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Research Article

On Metric Dimension of Subdivided Honeycomb Network and Aztec Diamond Network

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This paper investigates the metric dimensions of the polygonal networks, particularly, the subdivided honeycomb network, Aztec diamond as well as the subdivided Aztec diamond network. A polygon is any two-dimensional shape formed by straight lines. Triangles, quadrilaterals, pentagons, and hexagons are all representations of polygons. For instance, hexagons help us in many models to construct honeycomb network, where n is the number of hexagons from a central point to the borderline of the network. A subdivided honeycomb network $(\mathrm{SHCN}(n))$ is obtained by adding additional vertices on each edge of $\mathrm{HCN}(n)$. An Aztec diamond network $(\mathrm{AZN}(n))$ of order n is a lattice comprises of unit squares with center (a, b) satisfying $|a| + |b| \leq n$. The subdivided Aztec diamond network $(\mathrm{SAZN}(n))$ is obtained by adding additional vertices to each edge of $\mathrm{AZN}(n)$. In this work, our main aim is to establish the results to show that the metric dimensions of $\mathrm{SHCN}(n)$ and $\mathrm{AZN}(n)$ are 2 and 3 for $n = 1$ and $n \geq 2$, respectively. In the end, some open problems are listed with regard to metric dimensions for k-subdivisions of $\mathrm{HCN}(n)$ and $\mathrm{AZN}(n)$.

1. Introduction

Graph theory is used as an important mean for modeling real-world problems, including physicochemical property testing [1]. Inspired by the problem of evaluating the position of an individual across the defined network precisely, Slater [2] presented the concept of the metric dimension of a graph, where the metric generators are referred to as locating sets. Coming after Slater's concept, Harary and Melter [3] extended the work on metric dimension by defining metric generators as resolving sets. It has many applications in different fields of life, for example, image processing, network theory, pattern recognition, optimization, and robot navigation. Throughout the graph, a traveling point can be identified after measuring the length between the point and sound stations that have been precisely located in the graph.

The mathematical illustration of various chemical structures is of vital importance for the chemists to discover drugs. A labeled graph is used to describe the composition of a chemical compound, edges indicate atomic bonds and vertices represent atoms [4, 5]. In a navigation network, a robot that needs to find its current position during navigation in space is modeled by a graph. It may send signals in order to measure its distance from each of a set of defined destinations. Here, the problem is to measure the minimum number of destinations with their locations, as the robots can likely decide their positions. The set of nodes representing destinations and the number of destinations are known as the metric basis and the metric dimension of the graph, respectively.

The metric dimension is formally initiated after considering a connected graph $G = (V, E)$, carrying set V of vertices/nodes and set E of edges. Let $v_1, v_2 \in V$ be two distinct vertices, then the length of the shortest $(v_1 v_2)$-path denotes the distance for them that is symbolized by $d(v_1, v_2)$. A set $N_k(v) = \{u \mid d(u, v) = k\}$ is the k-neighborhood of vertex $v \in V$, where k is a positive integer. If

$M = \{m_1, m_2, \ldots, m_k\}$ is the ordered subset of vertices and $v \in G$, then $r(v \mid M) = \{d(v, m_1), d(v, m_2), \ldots, d(v, m_k)\}$ is called the *code* of v in relation with M. If there exist separate codes for two vertices in G, then M is the *resolving set* [3] (or *locating set* [6]) for G. The minimum cardinality of a resolving (or locating) set refers as *metric dimension* of G symbolized by $\dim(G)$. While a locating (or resolving) set with a minimum number of vertices is called *basis* for G [7].

Raj et al. studied metric dimensions of different chemical networks as well as star of David network $SDN(n)$ in [8, 9]. If the vertices of a connected graph are changed, then the metric dimension of the graph will also be changed and become infinite when the number of vertices is infinite; this is called an unbounded metric dimension. Similarly, the metric dimension remains finite when changes in the number of vertices is finite and is called bounded metric dimension. Finally, if the metric dimension remains the same as the number of all vertices in a connected graph G, then it is called a constant metric dimension [10]. The metric dimension of a path graph is 1 in [5]; cycles have a metric dimension 2 for every $n \geq 3$. The rooted product of two graphs F and J is stated as follows: take $u = |V(F)|$ copies of J, and for each vertex u_j of F, identify u_i with the root node of the j^{th} copy of J. Godsil and McKay [11], the rooted product of Harary graphs $H_{(m,n)}$, Jahangir graphs, antiprism A_n, and generalized Petersen graphs $P(n, 2)$ by path and cycle would be calculated as well as metric dimensions of line graph of certain families of graphs would be determined. It is also of interest to determine the rooted product of graphs and then find out the metric dimension of the rooted product of graphs by path and cycle. Imran et al. [12] have established some results of the metric dimension for some gear graphs. Manuel et al. [13] determined the constant metric dimension of honeycomb networks. Moreover, in [14–16], we determined the metric dimensions and edge metric dimensions for honeycomb, hex-derived, and hexagonal networks. Zahid et al. determined the edge metric dimension of the wheel graph, k multiwheel graph, and Cayley graphs and its barycentric subdivisions in [17–19]. After gaining some idea of Manuel, the metric dimension of the subdivided honeycomb network would be determined.

2. Honeycomb Network

In this section, firstly, the structural introduction of $\mathrm{HCN}(n)$ [13] and $\mathrm{SHCN}(n)$ is given. Secondly, we have established some results and showed that the metric dimensions of $\mathrm{SHCN}(n)$ for $n = 1$ and $n \geq 2$ are 2 and 3, respectively.

