s,t)dtds\\ &\quad+\sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{1i}}^{\chi_{1i+1}}\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega^{\varpi_1}_{\nu_1}(\chi_{1\imath+1} - s)\quad\Omega^{\varpi_2}_{\nu_2}\left(\chi_{2J+1} - t\right)dtds\\ &\quad\left((\chi_{1i+1/2} - s)\partial^2_{\chi_1\chi_1}\mathscr{U}(s,t) + \left(\chi_{2j+1/2} - t\right)\partial^2_{\chi_2\chi_2}\mathscr{U}(s,t) + \mathscr{O}(\Delta)\right) - \mathscr{F}\left(\chi_{1\imath+1}, \chi_{2J+1}, \overline{\mathscr{U}}_{\imath+1,\mathrm{J}+1}\right) + \mathscr{O}(\Delta)\\ &= \sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{1i}}^{\chi_{1i+1}}\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega^{\varpi_1}_{\nu_1}(\chi_{1\imath+1} - s)\quad\Omega^{\varpi_2}_{\nu_2}\left(\chi_{2J+1} - t\right)dtds\\ &\quad\left((\chi_{1i+1/2} - s)\partial^2_{\chi_1\chi_1}\mathscr{U}(s,t) + \left(\chi_{2j+1/2} - t\right)\partial^2_{\chi_2\chi_2}\mathscr{U}(s,t) + \mathscr{O}(\Delta)\right).\end{aligned} \tag{45}$$

Hence, we get for any $1 \le \imath \le \mathrm{N}$ and $1 \le i + 1 \le M$.

$$\begin{aligned}\left|R_{\imath+1,\mathrm{J}+1}\right| &\le \kappa_1\sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{1i}}^{\chi_{1i+1}}(\chi_{1i+1/2} - s)\Omega^{\varpi_1}_{\nu_1}(\chi_{1\imath+1} - s)\quad\mathrm{ds}\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega^{\varpi_2}_{\nu_2}\left(\chi_{2J+1} - t\right)dt\\ &\quad+\kappa_1\sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{2j}}^{\chi_{2j+1}}\left(\chi_{2j+1/2} - t\right)\Omega^{\varpi_2}_{\nu_2}\left(\chi_{2J+1} - t\right)\quad\mathrm{dt}\int_{\chi_{1i}}^{\chi_{1i+1}}\Omega^{\varpi_1}_{\nu_1}(\chi_{1\imath+1} - s)ds + \mathscr{O}(\Delta),\end{aligned} \tag{46}$$

with $\kappa_1 = \max(\|\partial^2_{\chi_1\chi_1}\mathscr{U}\|L^\infty(J), \|\partial^2_{\chi_2\chi_2}\mathscr{U}\|L^\infty(J))$. Moreover, we have

$$\begin{aligned}\left|\int_{a_1}^{\chi_{1\imath+1}}\Omega^{\varpi_1}_{\nu_1}(\chi_{1\imath+1} - s)\,ds\right| &= \frac{\nu_1^{\varpi_1} - 1}{\Gamma(1 - \varpi_1)}\int_0^{\chi_{1\imath+1} - a_1}\chi_1^{-\varpi_1}\exp\left(-\frac{\overline{\nu}_1}{\nu_1}\chi_1\right)d\chi_1\\ &= \frac{\nu_1^{\varpi_1 - 1}}{\Gamma(1 - \varpi_1)}\int_0^{(/)(\chi_{1\imath+1} - a_1)}\eta^{-\varpi_1}e^{-\eta}\,d\eta\\ &\le \overline{\nu}_1^{\varpi_1 - 1},\end{aligned} \tag{47}$$

$$\left|\int_{a_2}^{\chi_{2J+1}} \Omega_{\nu_2}^{\varpi_2}\left(\chi_{2J+1}-t\right) \quad dt\right| \leq \overline{\nu}_2^{\varpi_2-1}. \tag{48}$$

It follows

$$\begin{aligned}\left|R_{\imath+1,\mathrm{J}+1}\right| &\leq \kappa_1 \overline{\nu}_2^{\varpi_2-1} \sum_{i=1}^{\imath} \int_{\chi_{1i}}^{\chi_{1i+1}} (\chi_{1i+1/2}-s)\Omega_{\nu_1}^{\varpi_1}(\chi_{1\imath+1}-s)\,ds \\ &+ \kappa_1 \overline{\nu}_1^{\varpi_1-1} \sum_{j=1}^{J} \int_{\chi_{2j}}^{\chi_{2j+1}} \left(\chi_{2j+1/2}-t\right)\Omega_{\nu_2}^{\varpi_2}\left(\chi_{2J+1}-t\right) dt + \mathcal{O}(\Delta).\end{aligned} \tag{49}$$

