

**Research Article**

**An Impulse-Quantizing Synchronization Approach for Mixed Continuous-Discrete Complex Networks**

Na Hu, Luyan Li, Honghua Bin, and Zhenkun Huang

School of Science, Jimei University, Xiamen 361021, China

Correspondence should be addressed to Zhenkun Huang; hzk974226@jmu.edu.cn

Received 19 October 2022; Revised 4 April 2023; Accepted 29 September 2023; Published 13 October 2023

Academic Editor: Ya Jia

Copyright © 2023 Na Hu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper presents synchronization of mixed continuous-discrete complex network via impulse-quantizing approach. A delay-partitioning group strategy is proposed and impulse-quantizing control is designed to derive theoretical criteria ensuring scale-type synchronization of complex networks. Our results show that synchronization of mixed continuous-discrete complex networks can be realized by controlling delay-partitioning subgroup nodes with impulsive quantization. The theoretical results give scale-limited sufficient conditions for quantized synchronization relying on control gains, impulsive intervals, and delays. A numerical simulation is given to demonstrate the effectiveness of the theoretical results.

1. Introduction

A complex network is a dynamical system composed of a large number of nodes with various interconnection and active interaction. Since complex networks have intrinsic characteristics of dynamic evolution, connection diversity, and structural complexity, many researchers from different disciplines have paid increasing attention to complex networks in the past few decades [1–3]. Meanwhile, the synchronization control of complex networks has been extensively studied due to its broad and cross-border applications in the fields of social networks, power grids, digital encryption communications, brain science, electronics, and so on [4]. By synthetically employing the control theory, various synchronization control strategies have been implemented for complex networks [5–9].

In recent ten years, many scholars have obtained a large number of constructive results on the synchronization of complex networks, and put forward many effective methods, such as feedback control [10], pinning control [11, 12], and impulsive control [13]. In fact, in the real world, there are often interference factors such as channel congestion, frequency change, and delays [14, 15]. Therefore, it is important to focus on the interactions of nodes in complex networks. In [16], the authors proposed that a complex network can be considered as the composition system with the two coupled dynamic subsystems: the nodes subsystem (NS) and the links subsystem (LS). The links synchronization is defined and synthesizes the adaptive control scheme to realize it. In [17], the authors described the dynamic behaviour of LS with the outgoing links vector at every node, and a double tracking control for the directed complex dynamic network via the state observer of outgoing links is presented. Coupling configurations between network nodes will have impulsive discontinuity, that is, the topology of the network is dynamic and may subject to instantaneous transmission. In [18], finite-time synchronization problem of uncertain nonlinear complex networks with time-varying delay is studied.

The traditional synchronous control often relies on the state or output feedback continuous signal, but in reality, the control system is based on digital equipment such as computers with limited accuracy. The signals between network nodes and controllers are transmitted through the network with limited capacity, and the feedback control signal usually needs to be quantified before transmission [19–24]. The quantization errors will affect the synchronization of the network. In [25], quantization and cyclic protocols are used to solve the problem of limited communication channel capacity, and then intermittent control strategy is used to improve the efficiency of communication.
channel and controller. In [26], quantization and trigger errors are combined to discuss the synchronization of Lur’e form driven response system in finite channel. Due to the solution, space of the high-dimensional dynamic system described by the dynamic network with quantized signal is more complex than the general continuous or discrete dynamic system, and its theoretical analysis is also more attractive and challenging.

However, in many network systems, the interaction among subsystems would occur at any different time domains including discrete-time sequences or continuous time intervals, respectively [27, 28]. To avoid adopting separate dynamical analysis, it makes sense to discuss these systems on time scales [28, 29] which can unify continuous and discrete dynamics under a unified framework. In [30], based on the time scale theory of calculus and linear matrix inequality (LMI), some sufficient conditions are obtained to ensure the global synchronization of the complex networks with delays. In [31], the synchronization problem of linear dynamical networks on time scale was dealt through node-based pinning control. Inspired by existing ones [31–33], we introduce some basic theories and present the main structure of this paper is as follows. In Section 2, we introduce some basic theorems and present the quantized synchronization problem of mixed continuous-discrete complex networks. In Section 3, quantization synchronization criteria of complex networks are established. In Section 4, the effectiveness of the proposed control strategy is illustrated by numerical simulations. Finally, Section 5 summarizes this paper.