There is a range of designs in which polygons play a role to construct a honeycomb network $\mathrm{HCN}(n)$, where n denotes the number of hexagons from the center to the boundary of the network. Given $\mathrm{HCN}(1)$, we will have to add a layer of six hexagons to the exterior boundary of $\mathrm{HCN}(1)$ in order to construct $\mathrm{HCN}(2)$. Consequently, after coating $\mathrm{HCN}(n-1)$ with $6(n-1)$ hexagons, we get $\mathrm{HCN}(n)$. While $\mathrm{SHCN}(n)$ is obtained by adding additional vertices to each edge of $\mathrm{HCN}(n)$. The honeycomb network is very useful in navigation, computer graphics, image processing, and cell phone.

The diagrams in Fig (1) are examples of $\mathrm{SHCN}(2)$ with 1 subdivision.

Theorem 1. *If $G \cong SHCN(1)$, then the metric dimension of G is 2.*

Proof. $\mathrm{SHC}(1)$ is a cycle with 12 vertices; it is not a path so its metric dimension is not 1 [4], and as it is C_{12}, it has metric dimension 2. □

Theorem 2. *If $G \cong \mathrm{SHCN}(n)$, then G has a metric dimension greater than 2 for $n \geq 2$.*

Proof. Here, we have to show that G does not have any resolving set M with two vertices. On the contrary, suppose that G has metric dimension equal to 2.

For $M = \{a_1, a_i\}$, it implies that $r(v_i \mid M) = r(u_i \mid M)$. Hence, M is not a resolving set for the graph.

For $M = \{v_1, v_{8n-3}\}$, it implies that $r(c_i^1 \mid M) = r(u_{i+2} \mid M)$. Hence, M is not a resolving set.

For $M = \{d_1^i, d_i^1\}$, it implies that $r(t_i^1 \mid M) = r(u_{i+2} \mid M)$. Hence, M is not a resolving set.

For $M = \{c_1^j, c_i^j\}$, it implies that $r(t_3^j \mid M) = r(t_7^j \mid M)$. Hence, M is not a resolving set.

For $M = \{u_1, u_{8n-3}\}$, it implies that $r(t_i^1 \mid M) = r(v_{i+2} \mid M)$. Hence, M is not a resolving set.

For $M = \{t_1^j, t_i^j\}$, it implies that $r(c_3^j \mid M) = r(c_7^j \mid M)$. Hence, M is not a resolving set.

For $M = \{v_1, u_1\}$, it implies that $r(d_1^1 \mid M) = r(u_4 \mid M)$. Hence, M is not a resolving set.

For $M = \{v_1, a_1\}$, it implies that $r(a_1^1, \mid M) = r(v_4 \mid M)$. Hence, M is not a resolving set.

For $M = \{u_1, a_1\}$, it implies that $r(d_1^1 \mid M) = r(u_4 \mid M)$. Hence, M is not a resolving set.

For $M = \{v_1, c_1^1\}$, it implies that $r(u_5 \mid M) = r(v_7 \mid M)$. Hence, M is not a resolving set.

For $M = \{u_1, c_1^1\}$, it implies that $r(u_{12} \mid M) = r(t_8^1 \mid M)$. Hence, M is not a resolving set.

Therefore, with two vertices, there is no resolving set M for $\mathrm{SHCN}(n)$, $n \geq 2$, so its metric dimension is greater than 2. □

Theorem 3. *If $G \cong \mathrm{SHCN}(n)$, $n \geq 2$, then the metric dimension of G is 3.*

Proof. The $\mathrm{SHCN}(n)$ with one vertex between every two vertices has a vertex set,

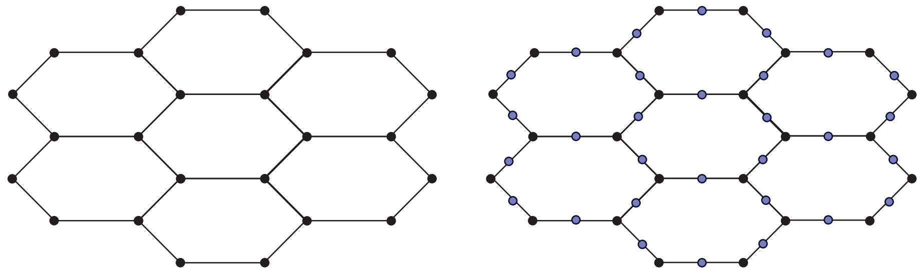

Figure 1: Honeycomb HCN(2) and subdivided honeycomb network SHCN(2) with 1 subdivision.

$$\begin{aligned} V(SHCN(n)) = &\{v_i\colon 1 \le i \le 8n-3\} \cup \{a_i\colon 1 \le i \le 2n\} \cup \left\{a_i^j\colon 1 \le i \le 2n-j+1, 1 \le j \le n-1\right\} \\ &\cup \left\{d_i^j\colon 1 \le i \le 2n-j+1, 1 \le j \le n-1\right\} \cup \left\{c_i^j\colon 1 \le i \le 8n-4j-3, 1 \le j \le n-1\right\} \cup \{u_i\colon 1 \le i \le 8n-3\} \\ &\cup \left\{t_i^j\colon 1 \le i \le 8n-4j-3, 1 \le j \le n-1\right\}. \end{aligned} \quad (1)$$

Now, let $M = \{v_1, u_1, v_{8n-3}\}$ be the resolving set for the above graph.