Now, using the estimate,

$$\frac{1}{\Gamma(1-\mu)} \sum_{i=1}^{\imath} \left|\int_{\chi_{1i}}^{\chi_{1i+1}} (\chi_{1i+1/2}-s)(\chi_{1\imath+1}-s)^{-\mu} ds\right| \leq \kappa_2 \quad \Delta^{2-\mu}, \tag{50}$$

for any value of $\mu \in (0,1)$, with $\kappa_2 > 0$ is a constant that do not depend on Δ (see [24]), we deduce

$$\begin{aligned}&\left|\sum_{i=1}^{\imath} \int_{\chi_{1i}}^{\chi_{1i+1}} (\chi_{1i+1/2}-s)\Omega_{\nu_1}^{\varpi_1}(\chi_{1\imath+1}-s)\,ds\right| \\ &= \left|\frac{\nu_1^{\varpi_1-1}}{\Gamma(1-\varpi_1)} \sum_{i=1}^{\imath} \int_{\chi_{1i}}^{\chi_{1i+1}} (\chi_{1i+1/2}-s)(\chi_{1\imath+1}-s)^{-\varpi_1} \exp\left(-\frac{\overline{\nu}_1}{\nu_1}(\chi_{1\imath+1}-s)\right) ds\right| \\ &\leq \frac{\nu_1^{\varpi_1-1}}{\Gamma(1-\varpi_1)} \sum_{i=1}^{\imath} \left|\int_{\chi_{1i}}^{\chi_{1i+1}} (\chi_{1i+1/2}-s)(\chi_{1\imath+1}-s)^{-\varpi_1} ds\right| \\ &\leq \kappa_3 \Delta^{2-\varpi_1},\end{aligned} \tag{51}$$

with $\kappa_3 = \kappa_2 \quad \nu_1^{\varpi_1-1}$. The same estimate holds for the second term in the right-hand side of (49). Finally, the result follows from (49) and (51). □

Theorem 16. *Assume $\mathcal{F}$ is L-Lipschitz with respect to its third variable. Then, scheme (41)-(42) is convergent.*

Proof. All we need is to prove that scheme (41)-(42) is stable with respect to perturbations. Let $(u_{i,j})_{1\leq i\leq M+1, 1\leq j\leq N+1}$ be the solution of (41)-(42) and let $(\theta_{i,j})_{1\leq i\leq N+1, 1\leq i\leq M+1}$ be a solution of system (41)-(42) with additional perturbations: for all $1 \leq \imath \leq \mathrm{N}$ and $1 \leq J \leq M$,

$$\begin{aligned}&\sum_{i=1}^{\imath}\sum_{j=1}^{J} \int_{\chi_{1i}}^{\chi_{1i+1}} \int_{\chi_{2j}}^{\chi_{2j+1}} \Omega_{\nu_1}^{\varpi_1}(\chi_{1\imath+1}-s) \quad \Omega_{\nu_2}^{\varpi_2}\left(\chi_{2J+1}-t\right) \quad dt\,ds \Big(\Delta^2 \quad \overline{\nu}_1\overline{\nu}_2\theta_{i,j} + \Delta\nu_1\overline{\nu}_2\left(\theta_{i+1,j}-\theta_{i,j}\right) + \Delta\nu_2\overline{\nu}_1\left(\theta_{i,j+1}-\theta_{i,j}\right) \\ &\quad + \nu_1\nu_2\left(\theta_{i+1,j+1}-\theta_{i+1,j}-\theta_{i,j+1}+\theta_{i,j}\right)\Big) - \Delta^2 \quad \mathcal{F}\left(\chi_{1\imath+1}, \chi_{2J+1}, \theta_{\imath,\mathrm{J}}\right) = \Delta^2 \quad \epsilon_{\imath+1,\mathrm{J}+1},\end{aligned} \tag{52}$$

$$\theta_{\imath,1} = \mathcal{U}(\chi_{1\imath}, a_2), 1 \leq \imath \leq \mathrm{N}+1;\ \theta_{1,\mathrm{J}} = \mathcal{U}\left(\mathrm{a}_1, \chi_{2\mathrm{J}}\right), 1 \leq \mathrm{J} \leq \mathrm{M}+1, \tag{53}$$

where $\epsilon_{\imath+1,\mathrm{J}+1} \geq 0$ denotes the perturbations. Define the error term by $e_{i,j} := \theta_{i,j} - u_{i,j}$, and then (41) and (52) yield

$$\begin{aligned}\Upsilon_{\imath+1,J+1} &:= \frac{1}{\Delta^2}\sum_{i=1}^{\imath}\sum_{j=1}^{J}\int_{\chi_{1i}}^{\chi_{1i+1}}\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega_{\nu_1}^{\varpi_1}\left(\chi_{1\imath+1}-s\right)\quad\Omega_{\nu_2}^{\varpi_2}\left(\chi_{2J+1}-t\right)dtds\\ &\quad\cdot\left(\Delta^2\,\overline{\nu}_1\overline{\nu}_2 e_{i,j}+\Delta\nu_1\overline{\nu}_2\left(e_{i+1,j}-e_{i,j}\right)+\Delta\nu_2\overline{\nu}_1\left(e_{i,j+1}-e_{i,j}\right)\right.\\ &\quad\left.+\,\nu_1\nu_2\left(e_{i+1,j+1}-e_{i+1,j}-e_{i,j+1}+e_{i,j}\right)\right)\\ &=\epsilon_{\imath+1,J+1}+\mathcal{F}\left(\chi_{1\imath+1},\chi_{2J+1},\theta_{\imath,J}\right)-\mathcal{F}\left(\chi_{1\imath+1},\chi_{2J+1},u_{\imath,J}\right)\\ &\le\epsilon_{\imath+1,J+1}+L\left|e_{\imath,J}\right|.\end{aligned}\tag{54}$$