2. Preliminaries and Model

In this section, we give some basic definitions and related Lemmas about time scale. For the theory of time scale, we refer to the monograph [35].

Let T be a time scale (i.e., a nonempty closed subset of R). The forward jump operator ρ: T → T is defined by ρ(t) = inf{s ∈ T: s > t} for all t ∈ T, while the backward jump operator p: T → T is defined by ρ(t) = sup{s ∈ T: s < t} for all t ∈ T. If σ(t) > t, we say that t is right-scattered, while if ρ(t) < t we say that t is left-scattered. Also, if σ(t) = t, we say that t is right-dense, while if ρ(t) = t we say that t is left-dense. Define Tσ = T − {M}, when T has a left-scattered maximum M, otherwise Tσ = T. The graininess function μ: T → [0, ∞) is defined by μ(t) = σ(t) − t.

Definition 1 (see [35]). Let f: T → R and t ∈ Tσ. fβ(t) is said to be the delta derivative of f at t, given any ε > 0, if there is a neighborhood U of t such that

$$\lim_{s \to t} \frac{|f(\sigma(s)) - f(s) - f^\beta(\sigma(s) - s)|}{\varepsilon} = 0$$

Definition 2 (see [35]). A function f: T → R is rd-continuous if it is continuous at right-dense points in T and its left-sided limits exist at left-dense points in T. The set of all rd-continuous functions f: T → R will be denoted by Crd. A function p: T → R is regressive provided 1 + μ(t)p(t) ≠ 0 for all t ∈ Tσ. Denote by $\mathcal{R}$ the set of all regressive, rd-continuous functions: $f: T → R$ and $\mathcal{R}^\tau = \{ p \in \mathcal{R}: 1 + μ(t)p(t) > 0, \forall t ∈ T \}$.

Definition 3 (see [35]). If $p \in \mathcal{R}$, the exponential function $e_p(t, s)$ is the solution of $\xi = \exp\left(\int_{\xi}^{\tau} f(\tau) d\tau\right)$ for $t, s \in T$, where the cylinder transformation $\xi_h(z) = \begin{cases} \log(1 + h^2)/h, & h ≠ 0 \\
0, & h = 0 \end{cases}$ and Log is the natural logarithm function.

Remark 4. Let $a ∈ \mathcal{R}$ be constant. If $T = \mathbb{Z}$, then $e_a(t, t_0) = (1 + a)^{t - t_0}$ for all $t \in T$. If $T = \mathbb{R}$, then $e_a(t, t_0) = e^{a(t - t_0)}$ for all $t \in T$. If $\alpha ≥ 0$, then $e_\alpha(t, s) ≥ 1$ for $t ≥ s$ and $t, s \in T$.

Lemma 5 (see [35]). If $f \in C_{rd}$ and $t \in T^\sigma$, then

$$\int_{\tau}^{t} f(\tau) d\tau = μ(t)f(t).$$

Lemma 6 (see [35]). Let $f \in C_{rd}$ and $p \in \mathcal{R}$. For all $t \in T$, the dynamic inequality $y^\Delta(t) ≤ p(t)y(t) + f(t)$ implies that $y(t) ≤ y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(t))f(\tau)d\tau$.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\| \cdot \|$. Let $Z^+$ denote the set of positive integer numbers, N is the set of natural numbers, $\mathbb{R}^{n \times n}_{sym}$ is the set of all $n \times n$ real matrices. The superscript $T$ stands for the transpose of a matrix. $I$ is an appropriately dimensioned identity matrix. The notion $X ≽ Y$ (respectively, $X ≼ Y$) means that the matrix $X − Y$ is positive semidefinite (respectively, positive definite).

Lemma 7 (see [35]). If $p \in \mathcal{R}$, $c \in T$, and $f, g: T → R$ are differentiable at $t \in T^\sigma$, then

$$\left[e_p(c, \cdot)\right]^\Delta = -p\left[e_p(c, \cdot)\right]^\Delta,$$

$$\left(f^\Delta g\right)(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$

$$\left(e_p(c, \cdot)\right)^\Delta = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t)$$

(2)
In this paper, assume that \( Q: \mathbb{R} \rightarrow \mathbb{R} \) is a logarithmic quantization function, the set of logarithmic quantization levels is described by the following equation:

\[
U = \left\{ \pm u_i, u_i = \rho^j u_0, i = \pm 1, \pm 2, \cdots \right\} \cup \{ \pm u_0 \} \cup \{ 0 \}, 0 < \rho < 1, u_0 > 0.
\]  

(3)

The associated quantizer is defined as follows:

\[
Q(a) = \begin{cases} 
q_j & \text{if } \frac{1}{1 + \delta} u_i < a \leq \frac{1}{1 - \delta} u_i, a > 0, \\
0 & \text{if } a = 0, \\
-Q(-a) & \text{if } a < 0,
\end{cases}
\]

(4)

where \( \delta = (1 - \rho)/(1 + \rho) \) and \( Q(a) = (1 + \Theta)a \) with \( \Theta \in [-\delta, \delta] \).