$$\begin{aligned} r\left(v_i \middle| M\right) &= (i-1, i+1, 8n-i-3), 1 \le i \le 8n-3, \\ r\left(a_i \middle| M\right) &= (4i-3, 4i-3, 8n-4i+1), 1 \le i \le 2n, \\ r\left(a_i^j \middle| M\right) &= (4(i+j)-5, 4(i+j)-3, 8n-4i-1), 1 \le i \le 2n-j+1, 1 \le j \le n-1, \\ r\left(d_i^j \middle| M\right) &= (4(i+j)-3, 4(i+j)-5, 8n-4i+1), 1 \le i \le 2n-j+1, 1 \le j \le n-1, \\ r\left(c_i^j \middle| M\right) &= (4j+i-1, 4j+i+1, 8n-i-3), 1 \le i \le 8n-4j-3, \ 1 \le j \le n-1, \\ r\left(c_i^j \middle| M\right) &= (4j+i-1, 4j+i+1, 8n-i-3), 1 \le i \le 8n-4j-3, 1 \le j \le n-1, \\ r\left(u_i \middle| M\right) &= (i+1, i-1, 8n-i-1), 1 \le i \le 8n-3, \\ r\left(t_i^j \middle| M\right) &= (4j+i+1, 4j+i-1, 8n-i-1), 1 \le i \le 8n-4j-3, 1 \le j \le n-1. \end{aligned} \quad (2)$$

Let v_l and v_p be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(v_l | M) = r(v_p | M) \Rightarrow (l-1, l+1, 8n-l-3) = (p-1, p+1, 8n-p-3) \Rightarrow l = p$, $l = p$, $l = p$ which is contradiction.

Let a_l^j and a_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(a_l^j | M) = r(a_p^j | M) \Rightarrow (4(l+j)-5, 4(l+j)-3, 8n-4l-1) = (4(p+j)-5, 4(p+j)-3, 8n-4p-1) \Rightarrow l = p$, $l = p$, $l = p$ which is contradiction.

Let d_l^j and d_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(d_l^j | M) = r(d_p^j | M) \Rightarrow (4(l+j)-3, 4(l+j)-5, 8n-4l+1) = (4(p+j)-3, 4(p+j)-5, 8n-4p+1) \Rightarrow l = p$, $l = p$, $l = p$ which is contradiction.

Let u_l and u_p be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(u_l | M) = r(u_p | M) \Rightarrow (l+1, l-1, 8n-l-1) = (p+1, p-1, 8n-p-1) \Rightarrow l = p$, $l = p$, $l = p$ which is contradiction.

Similarly, now that we will consider two different vertices from opposite sides, we will again get contradiction.

Let v_l and a_p be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(v_l | M) = r(a_p | M) \Rightarrow (l-1, l+1, 8n-l-3) = (4p-3, 4p-3, 8n-4p+1) \Rightarrow l = 2(2p-1), l = 4(p-1), l = 4(p-1)$ which is contradiction.

Let v_l and a_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(v_l | M) = r(a_p^j | M) \Rightarrow (l-1, l+1, 8n-l-3) = (4(p+j)-5, 4(p+j)-3, 8n-4p-1) \Rightarrow l = 4(p+j-1)$, $l = 4(p+j-1), l = 2(2p-1)$ which is contradiction.

Let v_l and c_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(v_l | M) = r(c_p^j | M) \Rightarrow (l-1, l+1, 8n-l-3) = (4j+p-1, 4j+p+1, 8n-p-3) \Rightarrow l = 4j+p, l = 4j+p, l = p$ which is contradiction.

Let a_l^j and d_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(a_l^j | M) = r(d_p^j | M) \Rightarrow (4(j+l)-5, 4(j+l)-3, 8$

$n-4l-1) = (4(p+j)-3, 4(p+j)-5, 8n-4p+1) \Rightarrow l = 4p+2/4, l = 4p-2/4, l = 4p-2/4$ which is contradiction.

Let a_l^j and u_p be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(a_l^j \mid M) = r(u_p \mid M) \Rightarrow (4(j+l)-5, 4(j+l)-3, 8n-4l-1) = (p+1, p-1, 8n-p-1) \Rightarrow l = p-4j+6/4, l = p-4j+2/4, l = p/4$ which is contradiction.

Let a_l^j and t_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(a_l^j \mid M) = r(t_p^j \mid M) \Rightarrow (4(j+l)-5, 4(j+l)-3, 8n-4l-1) = (4j+p+1, 4j+p-1, 8n-p-1) \Rightarrow l = p+6/4, l = p+2/4, l = p/4$ which is contradiction.

Let d_l^j and c_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(d_l^j \mid M) = r(c_p^j \mid M) \Rightarrow (4(j+l)-3, 4(j+l)-5, 8n-4l+1) = (4j+p-1, 4j+p+1, 8n-p-3) \Rightarrow l = p+2/4, l = p+6/4, l = p+4/4$ which is contradiction.

Let c_l^j and t_p^j be two distinct vertices from $V(\mathrm{SHCN}(n))$, then $r(c_l^j \mid M) = r(t_p^j \mid M) \Rightarrow (4j+l-1, 4j+l+1, 8n-l-3) = (4j+p-1, 4j+p+1, 8n-p-1) \Rightarrow l = p+2, l = p-2, l = p-2$ which is contradiction.

If we consider the following options of two vertices and continue the process in this way, we will get contradiction, a_l^j, t_p^j, c_l^j, d_p^j, t_l^j, a_p^j, u_l, d_p^j, u_l, t_p^j, u_l, a_p^j, v_l, t_p^j, v_l, d_p^j, c_l^j, v_p, a_l^j, c_p^j, c_l^j, v_p, c_l^j, and a_p^j.

So, every vertex has distinct representation with respect to M, so M is a resolving set for $\mathrm{SHCN}(n)$, $n \geq 2$. □

The following Figure 2 represents the justification for the abovementioned Theorems 2 and 3.