In addition, we have for all $1\le i\le\imath$,

$$\int_{\chi_{1i}}^{\chi_{1i+1}}\Omega_{\nu_1}^{\varpi_1}\left(\chi_{1\imath+1}-s\right)\quad ds=\overline{\nu}_1^{\varpi_1-1}\left(\kappa_{\imath+1-i}^{1}-\kappa_{\imath-i}^{1}\right),\tag{55}$$

and for all $1\le j\le J$,

$$\int_{\chi_{2j}}^{\chi_{2j+1}}\Omega_{\nu_2}^{\varpi_2}\left(\chi_{2J+1}-t\right)\quad dt=\overline{\nu}_2^{\varpi_2-1}\left(\kappa_{J+1-j}^{2}-\kappa_{J-j}^{2}\right),\tag{56}$$

where $\kappa_{\imath+1-r}^{1}=\varrho(\omega_1(\chi_{1\imath+1}-\chi_{1r}),1-\varpi_1)$ and $\kappa_{J+1-r}^{2}=\varrho(\omega_2(\chi_{2J+1}-y_r),1-\varpi_2)$ (with $\kappa_0^1=\kappa_0^2=0$), $\omega_i:=\overline{\nu}_i/\nu_i$, and ϱ is defined by $\varrho(z,\zeta):=1/\Gamma(\zeta)\int_0^z t^{\zeta-1}e^{-t}\quad dt$). Denoting $C_0=\Delta^2\quad\overline{\nu}_1^{1-\varpi_1}\overline{\nu}_2^{1-\varpi_2}$, then

$$\begin{aligned}C_0\Upsilon_{\imath+1,J+1}&=\Delta^2\overline{\nu}_1\overline{\nu}_2\sum_{i=1}^{\imath}\sum_{j=1}^{J}\left(\kappa_{\imath-i+1}^{1}-\kappa_{\imath-i}^{1}\right)\left(\kappa_{J-j+1}^{2}-\kappa_{J-j}^{2}\right)e_{i,j}\\ &\quad+\Delta\nu_1\overline{\nu}_2\sum_{i=1}^{\imath}\sum_{j=1}^{J}\left(\kappa_{\imath-i+1}^{1}-\kappa_{\imath-i}^{1}\right)\left(\kappa_{J-j+1}^{2}-\kappa_{J-j}^{2}\right)\left(e_{i+1,j}-e_{i,j}\right)\\ &\quad+\Delta\nu_2\overline{\nu}_1\sum_{i=1}^{\imath}\sum_{j=1}^{J}\left(\kappa_{\imath-i+1}^{1}-\kappa_{\imath-i}^{1}\right)\left(\kappa_{J-j+1}^{2}-\kappa_{J-j}^{2}\right)\left(e_{i,j+1}-e_{i,j}\right)\\ &\quad+\nu_1\nu_2\sum_{i=1}^{\imath}\sum_{j=1}^{J}\left(\kappa_{\imath-i+1}^{1}-\kappa_{\imath-i}^{1}\right)\left(\kappa_{J-j+1}^{2}-\kappa_{J-j}^{2}\right)\left(e_{i+1,j+1}-e_{i+1,j}-e_{i,j+1}+e_{i,j}\right).\end{aligned}\tag{57}$$

Using some index changes, one may find

$$C_0\Upsilon_{\imath+1,J+1}=S_{\imath+1,J+1}+C_4\left(\kappa_1^1-\kappa_0^1\right)\left(\kappa_1^2-\kappa_0^2\right)e_{\imath+1,J+1}\tag{58}$$

with

$$\begin{aligned}S_{\imath+1,J+1}&=C_1\sum_{i=1}^{\imath}\sum_{j=1}^{J}\left(\kappa_{\imath-i+1}^{1}-\kappa_{\imath-i}^{1}\right)\left(\kappa_{J-j+1}^{2}-\kappa_{J-j}^{2}\right)e_{i,j}\\ &\quad+C_2\sum_{i=2}^{k+1}\sum_{j=1}^{J}\left(\kappa_{\imath-i+2}^{1}-\kappa_{\imath-i+1}^{1}\right)\left(\kappa_{J-j+1}^{2}-\kappa_{J-j}^{2}\right)e_{i,j}\\ &\quad+C_3\sum_{i=1}^{k}\sum_{j=2}^{J+1}\left(\kappa_{\imath-i+1}^{1}-\kappa_{\imath-i}^{1}\right)\left(\kappa_{J-j+2}^{2}-\kappa_{J-j+1}^{2}\right)e_{i,j}\\ &\quad+C_4\sum_{i=2}^{k}\sum_{j=2}^{J+1}\left(\kappa_{\imath-i+2}^{1}-\kappa_{\imath-i+1}^{1}\right)\left(\kappa_{J-j+2}^{2}-\kappa_{J-j+1}^{2}\right)e_{i,j}\\ &\quad+C_4\sum_{j=2}^{J}\left(\kappa_1^1-\kappa_0^1\right)\left(\kappa_{J-j+2}^{2}-\kappa_{J-j+1}^{2}\right)e_{\imath+1,j}\end{aligned}\tag{59}$$

$$\begin{aligned} C_1 &= \Delta^2 - \Delta(1+\Delta)(\nu_1+\nu_2) + (1+\Delta)^2\nu_1\nu_2, \\ C_2 &= \nu_1(\Delta - (1+\Delta)\nu_2), \\ C_3 &= \nu_2(\Delta - (1+\Delta)\nu_1), \\ C_4 &= \nu_1\nu_2. \end{aligned} \tag{60}$$