Consider the following mixed continuous-discrete complex dynamic networks (CDNs) with \( N \) identical nodes on time scale \( T \) as follows:

\[
x_i^\Delta(t) = Ax_i(t) + Bf(x_i(t)) + c \sum_{j=1}^{N} g_{ji} \Gamma x_j(t), i = 1, 2, \ldots, N,
\]

(5)

where \( x_i(t) \in \mathbb{R}^n \) denotes the state vector of the \( i \) th node, \( c > 0 \) is the coupling strength, \( f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^T \) is a nonlinear function, \( A, B \in \mathbb{R}^{mn \times n} \) are constant matrices, \( G = (g_{ij})_{N \times N} \) represents the network connection topology, which is defined as follows: if there is a connection between the \( i \) th node and the \( j \) th node \( (i \neq j) \), then \( g_{ij} = g_{ji} = 1 \); otherwise, \( g_{ij} = g_{ji} = 0 \), and the diagonal elements are defined as \( g_{ii} = -\sum_{j=1,j \neq i}^{N} g_{ij} \).

\( \Gamma = \text{diag} \{ y_1, y_2, \ldots, y_n \} > 0 \) is the inner coupling matrix between nodes. For system (5), we introduce an isolated node as synchronization target, which is described as follows:

\[
y^\Delta(t) = Ay(t) + Bf(y(t)),
\]

(6)

where \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \) and \( f(y(t)) = \left( f_1(y_{11}(t)), f_2(y_{22}(t)), \ldots, f_n(y_{nn}(t)) \right)^T \).

Consider system (5) with feedback control as follows:

\[
x_i^\Delta(t) = Ax_i(t) + Bf(x_i(t)) + c \sum_{j=1}^{N} g_{ji} \Gamma x_j(t) + u_i(t), i = 1, 2, \ldots, N,
\]

(7)

where the impulse-quantizing controller is designed as follows:

\[
u_i(t) = \sum_{k=1}^{\infty} Q(q_j e_i(t) + q_j e_i(t) + \overline{q}_j e_i(t - \tau_j)) \delta(t - t_k), \quad i \in N_j.
\]

(8)

Each \( e_i(t) = x_i(t) - y(t) \) is an error vector of \( i \) th node, \( \delta(\cdot) \) is the Dirac delta function, \( q_j \) and \( \overline{q}_j \) are the designed impulse control gain.

\( N_j \) denotes a delay-partitioning subgroup which allows \( \tau_j \)-delay impulse imposed on all nodes in \( N_j, \ N_1 \cup N_1 \cup \cdots \cup N_m = N, \) and \( N_i \cap N_j = \emptyset, i \neq j. \)

As \( i \in N_j, \)

\[
\begin{cases}
\Delta x_i(t) = Ax_i(t) + Bf(x_i(t)) + c \sum_{j=1}^{N} g_{ji} \Gamma x_j(t), \quad t \neq t_k, \\
\Delta x_i(t) = Q(q_j e_i(t) + \overline{q}_j e_i(t - \tau_j)), \quad t = t_k,
\end{cases}
\]

(9)

where \( \Delta x_i(t_k) = x_i(t_k) - x_i(t_k) \) and \( x_i(t_k) - x_i(t_k) \).
As $i \in N_j$,\[\begin{align*}
e^i(t) &= A \eta_i(t) + B f(\eta_i(t)) + c \sum_{j=1}^{N} g_{ij} T \eta_j(t), \quad t \neq t_k, \\
\Delta \eta_i(t) &= Q(q_{ij} \eta_i(t) + \bar{q}_{ij} \eta_i(t - \tau_j)), \quad t = t_k,\end{align*}\]where $f(\eta_i(t)) = f(x_i(t)) - f(y(t))$.