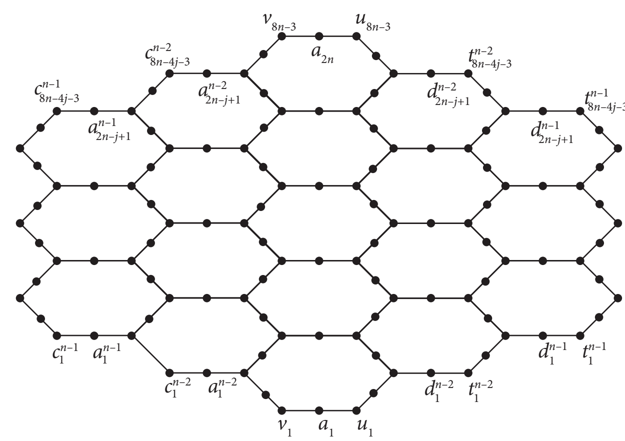

Figure 2: Structure of subdivided honeycomb network $\mathrm{SHCN}(n)$.

3. Aztec Diamond and Subdivided Aztec Diamond Network

In this section, firstly, the structural introduction of $\mathrm{AZN}(n)$ is given. Secondly, we have established some results and showed that the metric dimensions of $\mathrm{AZN}(n)$ and $\mathrm{SAZN}(n)$ for $n = 1$ and $n \geq 2$ are 2 and 3, respectively. The area derived from staircase shapes of height n when glued together by the straight edges is known as the Aztec diamond network $\mathrm{AZN}(n)$ [20] of order n. So, we can define it as a lattice comprises of unit squares centered at (a, b) such that $|a| + |b| \leq n$. AZN (n) with order n is composed of $2n(n+1)$ number of unit squares. An AZN (n) with different proportions is portrayed and further studied in [21, 22]. The diagrams in the following Figure 3 depict AZN (1), AZN (2), and AZN (3), respectively. The subdivided Aztec diamond network ($\mathrm{SAZN}(n)$) is obtained by adding additional vertices on each edge of $\mathrm{AZN}(n)$.

Theorem 4. *If $G \cong \mathrm{AZN}(1)$, then the metric dimension of G is 2.*

Proof. An $\mathrm{AZN}(n)$ has a vertex set

$$V(\mathrm{AZ}(n)) = \{v_i\colon 1 \leq i \leq 3\} \cup \{a_i\colon 1 \leq i \leq 3\} \cup \{u_i\colon 1 \leq i \leq 3\}. \quad (3)$$

Now, let $M = \{v_1, u_1\}$ be the resolving set for the abovementioned graph.

$$r(v_i \mid M) = (i-1, i+1), 1 \leq i \leq 3,$$
$$r(a_i \mid M) = (i, i), 1 \leq i \leq 3, \quad (4)$$
$$r(u_i \mid M) = (i+1, i-1), 1 \leq i \leq 3.$$

Let v_l and v_p be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(v_l \mid M) = r(v_p \mid M) \Rightarrow (l-1, l+1) = (p-1, p+1) \Rightarrow l = p$, $l = p$ which is contradiction.

Let a_l and a_p be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(a_l \mid M) = r(a_p \mid M) \Rightarrow (l, l) = (p, p) \Rightarrow l = p$, $l = p$ which is contradiction.

Let u_l and u_p be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(u_l \mid M) = r(u_p \mid M) \Rightarrow (l+1, l-1) = (p+1, p-1) \Rightarrow l = p$, $l = p$ which is contradiction.

Let v_l and a_p be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(v_l \mid M) = r(a_p \mid M) \Rightarrow (l-1, l+1) = (p, p) \Rightarrow l = p+1$, $l = p-1$ which is contradiction.

Let v_l and u_p be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(v_l \mid M) = r(u_p \mid M) \Rightarrow (l-1, l+1) = (p+1, p-1) \Rightarrow l = p+2$, $l = p-2$ which is contradiction.

Let a_l and u_p be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(a_l \mid M) = r(u_p \mid M) \Rightarrow (l, l) = (p+1, p-1) \Rightarrow l = p+1$, $l = p-1$ which is contradiction.

Hence, every vertex has a distinct representation with respect to M, so M is a resolving set for $\mathrm{AZN}(n)$ and $n = 1$. □

Theorem 5. *If $G \cong \mathrm{AZN}(n)$, then G has a metric dimension greater than 2 for $n \geq 2$.*

Proof. Suppose on contrary $\mathrm{AZN}(n)$ has D as its resolving set with cardinality 2. Let $M = \{v_1, v_{2n+1}\}$ be a resolving set. Here, $r(c_i^1 \mid M) = r(a_{i+1} \mid M), i = 1, 2, \ldots, 2n$, which is contradiction.

Let $M = \{u_1, u_{2n+1}\}$ be a resolving set. Here, $r(p_i^1 \mid M) = r(a_{i+1} \mid M), i = 1, 2, \ldots, 2n$, which is contradiction.

Let $M = \{a_i, a_{2n+1}\}$ be a resolving set. Here, $r(v_i \mid M) = r(u_i \mid D)$ which is contradiction.