Now, we need the following results. □

Lemma 17. *Let* $D_i = \nu_i^{1-\varpi_i}/\Gamma(2-\varpi_i)e^{-\omega_i(b_i-a_i)}$ *and* $E_i = \nu_i^{1-\varpi_i}/\Gamma(2-\varpi_i)$, $i=1,2$. *Then, we have*

$$\kappa_1^i \geq D_i\Delta^{1-\varpi_i}, i = 1,2, \tag{61}$$

and $\forall n \in \mathbb{N}$.

$$\kappa_n^i \leq E_i\Delta^{1-\varpi_i}, i = 1,2. \tag{62}$$

Proof. The first inequality is a direct consequence of the estimate $\varrho(z,\zeta) \geq e^{-\mathcal{Z}}/\Gamma(1+\zeta)z^{\zeta}$ for any $z \in [0,\mathcal{Z}]$. Using the identity $b^t - a^t \leq (b-a)^t$ for any $a \leq b$ and $t \in [0,1]$, we obtain for $1 \leq i \leq \imath + 1$,

$$\begin{aligned} \kappa_{\imath+1-i}^1 &= \frac{\nu_1^{\varpi_1-1}}{\Gamma(1-\varpi_1)}\int_{\chi_{1i}}^{\chi_{1i+1}} (\chi_{1\imath+1}-s)^{-\varpi_1}e^{-\omega_1(\chi_{1\imath+1}-s)} \quad ds \\ &\leq \frac{\nu_1^{\varpi_1-1}}{\Gamma(2-\varpi_1)}\left((\chi_{1\imath+1}-\chi_{1i})^{1-\varpi_1} - (\chi_{1\imath+1}-\chi_{1i+1})^{1-\varpi_1}\right) \\ &\leq \frac{\nu_1^{\varpi_1-1}\Delta^{1-\varpi_1}}{\Gamma(2-\varpi_1)}. \end{aligned} \tag{63}$$

We obtain the following from (54) and (58) and Lemma 17:

$$\nu_1\nu_2 D_1 D_2 \Delta^{2-(\varpi_1+\varpi_2)}e_{\imath+1,J+1} \leq C_0\left(\epsilon_{\imath+1,J+1} + L\left|e_{\imath,J}\right|\right) - S_{\imath+1,J+1}, \tag{64}$$

yielding

$$\left|e_{\imath+1,J+1}\right| \leq C_0 D_3\Delta^{\varpi_1+\varpi_2-2}\left(\epsilon_{\imath+1,J+1} + L\left|e_{\imath,J}\right|\right) + D_3\Delta^{\varpi_1+\varpi_2-2}\left|S_{\imath+1,J+1}\right|, \tag{65}$$

with $D_3 = (\nu_1\nu_2 D_1 D_2)^{-1}$. Let us notice that $C_1 \geq 0$, $C_2 \leq 0$, $C_3 \leq 0$, and $C_4 \geq 0$ if Δ is sufficiently small (a sufficient condition is $\Delta \leq \min(\nu_1/1-\nu_1, \nu_2/1-\nu_2)$). Thus, we deduce using the discrete Gronwall inequality (see Lemma 4.4 in [2]).

$$\begin{aligned} \left|e_{\imath+1,J+1}\right| &\leq C_0 D_3\Delta^{\varpi_1+\varpi_2-2}\left(\epsilon_{\imath+1,J+1} + L\left|e_{\imath,J}\right|\right) \\ &\quad \times \exp\left\{D_3\Delta^{\varpi_1+\varpi_2-2}\left(C_1\sum_{i=1}^{\imath}\sum_{j=1}^{J}\left(\kappa_{\imath-i+1}^1 - \kappa_{\imath-i}^1\right)\left(\kappa_{J-j+1}^2 - \kappa_{J-j}^2\right)\right.\right. \\ &\quad - C_2\sum_{i=2}^{\imath+1}\sum_{j=1}^{J}\left(\kappa_{\imath-i+2}^1 - \kappa_{\imath-i+1}^1\right)\left(\kappa_{J-j+1}^2 - \kappa_{J-j}^2\right) \\ &\quad - C_3\sum_{i=1}^{\imath}\sum_{j=2}^{J+1}\left(\kappa_{\imath-i+1}^1 - \kappa_{\imath-i}^1\right)\left(\kappa_{J-j+2}^2 - \kappa_{J-j+1}^2\right) \\ &\quad + C_4\sum_{i=2}^{\imath}\sum_{j=2}^{J+1}\left(\kappa_{\imath-i+2}^1 - \kappa_{\imath-i+1}^1\right)\left(\kappa_{J-j+2}^2 - \kappa_{J-j+1}^2\right) \\ &\quad \left.\left. + C_4\sum_{j=2}^{J}\left(\kappa_1^1 - \kappa_0^1\right)\left(\kappa_{J-j+2}^2 - \kappa_{J-j+1}^2\right)\right)\right\} \\ &\leq C_0 D_3\Delta^{\varpi_1+\varpi_2-2}\left(\epsilon_{\imath+1,J+1} + L\left|e_{\imath,J}\right|\right) \\ &\quad \times \exp\left\{D_3\Delta^{\varpi_1+\varpi_2-2}\left[(C_1 - C_2 - C_3 + C_4)\kappa_\imath^1\kappa_J^2 + C_4\kappa_1^1\kappa_J^2\right]\right\}. \end{aligned} \tag{66}$$