**Remark 8.** Due to the limitation of network bandwidth, data transmission between nodes in networks will arouse delays [36] and needs to be quantified, and the quantization will affect the performance of the system [24, 32]. Therefore, an impulse-quantizing multigroup strategy is proposed for equation (9) and it has a very important impact on the system synchronization.

**Definition 9.** System (7) is said to achieve impulse-quantizing synchronization with system (6), if
\[
\lim_{t \to +\infty} \|x_i(t) - y(t)\| = 0 \text{ for all } i = 1, 2, \ldots, N. \tag{11}\]

**Lemma 10.** Given any vector $x, y$, a positive definite matrix $H$, and $\varepsilon$ is a positive definite constant and satisfies $\varepsilon > 0$, then the following inequality holds:
\[
\begin{pmatrix} 0 & \varepsilon \ast & \varepsilon \ast \\ \ast & -Q_1 & 0 \\ \ast & \ast & -Q_2 \end{pmatrix} \leq 0,
\]

where $\Xi := \mu A^T P A + \mu L^T Q L + \mu \frac{1}{2} c u L^T Q L B P + \mu (c c^T + \beta_1^2 A^T + \beta_2^2 B^T P + PA)$ and $\Xi := \Xi A^T P + PA + c(1 + \beta_1^2) P + L^T Q L + \Xi - \alpha P$. (A1): $d_M := \max_{i_1(1, -\varepsilon)} ||(1 + \varepsilon - \tau_i)^{-1} ||^2, i = 1, 2, \ldots, m$ and $d_M = \max_{i \in [1, -\varepsilon]} ||(1 + \varepsilon - \tau_i)^{-1} ||^2, i = 1, 2, \ldots, m$.

Then, system (9) can achieve impulse-quantizing synchronization with system (6).

**Proof.** Set $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T$ and $f(e(t)) = (f^T(e_1(t)), f^T(e_2(t)), \ldots, f^T(e_n(t)))^T$, then system (10) can be rewritten as follows:
\[
\begin{align*}
e^i(t) &= (I_N \otimes A)e(t) + (I_N \otimes B)f(e(t)) + c(G \otimes \Gamma)e(t), \quad t \neq t_k, \\
\Delta \eta_i(t) &= Q(q_{ij} \eta_i(t) + \bar{q}_{ij} \eta_i(t - \tau_j)), \quad t = t_k. \tag{15}\end{align*}\]

Consider Lyapunov function $V(t) = e^T(t) (I_N \otimes P) e(t)$. Calculating the $\Delta$-derivative of $V(t)$ over impulse interval $t \in k \in Z_+$ along the trajectory of system (10), we get the following equation:
\[ V^\Delta (t) = \sum_{i=1}^{N} \left[ (e_i^T)^\Delta P e_i + (e_i^T)^\delta P e_i^\Delta \right] \]
\[ = \sum_{i=1}^{N} \left[ (e_i^T)^\Delta P e_i + (e_i + \mu e_i^T)^T P e_i^\Delta \right] \]
\[ = \sum_{i=1}^{N} \left[ e_i^T (A^T P + PA + \mu A^T PA) e_i + 2e_i^T PB f(e_i) + 2\mu e_i^T A^T PB f(e_i) \right. \]
\[ + \mu f^T (e_i)B^T PB f(e_i) + 2c \sum_{j=1}^{N} g_{ij} e_j^T P e_j + 2c \mu \sum_{j=1}^{N} g_{ij} e_j^T A^T P e_j \]
\[ + 2c\mu \sum_{j=1}^{N} g_{ij} f^T (e_j)B^T P e_j + c^2 \mu \sum_{j=1}^{N} g_{ij} f_j^T T_i P e_j \sum_{j=1}^{N} g_{ij} e_j \right] \]

It can be obtained from Lemma 10 that

\[ 2e_i^T PB f(e_i) + 2\mu e_i^T A^T PB f(e_i) \leq e_i^T PBQ^{-1} B^T P e_i + f^T (e_i)Q_1 f(e_i) + \mu e_i^T PBQ_2^{-1} B^T PA e_i + \mu f^T (e_i)Q_2 f(e_i) \leq e_i^T (PBQ^{-1} B^T P + L^T Q_1 L + \mu A^T PBQ_2^{-1} B^T PA + \mu L^T Q_2 L) e_i \]