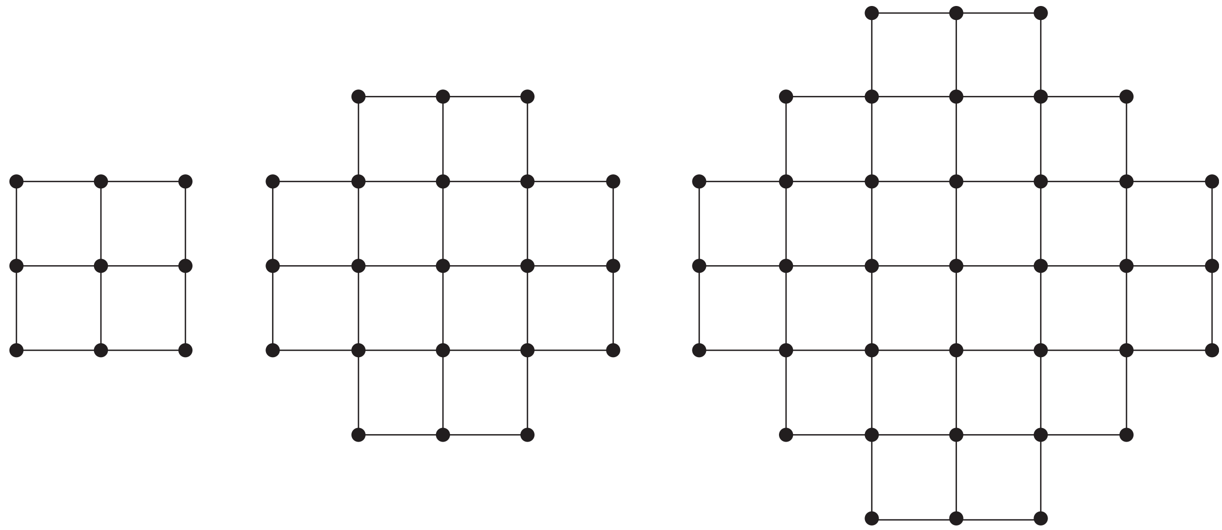

Figure 3: Structure of Aztec diamond networks AZN(1), AZN(2), and AZN(3) (from left to right), respectively.

Let $M = \{c_1^j, c_i^j\}$ be a resolving set. Here, $r(v_1 | M) = r(a_2 | M)$ which is contradiction.

Let $M = \{p_1^j, p_i^j\}$ be a resolving set. Here, $r(u_1 | M) = r(a_2 | M)$ which is contradiction.

Let $M = \{u_i, v_{2n+1}\}$ be a resolving set. Here, $r(v_i | M) = r(a_{i+1} | M), i = 1, 2, \ldots, 2n$, which is contradiction.

Let $M = \{v_{2n+1}, p_1^j\}$ be a resolving set. Here, $r(v_{i+1} | M) = r(a_{i+2} | M)$ which is contradiction.

Let $M = \{u_{2n+1}, c_1^j\}$ be a resolving set. Here, $r(u_{i+1} | M) = r(a_{i+2} | M), i = 1, 2, \ldots, 2n-1$, which is contradiction.

Let $M = \{a_1, v_{2n+1}\}$ be a resolving set. Here, $r(v_i | M) = r(a_{i+1} | M), i = 1, 2, \ldots, 2n$, which is contradiction.

Let $M = \{a_1, u_{2n+1}\}$ be a resolving set. Here, $r(u_i | M) = r(a_{i+1} | M), i = 1, 2, \ldots, 2n$, which is contradiction.

Let $M = \{a_1, c_1^j\}$ be a resolving set. Here, $r(v_i | M) = r(a_{i+1} | M), i = 1, 2, \ldots, 2n-1$, which is contradiction.

Let $M = \{a_1, p_1^j\}$ be a resolving set. Here, $r(u_i | M) = r(a_{i+1} | M), i = 1, 2, \ldots, 2n-1$, which is contradiction.

Thus, there is no resolving set with two basis elements. It implies that $\dim(G) > 2$ for $n \geq 2$. □

Theorem 6. *If $G \cong \mathrm{AZN}(n)$, $n \geq 2$, then the metric dimension of G is 3.*

Proof. The AZN(n) has a vertex set

$$V(AZ(n)) = \{v_i\colon 1 \leq i \leq 2n+1\} \cup \{u_i\colon 1 \leq i \leq 2n+1\} \cup \{a_i\colon 1 \leq i \leq 2n+1\} \cup \left\{c_i^j\colon 1 \leq i \leq 2n-2j+1, 1 \leq j \leq n-1\right\} \cup \left\{p_i^j\colon 1 \leq i \leq 2n-2j+1, 1 \leq j \leq n-1\right\}. \quad (5)$$

Now, let $M = \{v_1, u_1, v_{2n+1}\}$ be the resolving set for the abovementioned graph.

$$\begin{aligned} r\left(v_i \mid M\right) &= (i-1, i+1, 2n-i+1), 1 \leq i \leq 2n+1, \\ r\left(u_i \mid M\right) &= (i+1, i-1, 2n-i+3), 1 \leq i \leq 2n+1, \\ r\left(a_i \mid M\right) &= (i, i, 2n-i+2), 1 \leq i \leq 2n+1, \\ r\left(c_i^j \mid M\right) &= (2j+i-1, 2j+i+1, 2n-i+1), 1 \leq i \leq 2n-2j+1, \\ r\left(p_i^j \mid M\right) &= (2j+i+1, 2j+i-1, 2n-i+3), 1 \leq i \leq 2n-2j+1. \end{aligned} \quad (6)$$

Let v_x and v_y be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(v_l|M) = r(v_p|M) \Rightarrow (x-1, x+1, 2n-x+1) = (y-1, y+1, 2n-y+1) \Rightarrow x = y,\ x = y,\ x = y$ which is contradiction.

Let c_x^j and c_y^j are two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(c_x^j|M) = r(c_y^j|M) \Rightarrow (2j+x-1, 2j+x+1, 2n-x+1) = (2j+y-1, 2j+y+1, 2n-y+1) \Rightarrow x = y,\ x = y,\ x = y$ which is contradiction.