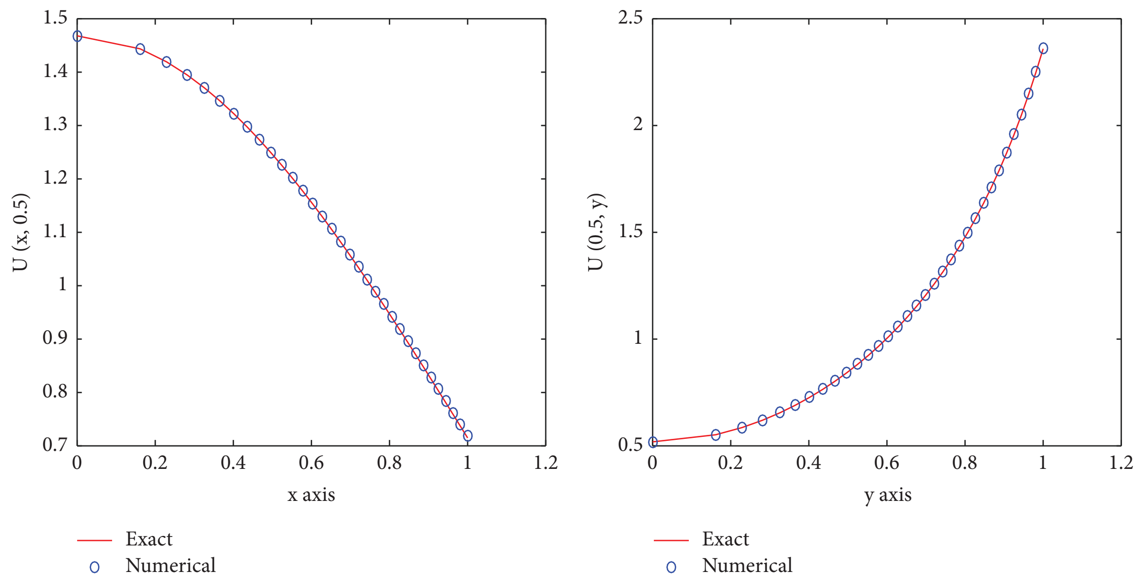

Figure 1: Comparison of the numerical and the exact solutions for example 1. The parameters in use are $(\varpi_1, \varpi_2) = (0.5, 0.5)$ and $(\nu_1, \nu_2) = (0.7, 0.4)$.

Table 1: L^∞-errors and convergence orders for example 1 with different parameters ν_1 and ν_2.

Δ	$(\nu_1;\nu_2) = (0.7;0.4)$		$(\nu_1;\nu_2) = (0.2;0.6)$		$(\nu_1;\nu_2) = (0.3;0.3)$	
	$\max_{i,j}\lvert\overline{\mathcal{U}}_{i,j} - u_{i,j}\rvert$	Order	$\max_{i,j}\lvert\overline{\mathcal{U}}_{i,j} - u_{i,j}\rvert$	Order	$\max_{i,j}\lvert\overline{\mathcal{U}}_{i,j} - u_{i,j}\rvert$	Order
1/4	3.9183 (−02)	—	1.9803 (−01)	—	1.0711 (−01)	—
1/8	1.8161 (−02)	1.1094	8.7458 (−02)	1.1791	4.9137 (−02)	1.1241
1/16	8.7656 (−03)	1.0509	4.1779 (−02)	1.0658	2.3606 (−02)	1.0577
1/32	4.3119 (−03)	1.0235	2.0391 (−02)	1.0349	1.1587 (−02)	1.0266

In addition, there exists a constant $C_5 > 0$ that depends on ν_1 and ν_2 only but not on Δ such that

$$C_1 - C_2 - C_3 + C_4 = \Delta^2 - \Delta(\Delta + 2)(\nu_1 + \nu_2) + (\Delta + 2)^2 \nu_1 \nu_2 \leq C_5 \Delta^2. \tag{67}$$