Similarly, according to condition \((A_1)\), we have the following equation:

\[ 2c \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} e_i^T P e_j + 2c \mu \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} e_i^T A^T P e_j + 2c \mu \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} f^T (e_j)B^T P e_j \]
\[ + c^2 \mu \sum_{i=1}^{N} \left( \sum_{j=1}^{N} g_{ij} f_j^T T_i P \sum_{j=1}^{N} g_{ij} e_j \right) \]
\[ = 2ce^T (G \otimes (P^T)) e + 2c\mu e^T (G \otimes (A^T P^T)) e + 2c\mu f^T (e) (G \otimes (B^T P^T)) e \]
\[ \leq ce^T (I_N \otimes P) e + ce^T (G^T G) \otimes (T^T P^T) e + c\mu e^T (G \otimes (P^T)) e \]
\[ + c\mu e^T (G^T G) \otimes (T^T A^T P^T) e + c\mu f^T (e) (I_N \otimes (PB)) f (e) \]
\[ + c\mu e^T (G^T G) \otimes (T^T B^T P^T) e + c^2 \mu e^T (G^T G) \otimes (T^T P^T) e \]
\[ \leq ce^T (I_N \otimes P) e + ce^T ((\beta_1 I_N) \otimes P) e + c\mu e^T (I_N \otimes (PA)) e \]
\[ + c^2 \mu e^T (G^T G) \otimes (T^T P^T) e + c\mu e^T (\beta_1 I_N) \otimes (A^T P) e + c\mu f^T (e) (I_N \otimes (PB)) f (e) \]
\[ + c\mu e^T (\beta_1 I_N) \otimes (B^T P) e + c^2 \mu e^T (\beta_1 I_N) \otimes P) e \]
\[ \leq \sum_{i=1}^{N} \left[ e_i^T (cP + c\beta_1 P + c\mu PA + c\beta_2 A^T P + c\mu L^T PBL + c\mu \beta_3 B^T P + c^2 \mu \beta_1 P) e_i \right] \]

It follows from equations (16)–(18) and condition \((A_2)\) that
Due to $1 + \mu(t) > 0$ for $t \in (t_k, t_{k+1})$, by Lemma 6, it implies that

$$V(t) \leq V(t_k^*)e_a(t, t_k).$$  \hspace{1cm} (20)

Next, there are two cases for us to show that

$$V(t_{k+1}) \leq V(t_k^*)e_a(t_{k+1}, t_k).$$  \hspace{1cm} (21)

Case 12. If $t_{k+1}$ is left-dense, then by the continuity of $V(t)$ and $e_a(t, t_k)$, we have the following equation:

$$V(t_{k+1}) = \lim_{t \rightarrow t_{k+1}^-} V(t) \leq \lim_{t \rightarrow t_{k+1}^-} V(t_k^*)e_a(t, t_k) = V(t_k^*)e_a(t_{k+1}, t_k).$$  \hspace{1cm} (22)

Hence, we get the following equations:

$$V(t_k^*) \leq \sum_{i \in N_1} \left[ (1 + \varepsilon_i) \left( 1 + \Theta q_1 + 1 \right)^2 e_i^T(t_k)Pe_i(t_k) + (1 + \varepsilon_1^{-1})(1 + \Theta)^2 q_i e_i^T(t_k - \tau_1)Pe_i(t_k - \tau_1) \right]$$

$$+ \sum_{i \in N_2} \left[ (1 + \varepsilon_2) \left( 1 + \Theta q_2 + 1 \right)^2 e_i^T(t_k)Pe_i(t_k) + (1 + \varepsilon_2^{-1})(1 + \Theta)^2 q_i e_i^T(t_k - \tau_2)Pe_i(t_k - \tau_2) \right] + \cdots$$

$$+ \sum_{i \in N_m} \left[ (1 + \varepsilon_m) \left( 1 + \Theta q_m + 1 \right)^2 e_i^T(t_k)Pe_i(t_k) + (1 + \varepsilon_m^{-1})(1 + \Theta)^2 q_i e_i^T(t_k - \tau_m)Pe_i(t_k - \tau_m) \right].$$  \hspace{1cm} (25)

When $t_1 - \tau \in [t_0, t_1]$, we have the following equation:

$$V(t) \leq \text{He}^{-\gamma(t - t_0)}, t \in [t_0, t_1].$$  \hspace{1cm} (26)
where $H = e^{\psi(t_1 - t_0)} \sup_{t \in [t_0, t_1]} |V(t)|$.