Let p_x^j and p_y^j be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(p_x^j|M) = r(p_y^j|M) \Rightarrow (2j+x+1, 2j+x-1, 2n-i+3) = (2j+y+1, 2j+y-1, 2n-y+3) \Rightarrow x = y,\ x = y,\ x = y$ which is contradiction.

Now, we will consider those vertices that are on opposite sides, as follows:

Let v_x and u_y be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(v_x|M) = r(u_y|M) \Rightarrow (x-1, x+1, 2n-x+1) = (y+1, y-1, 2n-y+3) \Rightarrow x = y+2,\ x = y-2,\ x = y-2$ which is contradiction.

Let v_x and p_y^j be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(v_x|M) = r(p_y^j|M) \Rightarrow (x-1, x+1, 2n-x+1) = (2j+y+1, 2j+y-1, 2n-y+3) \Rightarrow x = 2j+y+2, x = 2j+y-2, x = y-2$ which is contradiction.

Let u_x and a_y be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(u_x|M) = r(a_y|M) \Rightarrow (x=1, x-1, 2n-x+3) = (y, y, 2n-y+2) \Rightarrow x = y-1,\ x = y+1,\ x = y+1$ which is contradiction.

Let u_x and c_y^j be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(u_x|M) = r(c_y^j|M) \Rightarrow (x+1, x-1, 2n-x+3) = (2j+y-1, 2j+y+1, 2n-y+1) \Rightarrow x = 2j+y-2, x = 2j+y+2, x = y+2$ which is contradiction.

Let c_x^j and p_y^j be two distinct vertices from $V(\mathrm{AZN}(n))$, then $r(c_x^j|M) = r(p_y^j|M) \Rightarrow (2j+x-1, 2j+x+1, 2n-x+1) = (2j+y+1, 2j+y-1, 2n-y+3) \Rightarrow x = y+2,\ x = y-2,\ x = y-2$ which is contradiction.

Under this process, by choosing any other options of two vertices we will get contradiction as mentioned above.

Hence, every vertex has a distinct representation with respect to M, so M is a resolving set for $\mathrm{AZN}(n)$; $n \geq 2$. □

The following Figure 4 represents the justification for Theorem 5 and Theorem 6.

Theorem 7. *If $G \cong \mathrm{SAZN}(n)$, $n \geq 2$, then the metric dimension of G is 3.*

Proof. The SAZN(n) has a vertex set

$$\begin{aligned} V(SAZ(n)) = &\{v_i\colon 1 \le i \le 4n+1\} \cup \{u_i\colon 1 \le i \le 4n+1\} \cup \{c_i\colon 1 \le i \le 4n+1\} \cup \{d_i\colon 1 \le i \le 2n+1\} \cup \{f_i\colon 1 \le i \le 2n+1\} \\ &\cup \left\{a_i^j\colon 1 \le i \le 2n-2j+1 \text{ and } 1 \le j \le n-1\right\} \cup \left\{b_i^j\colon 1 \le i \le 4n-4j+1\right\} \\ &\cup \left\{g_i^j\colon 1 \le i \le 2n-2j+1 \text{ and } 1 \le j \le n-1\right\} \cup \left\{h_i^j\colon 1 \le i \le 4n-4j+1 \text{ and } 1 \le j \le n-1\right\}. \end{aligned} \tag{7}$$

Now, let $M = \{v_1, u_1, v_{2n+1}\}$ be the resolving set for the abovementioned graph. Then,

$$\begin{aligned} r(v_i|M) &= (i-1, i+3, 4n-i+1), 1 \le i \le 4n+1, \\ r(u_i|M) &= (i+3, i-1, 4n-i+5), 1 \le i \le 4n+1, \\ r(c_i|M) &= (i+1, i+1, 4n-i+3), 1 \le i \le 4n+1, \\ r(d_i|M) &= (2i-1, 2i+1, 4n-2i+3), 1 \le i \le 2n+1, \\ r(f_i|M) &= (2i+1, 2i-1, 4n-2i+5), 1 \le i \le 2n+1, \\ r\left(a_i^j|M\right) &= (4j+i-1, 4j+i+3, 4n-i+1), 1 \le i \le 2n-2j+1 \text{ and } 1 \le j \le n-1, \\ r\left(b_i^j|M\right) &= (4j+2i-3, 4j+2i+1, 4n-2i+1), 1 \le i \le 4n-4j+1 \text{ and } 1 \le j \le n-1, \\ r\left(g_i^j|M\right) &= (4j+2i+1, 4j+2i-3, 4n-2i+5), 1 \le i \le 2n-2j+1 \text{ and } 1 \le j \le n-1, \\ r\left(h_i^j|M\right) &= (4j+i+3, 4j+i-1, 4n-i+5), 1 \le i \le 4n-4j+1 \text{ and } 1 \le j \le n-1. \end{aligned} \tag{8}$$

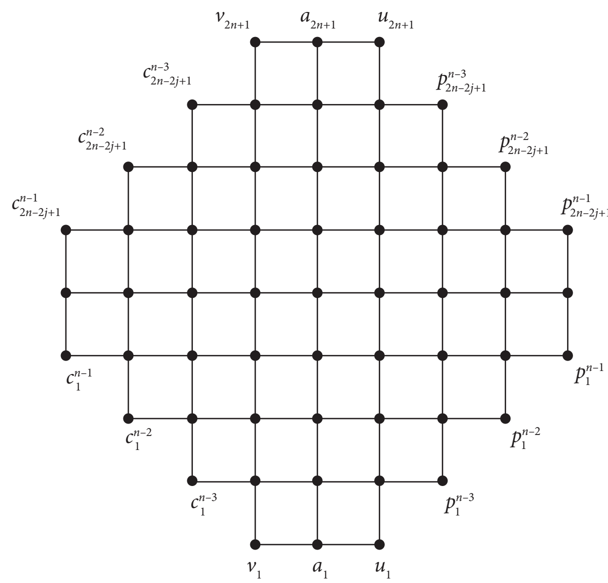

Figure 4: Aztec diamond AZN(n).