Using Lemma 17 once more, we deduce

$$\left|e_{I+1,J+1}\right| \leq D_4 \Delta^{\varpi_1+\varpi_2} \max_{\substack{1\leq i\leq I+1,\\ 1\leq j\leq J+1.}} \epsilon_{i,j} \tag{68}$$

with $D_4 = \overline{\nu}_1^{1-\varpi_1}\overline{\nu}_2^{1-\varpi_2} D_3 (1 + L) e^{D_3 E_1 E_2 (C_4 + C_5)}$ is a constant that do not depend on Δ. This achieves the proof. □

4.2. Numerical Tests

4.2.1. Example 1. We consider system (1)-(2) with the data as follows:

$$\begin{cases} f(\chi_1, \chi_2, \mathcal{U}) = 0, (\chi_1, \chi_2, \mathcal{U}) \in [0,1] \times [0,1] \times \mathbb{R}, \\ \varphi(\chi_1) = \cos(\pi\chi_1/2), \chi_1 \in [0,1], \\ \psi(\chi_2) = e^{\chi_2}, \chi_2 \in [0,1]. \end{cases} \tag{69}$$

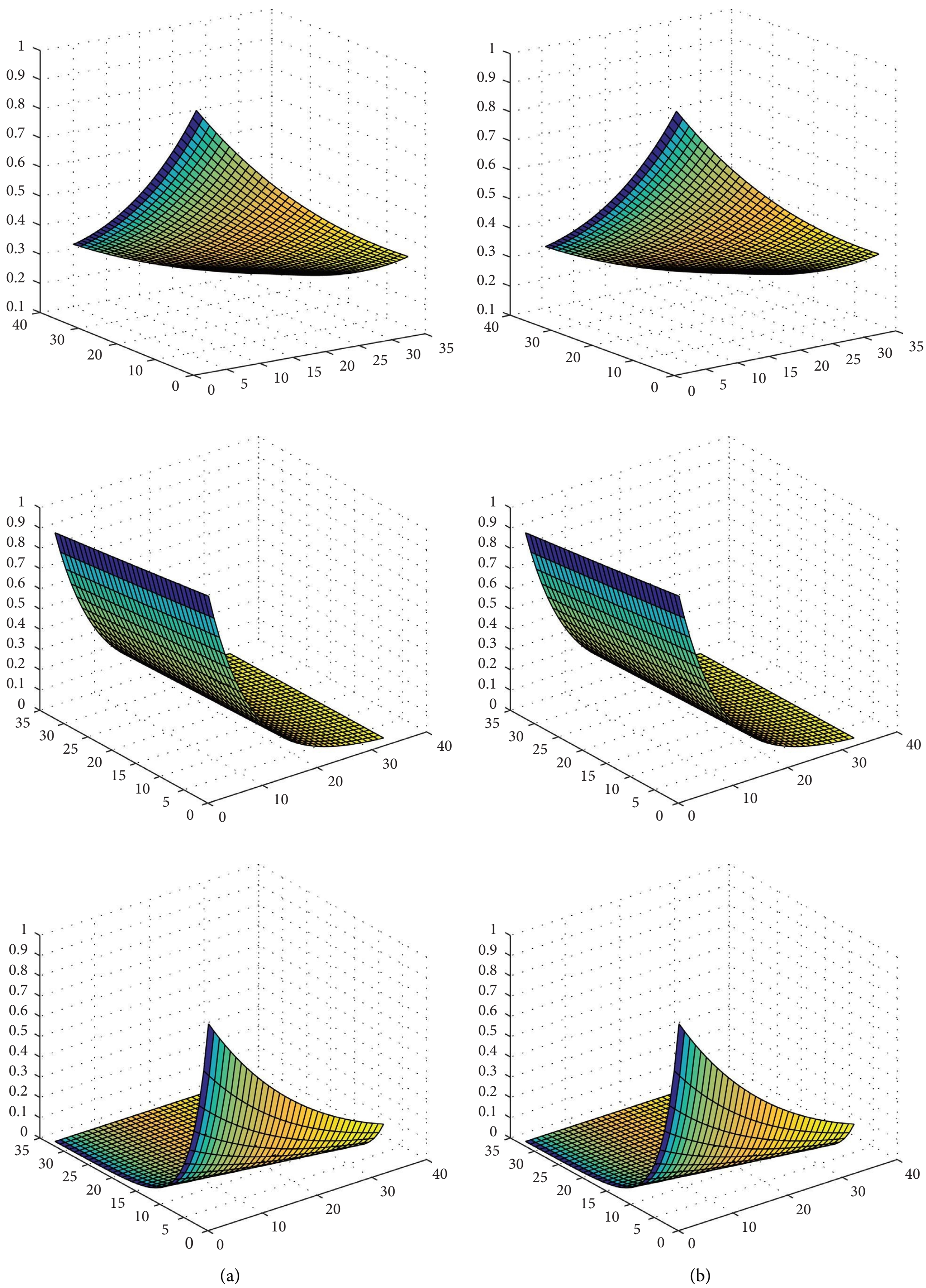

Figure 2: The numerical solutions (a) versus the exact solutions (b) relative to example 2. The parameters used are $(\varpi_1, \varpi_2) = (0.5, 0.2)$ and from top to bottom $(\nu_1; \nu_2) = (0.5; 0.5)$, $(\nu_1; \nu_2) = (0.9; 0.2)$, and $(\nu_1; \nu_2) = (0.1; 0.4)$.