Now, we shall show that
\[
V(t) \leq He^{-\psi(t-t_0)}, \; t \in (t_1, \infty).
\] (27)

Together with equation (20), one can get the following equation:
\[
V(t) \leq V(t_1)e_{a}(t_1, t) \leq He^{-\psi(t-t_0)} \leq He^{-\psi(t_0-t_0)} \leq He^{-\psi(t_0-t_0)}, \; t \in (t_1, t_2).
\] (29)

Thus, equation (27) holds for $t \in (t_1, t_2]$. Assume that equation (27) holds for $t \in (t_n, t_{n+1}]$, $n \in \mathbb{Z}_+$, which implies that equation (27) holds for $t \in (t_1, t_{n+1}]$. Next, we claim that equation (27) holds for $t \in (t_{n+1}, t_{n+2}]$. For each $i \in \{1, 2, \cdots, m\}$, there are two cases for us to estimate $V(t_{n+1} - t_i)$.

For $t \in (t_{n+1}, t_{n+2}]$, one gets from equations (20) and (32) that
\[
V(t_{n+1}) \leq d_MV(t_{n+1}) + \sum_{i=1}^{m} d_iV(t_{n+1} - t_i) \leq HdMe^{-\psi(t_{n+1} - t_0)} + H\sum_{i=1}^{m} d_ie^{-\psi(t_{n+1} - t_0)}
\] (32)

For $t \in (t_{n+1}, t_{n+2}]$, it follows from condition $(A_3)$ that
\[
V(t_{n+1}) \leq He^{-\psi(t_{n+1} - t_0)} \leq H\sum_{i=1}^{m} d_ie^{-\psi(t_{n+1} - t_0)}
\] (33)

Thus, equation (27) holds for $t \in (t_{n+1}, t_{n+2}]$. Therefore, by mathematical induction that equation (27) holds for all $t \in (t_1, \infty)$. Then,
\[
V(t) \leq He^{-\psi(t-t_0)}, \; t \in [t_0, \infty),
\] (34)

which implies $V(t) \rightarrow 0$ as $t \rightarrow \infty$, thus impulse-quantizing synchronization between system (9) and system (6) can be achieved.

Remark 16. System (7) is defined on hybrid time domains which include continuous time and discrete-time ones as its special cases. When $T = Z$, then $\mu(t) \equiv 1$ for all $t \in Z$, then $(A_2)$ reduces to the following equation:
\[
\begin{pmatrix}
\Xi & PB & A^TB \Xi \\
* & -Q_1 & 0 \\
* & * & -Q_2
\end{pmatrix} \leq 0,
\] (35)

where $\Xi_1 = A^TP + L^TQ_2L + cL^TBL + c((c\beta_1 + \beta_2)A^T + \beta_2B^T)P + PA$ and $\Xi_2 = A^TP + PA + c(1 + \beta_1)P + L^TQ_1L + \Xi_1 - \alpha P$. When $T = R$, then $\mu(t) \equiv 0$ for all $t \in R$, then $(A_2)$ reduces to the following equation:
\[
\begin{pmatrix}
\Xi & PB \\
* & -Q_1
\end{pmatrix} \leq 0,
\] (36)
discrete domain. Compared to the existing results in [31], our delay-partitioning group strategy avoids the complexity of incorporating a mechanism such as index set $D_k$ which reorders the states at impulsive instants. Additionally, based on the pinning control method, we have developed a quantized impulse effect that can enhance the control effect and lower the cost. There are quantitative control strategies that just take output measurements into account, we can refer to ones in [32]. However, introducing impulse-quantizing into the hybrid domains is still a vacancy in the existing literature and this paper is to fill the vacancy.

Remark 18. For the existing methods in [37, 38], the main differences and advantages of our impulse-quantizing approach for complex networks lie in two aspects: (1) In [37, 38], their control strategies are aimed at the continuous time domain and it will become inapplicable once the state synchronization arises in mixed time domain. (2) In [37, 38], networks information can be communicated without any loss. However, the amount of data that can be transmitted per unit time in complex networks is frequently constrained. To lessen the strain on the communication channel during transmission, the quantization effect [39] is introduced in the controller in this study.