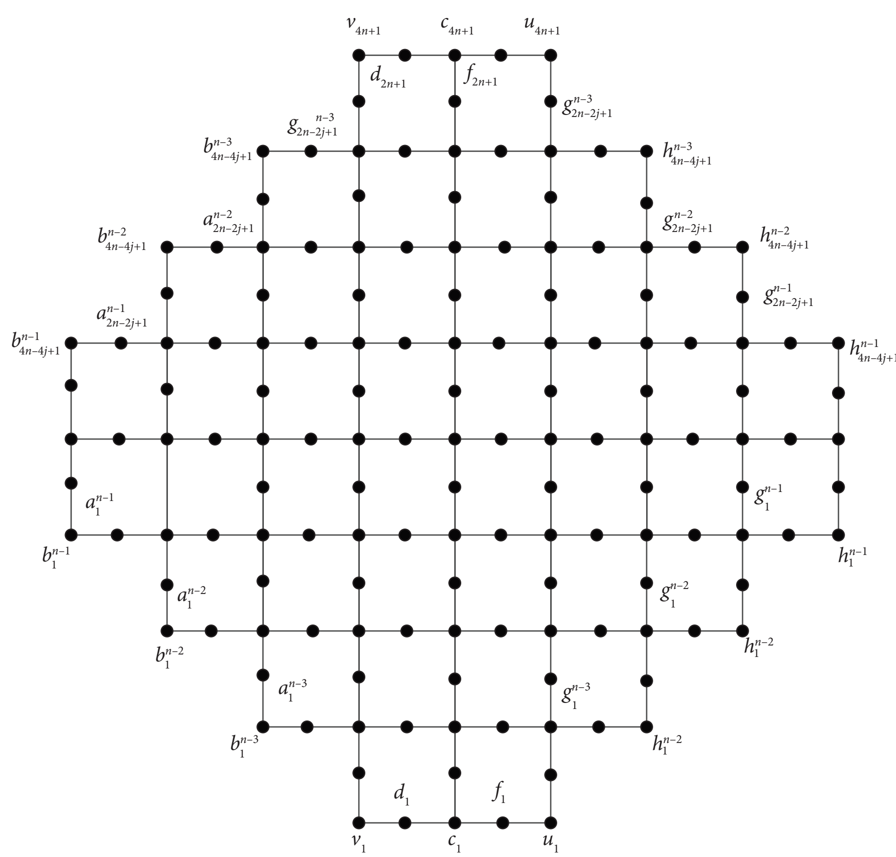

Figure 5: Subdivided SAZN(n).

Let v_l and v_p be two distinct vertices from $V(\mathrm{SAZN}(n))$, then $r(v_l \mid M) = r(v_p \mid M) \Rightarrow (l-1, l+3, 4n-l+1) = (p-1, p+3, 4n-p+1) \Rightarrow l = p,\quad l = p,\quad l = p$ which is contradiction.

Let c_l and c_p be two distinct vertices from $V(\mathrm{SAZN}(n))$, then $r(c_l \mid M) = r(c_p \mid M) \Rightarrow (l+1, l+1, 4n-l+3) = (p+1, p+1, 4n-p+3) \Rightarrow l = p,\quad l = p,\quad l = p$ which is contradiction.

Let a_l^j and a_p^j be two distinct vertices from V (SAZN (n)), then $r(a_l^j \mid M) = r(a_p^j \mid M) \Rightarrow (4j + l - 1, 4j + l + 3, 4n - l + 1) = (4j + p - 1, 4j + p + 3, 4n - p + 1) \Rightarrow l = p, l = p, l = p$ which is contradiction.

Now, we will consider those vertices which are from opposite sides, as follow.

Let v_l and u_p are two distinct vertices from $V(\mathrm{SAZN}(n))$, then $r(v_l \mid M) = r(u_p \mid M) \Rightarrow (l - 1, l + 3, 4n - l + 1) = (p + 3, p - 1, 4n - p + 5) \Rightarrow l = p + 4, \quad l = p - 4, \quad l = p - 4$ which is contradiction.

Let c_l and a_p^j be two distinct vertices from $V(\mathrm{SAZN}(n))$, then $r(c_l \mid M) = r(a_p^j \mid M) \Rightarrow (l + 1, l + 1, 4n - l + 3) = (4j + p - 1, 4j + p + 3, 4n - p + 1) \Rightarrow l = 4j + p - 2$, $l = 4j + p + 2$, $l = p + 2$ which is contradiction.

Let f_l and h_p^j be two distinct vertices from $V(\mathrm{SAZN}(n))$, then $r(f_l \mid M) = r(h_p^j \mid M) \Rightarrow (2l + 1, 2l + 1, 4n - 2l + 5) = (4j + p + 3, 4j + p - 1, 4n - p + 5) \Rightarrow l = 4j + p + 2/2, \quad l = 4j + p - 2/2$, $l = p/2$ which is contradiction.

Let b_l^j and g_p^j be two distinct vertices from $V(\mathrm{SAZN}(n))$, then $r(b_l^j \mid M) = r(g_p^j \mid M) \Rightarrow (4j + 2l - 3, 4j + 2l + 1, 4n - 2l + 1) = (4j + 2p + 1, 4j + 2p - 3, 4n - 2p + 5) \Rightarrow l = p + 2$, $l = p - 2$, $l = p - 2$ which is contradiction.