Table 2: L^{∞}-errors and convergence orders for example 2 with different parameters ν_1 and ν_2.

Δ	$(\nu_1;\nu_2)=(0.5;0.5)$		$(\nu_1;\nu_2)=(0.9;0.2)$		$(\nu_1;\nu_2)=(0.1;0.4)$	
	$\max_{i,j}\lvert\overline{\mathscr{U}}_{i,j}-u_{i,j}\rvert$	Order	$\max_{i,j}\lvert\overline{\mathscr{U}}_{i,j}-u_{i,j}\rvert$	Order	$\max_{i,j}\lvert\overline{\mathscr{U}}_{i,j}-u_{i,j}\rvert$	Order
1/4	2.6495 (−03)	—	5.1722 (−04)	—	1.9955 (−01)	—
1/8	5.8906 (−04)	2.1692	8.2121 (−05)	2.6549	1.6868 (−02)	3.5644
1/16	1.3937 (−04)	2.0795	1.7849 (−05)	2.2019	2.3908 (−03)	2.8187
1/32	3.3921 (−05)	2.0386	4.1988 (−06)	2.0878	5.1705 (−04)	2.2091

and $(\sigma_1,\sigma_2)(\xi)=(\sqrt{1+\xi^2},\ln(1+\xi))$. According to Lemma 10, the solution of system (1)-(2) is given by

$$\begin{aligned}\mathscr{U}(\chi_1,\chi_2)&=\exp\left(\frac{\nu_1-1}{\nu_1}\left(\sqrt{1+x^2}-1\right)\right)e^{\chi_2}+(1+y)^{\nu_2-1/\nu_2}\cos(\pi\chi_1/2)\\&\quad-\exp\left(\frac{\nu_1-1}{\nu_1}\left(\sqrt{1+x^2}-1\right)\right)(1+y)^{\nu_2-1/\nu_2}.\end{aligned}\tag{70}$$

Figure 1 shows the numerical solution obtained by using the FD scheme (41)-(42) versus the exact solution. The parameters used are $[a_1,b_1]=[a_2,b_2]=[0,1]$, $(\varpi_1,\varpi_2)=(0.5,0.5)$, and $(\nu_1,\nu_2)=(0.7,0.4)$. Clearly, the approximated solution fits very well with the theoretical one. Table 1 reports the L^{∞} difference of the solutions, as well as the orders of convergence for various values of (ν_1,ν_2). Notice that these errors go to zero when the path Δ tends to zero, which is in total agreement with our theoretical study.

4.2.2. Example 2. Consider system (1)-(2) with the data:

$$\begin{cases}f(\chi_1,\chi_2,\mathscr{U})=C\dfrac{\chi_1^{\ 2}\chi_2}{1+\chi_2}\quad\mathscr{U},(\chi_1,\chi_2,\mathscr{U})\in[0,1]\times[0,1]\times\mathbb{R},\\ \varphi(\chi_1)=e^{\lambda_1\chi_1},\chi_1\in[0,1],\\ \psi(\chi_2)=e^{\lambda_2\chi_2},\chi_2\in[0,1],\end{cases}\tag{71}$$

with $\lambda_i=\nu_i-1/\nu_i$, $i=1,2$, $C=2\overline{\nu}_1\overline{\nu}_2+\lambda_1\nu_1\overline{\nu}_2+\lambda_2\nu_2\overline{\nu}_1+\lambda_1\lambda_2\nu_1\nu_2/\nu_1^{1-\varpi_1}\quad\nu_2^{1-\varpi_2}\Gamma(2-\varpi_1)\Gamma(2-\varpi_2)$, $\sigma_1(\xi)=\xi^{2/1-\varpi_1}$, and $\sigma_2(\xi)=(2\xi/1+\xi)^{1/1-\varpi_2}$. One can check that the solution of (1)-(2) is given by $\mathscr{U}(\chi_1,\chi_2)=e^{\lambda_1\chi_1+\lambda_2\chi_2}$. We plotted in Figure 2 the exact versus the numerical solutions obtained by using the FD scheme (41)-(42) for various values of (ν_1,ν_2). The fractional order is set to $(\varpi_1,\varpi_2)=(0.5,0.2)$. One can notice that all the solutions are in good agreement. We also compute the errors in the L^{∞} norm between the solutions in Table 2. The obtained results clearly confirm the robustness of the proposed scheme.

5. Conclusion

The existence and uniqueness of solutions to 2D systems using the proportional fractional derivative operator with respect to other functions are introduced and studied in this work. We also use the finite differences method to study the numerical approximation of such operator, and we establish the convergence of the numerical scheme.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. He, Y. Wang, Y. Wei, M. Wang, X. Hu, and Q. Shi, "An adaptive central difference Kalman filter approach for state of charge estimation by fractional order model of lithium-ion battery," *Energy*, vol. 244, Article ID 122627, 2022.