Corollary 19. Under the condition of Theorem 11, when $\overline{\tau}_i = 0$, there exist some scalars $a > 0, \nu > 0, \beta_i > 0, (i = 1, 2, 3)$, $\epsilon_i (i = 1, \cdots, m)$, $P^T = P > 0, Q_1 > 0, Q_2 > 0$ that satisfy the following inequalities: 

\[
(B_1): (G^T G) \otimes (1^T P 1) \leq (\beta_1 I_N) \otimes P, (G^T G) \otimes (1^T A^T P 1) \leq (\beta_2 I_N) \otimes (A^T P), (G^T G) \otimes (1^T B^T P 1) \leq (\beta_3 I_N) \otimes (B^T P), \\
(B_2): \begin{pmatrix}
\Xi & PB \\
* & -Q_1 \\
* & -Q_2
\end{pmatrix} \leq 0,
\]

where $\Xi = \mu A^T PA + \mu L^T Q_2 L + c \mu L^T PBL + c \mu ((c^2 \beta_1 + \beta_2 A^T + \beta_3 B^T)P + PA)$ and $\Xi = A^T P + PA + c(1 + \beta_1)P + L^T Q_2 L + \Xi - aP.$

\[
(B_3): d_M e^{\nu(t_{k+1} - t_k)} + e_{\alpha_k}(t_{k+1} - t_k) \leq 1, d_M = \max_{\alpha_k \in [1, \cdots, m]} \left\{ 1 + \epsilon_k ((1 + \Theta)q_i + 1)^2 \right\}. 
\]

Then, system (9) can achieve impulse-quantizing synchronization with system (6).

4. Numerical Simulations

In the section, we present a numerical example to illustrate the proposed results. Consider the following mixed continuous-discrete complex networks on time scale $T = \cup_{j=0}^{\infty}[0.3j, 0.3j + 0.2]$:

\[
A = \begin{pmatrix}
0.2 & 0.4 & 1.2 \\
0 & 0.1 & 0.1 \\
0.3 & 1.1 & 0.1
\end{pmatrix}, 
B = \begin{pmatrix}
0.5 & 0.2 & 1 \\
0.2 & 0.4 & 0 \\
0.1 & 0.7 & 0.2
\end{pmatrix}, 
G = \begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}.
\]
For the phase space and state trajectories of system (39) with initial conditions $x_1(0) = [0.25, -0.45, -1.05]^T$, $x_2(0) = [0.1, -0.55, -0.85]^T$, and $x_3(0) = [0.8, 0.15, 0.02]^T$, we can refer to Figures 1(a) and 1(b). The graininess function of $T$ is given by the following equation:

$$\mu(t) = \begin{cases} 
0, & t \in \bigcup_{j=0}^{\infty} [0.3j, 0.3j + 0.2], \\
0.1, & t = 0.3j + 0.2, j \in \mathbb{Z}^+ 
\end{cases}$$

which implies that $\mu(t) \leq 0.1$ for all $t \in T$. 

---

Figure 1: The phase space and trajectories of $x_i(t), i = 1, 2, 3$ without impulsive control. (a) The phase space of $x_i(t), i = 1, 2, 3$. (b) The trajectories of $x_i(t), i = 1, 2, 3$.

Figure 2: The phase space and trajectories of $e_i(t), i = 1, 2, 3$ for Case 20. (a) The phase space of error variables. (b) The trajectories of error variables.
Choose \( N_1 = \{1, 2\}, N_2 = \{3\}, \alpha = 3, \nu = 1, \epsilon_i = 1 \) \((i = 1, 2)\), \( m = 2 \), and impulsive instants \( t_k = 0.1k \) for \( k \in \mathbb{N} \). It is easy to estimate the exponential function \( e_{\alpha} (t_{k+1} - t_k) \approx 1.35 \) and

\[
\left( d_M + \sum_{i=1}^{m} d_i e^{\alpha r_i} \right) e^{\alpha (t_{k+1} - t_k)} e_{\alpha} (t_{k+1}, t_k) < 0.7 < 1, \quad (42)
\]

which can satisfy \((A_3)\) in Theorem 11.