Let a_l^j and h_p^j be two distinct vertices from $V(\mathrm{SAZN}(n))$, then $r(a_l^j \mid M) = r(h_p^j \mid M) \Rightarrow (4j + l - 1, 4j + l + 3, 4n - l + 1) = (4j + p + 3, 4j + p - 1, 4n - p + 5) \Rightarrow l = p + 4, \quad l = p - 4$, $l = p - 4$ which is contradiction.

On continuing this process by choosing any other options of two vertices we will get contradiction as mentioned above.

Hence, every vertex has a distinct representation with respect to M, so M is a resolving set for $\mathrm{SAZN}(n)$; $n \geq 2$. □

The following Figure 5 represents the justification for the abovementioned Theorem 7.

4. Conclusions

In this paper, firstly, we investigated the structures of $\mathrm{HCN}(n)$, $\mathrm{SHCN}(n)$, $\mathrm{AZN}(n)$, and $\mathrm{SAZN}(n)$. Then, we established the results and showed that the metric dimensions of $\mathrm{SHCN}(n)$, $\mathrm{AZN}(n)$, and $\mathrm{SAZN}(n)$ are 2 for $n = 1$ and 3 for $n \geq 2$, respectively. We are raising the following problems for future perspective.

Open Problem 1. Determine the metric dimension of the subdivided honeycomb network $\mathrm{SHCN}(n)$ for k subdivision.

Open Problem 2. Determine the metric dimension of the subdivided Aztec diamond network $\mathrm{SAZN}(n)$ for k subdivision.

Data Availability

This study does not carry any external data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] S. Hayat, S. Wang, and J. B. Liu, "Valency-based topological descriptors of chemical networks and their applications," *Applied Mathematical Modelling*, vol. 60, pp. 164–178, 2018.
[2] P. J. Slater, "Leaves of trees," *congruent number*, vol. 14, no. 37, pp. 549–559, 1975.
[3] F. Harary and R. A. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, vol. 2, no. 1, pp. 191–195, 1976.
[4] A. Bowler, *Designs, Graphs, Codes and Their Links*, London Mathematical Society Student Texts, London, UK, 1992.
[5] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, "Resolvability in graphs and the metric dimension of a graph," *Discrete Applied Mathematics*, vol. 105, no. 1-3, pp. 99–113, 2000.
[6] M. E. Watkins, "A theorem on Tait colorings with an application to the generalized Petersen graphs," *Journal of Combinatorial Theory*, vol. 6, no. 2, pp. 152–164, 1969.
[7] P. S. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang, "On k-dimensional graphs and their bases," *Periodica Mathematica Hungarica*, vol. 46, no. 1, pp. 9–15, 2003.
[8] F. S. Raj and A. George, "On the metric dimension of silicate stars," *ARPN Journal of Engineering and applied sciences*, vol. 10, no. 5, pp. 2187–2192, 2015.
[9] F. Simonraj and A. George, "Embedding of poly honeycomb networks and the metric dimension of star of david network," *International Journal on Applications of Graph Theory In wireless Ad Hoc Networks And sensor Networks*, vol. 4, no. 4, pp. 11–28, 2012.
[10] I. Javaid, M. T. Rahim, and K. Ali, "Families of regular graphs with constant metric dimension," *Utilitas Mathematica*, vol. 75, no. 1, pp. 21–33, 2008.
[11] C. D. Godsil and B. D. McKay, "A new graph product and its spectrum," *Bulletin of the Australian Mathematical Society*, vol. 18, no. 1, pp. 21–28, 1978.
[12] S. Imran, M. K. Siddiqui, M. Imran et al., "Computing the metric dimension of gear graphs," *Symmetry*, vol. 10, no. 6, p. 209, 2018.
[13] P. Manuel, R. Bharati, I. Rajasingh, and C. Monica M, "On minimum metric dimension of honeycomb networks," *Journal of Discrete Algorithms*, vol. 6, no. 1, pp. 20–27, 2008.
[14] Z. Shao, P. Wu, E. Zhu, and L. Chen, "On metric dimension in some hex derived networks," *Sensors*, vol. 19, no. 1, p. 94, 2018.
[15] S. Abbas, Z. Raza, N. Siddiqui, F. Khan, and T. Whangbo, "Edge metric dimension of honeycomb and hexagonal networks for IoT," *Computers, Materials and Continua*, vol. 71, 2022.
[16] X. Zhang and M. Naeem, "Metric dimension of crystal cubic carbon structure," *Journal of Mathematics*, vol. 2021, Article ID 3438611, 8 pages, 2021.
[17] M. S. Bataineh, N. Siddiqui, and Z. Raza, "Edge Metric Dimension of K Multiwheel Graph," *Rocky Mountain Journal of Mathematics*, vol. 50, 2020.
[18] Z. Raza and M. S. Bataineh, "The comparative analysis of metric and edge metric dimension of some subdivisions of the wheel graph," *Asian-European Journal of Mathematics*, vol. 14, no. 04, Article ID 2150062, 2021.

[19] Z. Raza and N. Siddiqui, "The edge metric dimension of Cayley graphs and its barycentric subdivisions," *Nonlinear Functional Analysis and Applications*, vol. 24, no. 4, pp. 801–811, 2019.

[20] S. P. Eu and T. S. Fu, "A simple proof of the Aztec diamond theorem," 2004, https://arxiv.org/abs/1410.5590.

[21] M. Fendler and D. Grieser, "A new simple proof of the Aztec diamond theorem," *Graphs and Combinatorics*, vol. 32, no. 4, pp. 1389–1395, 2016.

[22] K. P. Kokhas, "Domino tilings of aztec diamonds and squares," *Journal of Mathematical Sciences*, vol. 158, no. 6, pp. 868–894, 2009.