[2] B. Ramakrishnan, M. E. Cimen, A. Akgul, C. Li, K. Rajagopal, and H. Kor, "Chaotic oscillations in a fractional-order circuit with a josephson junction resonator and its synchronization using fuzzy sliding mode control," *Mathematical Problems in Engineering*, vol. 2022, Article ID 6744349, 11 pages, 2022.

[3] L. Viviani, M. Di Paola, and G. Royer-Carfagni, "A fractional viscoelastic model for laminated glass sandwich plates under blast actions," *International Journal of Mechanical Sciences*, vol. 222, Article ID 107204, 2022.

[4] A. Zeb, P. Kumar, V. S. Erturk, and T. Sitthiwirattham, "A new study on two different vaccinated fractional-order COVID-19 models via numerical algorithms," *Journal of King Saud University Science*, vol. 34, no. 4, Article ID 101914, 2022.

[5] H. Arfaoui and A. Ben Makhlouf, "Some results for a class of two-dimensional fractional hyperbolic differential systems with time delay," *Journal of Applied Mathematics and Computing*, vol. 68, no. 4, pp. 2389–2405, 2022.

[6] A. Ben Makhlouf and D. Boucenna, "Ulam–Hyers–Rassias Mittag–Leffler stability for the Darboux problem for partial fractional differential equations," *Rocky Mountain Journal of Mathematics*, vol. 51, no. 5, pp. 1541–1551, 2021.

[7] A. Boutiara, "A coupled system of nonlinear Langevin Fractional q-Difference equations associated with two different fractional orders in Banach space," *Kragujevac Journal of Mathematics*, vol. 48, pp. 555–575, 2023.

[8] C. Derbazi, "Solvability for a class of nonlinear Caputo-Hadamard fractional differential equations with p-Laplacian operator in Banach spaces," *Facta Universitatis – Series: Mathematics and Informatics*, vol. 18, pp. 693–711, 2020.

[9] C. Derbazi, "Nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with Dirichlet Boundary conditions in Banach spaces," *Kragujevac Journal of Mathematics*, vol. 46, no. 6, pp. 841–855, 2022.

[10] F. Jarad, T. Abdeljawad, and J. Alzabut, "Generalized fractional derivatives generated by a class of local proportional derivatives," *The European Physical Journal- Special Topics*, vol. 226, no. 16-18, pp. 3457–3471, 2017.

[11] F. Jarad, M. Alqudah, and T. Abdeljawad, "On more general forms of proportional fractional operators," *Open Mathematics*, vol. 18, no. 1, pp. 167–176, 2020.

[12] F. Jarad, T. Abdeljawad, S. Rashid, and Z. Hammouch, "More properties of the proportional fractional integrals and derivatives of a function with respect to another function," *Advances in Differential Equations*, vol. 2020, no. 1, p. 303, 2020.

[13] R. Agarwal and S. Hristova, "Impulsive memristive cohen–grossberg neural networks modeled by short term generalized proportional Caputo fractional derivative and synchronization analysis," *Mathematics*, vol. 10, no. 13, p. 2355, 2022.

[14] A. Boutiara, M. K. Kaabar, Z. Siri, M. E. Samei, and X. G. Yue, "Investigation of the generalized proportional Langevin and sturm–liouville fractional differential equations via variable coefficients and antiperiodic boundary conditions with a control theory application arising from complex networks," *Mathematical Problems in Engineering*, vol. 2022, Article ID 7018170, 21 pages, 2022.

[15] R. Ali, A. Akgül, and M. I. Asjad, "Power law memory of natural convection flow of hybrid nanofluids with constant proportional Caputo fractional derivative due to pressure gradient," *Pramana*, vol. 94, no. 1, p. 131, 2020.

[16] A. K. Alzahrani, O. A. Razzaq, N. A. Khan, A. S. Alshomrani, and M. Z. Ullah, "Transmissibility of epidemic diseases caused by delay with local proportional fractional derivative," *Advances in Difference Equations*, vol. 2021, no. 1, p. 292, 2021.

[17] O. Moaaz, A. E. Abouelregal, and M. Alesemi, "Moore-gibson-thompson photothermal model with a proportional Caputo fractional derivative for a rotating magneto-thermoelastic semiconducting material," *Mathematics*, vol. 10, no. 17, p. 3087, 2022.

[18] A. Ben Makhlouf, M. Benjemaa, J. Boucenna, and M. A. Hammami, "Darboux Problem for proportional partial fractional differential equations, Chaos," *Solitons & Fractals*, vol. 166, 2023.

[19] S. Abbas and M. Benchohra, "Darboux problem for perturbed partial differential equations of fractional order with finite delay," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 597–604, 2009.

[20] S. Abbas and M. Benchohra, "Upper and lower solutions method for impulsive partial hyperbolic differential equations with fractional order," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 3, pp. 406–413, 2010.

[21] A. N. Vityuk and A. V. Golushkov, "Darboux problem for a differential equation with fractional derivative," *Nonlinear Oscillations*, vol. 8, no. 4, pp. 450–462, 2005.

[22] A. N. Vityuk and A. V. Mikhailenko, "On one class of differential equations of fractional order," *Nonlinear Oscillations*, vol. 11, no. 3, pp. 307–319, 2008.