---

**Figure 3:** The impulsive effect of \( e_i(t) \), \( i = 1, 2, 3 \), under different impulsive controllers. (a) The impulsive magnitudes of \( e_i(t) \), \( i = 1, 2, 3 \), for Case 20. (b) The impulsive quantized magnitudes of \( e_i(t) \), \( i = 1, 2, 3 \), for Case 21. (c) The impulsive magnitudes of \( e_i(t) \), \( i = 1, 2, 3 \), for Case 22. (d) The impulsive quantized magnitudes of \( e_i(t) \), \( i = 1, 2, 3 \) for Case 23.
Moreover, let $u_0 = 1, \delta = 0$, and $\rho = (9/11)$; all assumptions of Theorem 11 are satisfied.

$$P = \begin{pmatrix} 0.3282 & 0.0104 & 0.0415 \\ 0.0104 & 0.6148 & 0.1401 \\ 0.0415 & 0.01401 & 0.4438 \end{pmatrix}, Q_1 = \begin{pmatrix} 1.2586 \\ 0 \\ 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 1.2586 & 0 \\ 0 & 1.2586 & 0 \\ 0 & 1.2586 & 0 \end{pmatrix}.$$ (43)

Next, consider impulsive controller as follows:

$$u_i(t) = \sum_{k=1}^{\infty} Q(q_j e_i(t) + \overline{q}_j e_i(t - \tau_j)) \delta(t - t_k), \quad i \in N_j,$$ (44)

and choose $q_1 = -0.6, q_2 = -0.8, r_1 = 1, \text{and } r_2 = 2$, and there are four cases for impulsive control gains.

Case 20. When $\overline{q}_1 = 0, \overline{q}_2 = 0$, and $u_i(t) = \sum_{k=1}^{\infty} q_j e_i(t) \delta(t - t_k), \quad i \in N_j$, the error system (10) is impulsively synchronous. For phase space and trajectories of $e_i(t), \quad i = 1, 2, 3$, we can refer to Figures 2(a) and 2(b). Figure 3(a) shows the impulse magnitude of error variables.

Case 21. When $\overline{q}_1 = 0, \overline{q}_2 = 0$, and $u_i(t) = \sum_{k=1}^{\infty} Q(q_j e_i(t)) \delta(t - t_k), \quad i \in N_j$, the error system (10) can achieve impulse-quantizing synchronization. For phase space and trajectories of $e_i(t), \quad i = 1, 2, 3$, we can refer to Figures 4(a) and 4(b). Figure 3(b) shows the impulse quantized magnitude of error variables.

Case 22. When $\overline{q}_1 = -0.08, \overline{q}_2 = -0.06$, and

$$u_i(t) = \sum_{k=1}^{\infty} (q_j e_i(t) + \overline{q}_j e_i(t - \tau_j)) \delta(t - t_k), \quad i \in N_j,$$ (45)

the error system (10) can achieve impulsive synchronization. For phase space and trajectories of $e_i(t), \quad i = 1, 2, 3$, we can refer to Figures 5(a) and 5(b). Figure 3(c) shows the impulsive magnitude of error variables.

Case 23. When $\overline{q}_1 = -0.08, \overline{q}_2 = -0.06$, and

$$u_i(t) = \sum_{k=1}^{\infty} (Q q_j e_i(t) + \overline{q}_j e_i(t - \tau_j)) \delta(t - t_k), \quad i \in N_j,$$ (46)

the error system (10) can achieve impulse-quantizing synchronization. For phase space and trajectories of $e_i(t), \quad i = 1, 2, 3$, we can refer to Figures 6(a) and 6(b). Figure 3(d) shows the impulse quantized magnitude of error variables.

Comparing Case 20 and Case 21, it can be seen from Figures 2(b) and 4(b) that the error system achieves synchronization with or without the influence of the quantization effect when delayed impulses do not exist, but the quantization increases the error magnitude. Comparing Case 22 and Case 23, it can be seen from Figures 5(a) and 6(b) that the quantization effect increases the error amplitude when delayed impulses exist, and the quantization will make the error system reach the synchronization state faster than the simple impulsive control.
From the above analysis, it can be seen that impulse-quantizing synchronization proposed in this paper can be achieved more economically and effectively than existing ones.

5. Conclusion

In this paper, we studied impulse-quantizing synchronization problem of mixed continuous-discrete complex networks. A delay-partitioning group-based impulsive controller which can include delays and logarithmic quantizer is designed. Multigroup pinning control and impulsive quantization can be selected to flexibly realize synchronization according to different situations. Scale-type sufficient conditions concerning the upper bound of impulses and the communication delays have been derived and analyzed. Our simulations show the interesting synchronization schemes with or without impulsive and quantized control effects.
Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

This paper consists of part of the preprint [34] posted on Research Square.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding this study.

References


