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Research Article

Lyapunov Stability in the Cournot Duopoly Game

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This paper studies Lyapunov stability at a point and its application in the Cournot duopoly game. We first explore the relationship between Lyapunov stability at a point, nonsensitivity and non-Devaney chaos and find that a dynamical system is nonsensitive and non-Devaney chaotic if there is a point in this system such that it is Lyapunov stable at that point. We next prove a group of equivalent characterizations of Lyapunov stability at a point to deduce the composite theorems and product theorems of Lyapunov stability at a point, and then we prove three equivalent characterizations of the Cournot duopoly system to demonstrate that this system is Lyapunov stable at its unique nonzero fixed point (Cournot equilibrium point) when the unit costs of the Cournot double oligarchies satisfy certain conditions. Therefore, we conclude that the Cournot duopoly system is safe relative to both sensitivity and Devaney chaos. The robustness of our results are also verified by conducting numerical simulations in the Cournot duopoly game.

1. Introduction

A general concept of Lyapunov stability at a point was introduced by Ethan and Sergii in 2003 [1] through studies of Li-York sensitivity that link the Li-York versions of chaos with the notion of sensitivity to initial conditions. The Lyapunov stability (hereafter, L-stability) at a fixed point was formally defined by Clark Robinson in 2012 [2] for a state description of differential equation solutions of being close to an equilibrium point. In the following years, this concept emerged in various studies, and these studies mainly focused on different types of systems, for example, topologic dynamical systems, differential systems, neural network systems, and epidemiology systems (see References [3–6]). However, there is limited evidence of the nature and characterization of L-stability at a point in the literature.

To the best of our knowledge, this paper is the first to study the nature and characterization of L-stability at a point and its applications in the Cournot duopoly game. In Section 2, we explore the relationship between L-stability at a point, nonsensitive dependence on initial condition (hereafter, nonsensitivity), and nonchaos in the sense of Devaney (hereafter, non-Devaney chaos) by introducing the concept of sensitive dependence on initial condition (hereafter, sensitivity) at a point. First, we find that a system is nonsensitive at a point if and only if it is L-stable at this point (see Proposition 1). Next, the following two conclusions are deduced in Proposition 3: (1) a system is nonsensitive (i.e., nonglobally sensitive) if it is nonsensitive at a given point in the system and (2) a system is non-Devaney chaotic if the system is nonsensitive. Hence, we infer that a system is nonsensitive and non-Devaney chaotic as long as there is a point that is L-stable in this system. That is, we declare a system's nonsensitivity and non-Devaney chaos as long as we can find a point in the system that is L-stable. In Section 3, we first prove a group of equivalent characterizations of L-stability at a point (see Theorem 1). By using this theorem, we obtain the composite theorems (Theorem 2 and Corollary 1) and product theorems (Theorem 3 and Corollary 2) of L-stability at a point, and then three equivalent characterizations of L-stability at a point of the Cournot duopoly game system are obtained (Theorem 4). In Section 4, based on the previous theorems and the knowledge of differential calculus, we show that the system of the Cournot duopoly game is L-stable at its Cournot equilibrium point if the unit costs of two oligopolies are within a certain range (see Theorem 6). That is, the system of the Cournot duopoly game is safe relative to sensitivity and Devaney chaos if the condition of unit costs in Theorem 6 is satisfied. In Section 5, we conduct numerical simulations to show that our results are robust, as the sequences of the Cournot duopoly game generated by the iterations are either convergent or periodic if the condition of Theorem 6 is satisfied. It is well known that system security is very important for economics and other sciences. Therefore, our study might provide research methods for future research on the relative security of a system.

2. Relationship between L-Stability at a Point, Nonsensitivity, and Non-Devaney Chaos

Conventionally, a dynamical system (X, T) is a pair where X is a nonvoid compact metric space with metric ϱ and $T: X \longrightarrow Y$ is a surjective and continuous map. $\mathbb{N}$ denotes the set of all natural numbers (i.e., $\mathbb{N} = \{0, 1, 2, \ldots\}$), both ϵ and δ denote positive real numbers, where δ is usually a sufficiently small positive number and ϵ is any positive number. The symbol $B_{\varrho}(x, \delta)$ denotes the neighborhood centered on x with radial δ in a metric space X, where $\varrho(x, y)$ is the distance between x and y in X, and the symbol $\sup A$ denotes the smallest upper bound of a set A of some real numbers. Let A and B be two propositions, then, symbol $A \Rightarrow B$ denotes that B inevitably holds if A holds.

To elucidate the relationship between L-stability at a point, nonsensitivity, and non-Devaney chaos, we define the following precise concepts:

Definition 1 (see [1]). Assume that (X, T) is a dynamical system, $x \in X$, the system (X, T) is L-stable at x, if for any $\epsilon > 0$, there is $\delta > 0$ such that $\varrho_T(x, y) \leq \epsilon$ for each $y \in B_{\varrho}(x, \delta)$, where $\varrho_T(x, y) = \sup\{\varrho(T^n x, T^n y) \colon n \geq 0\}$.

Definition 2 (see [7]). The system (X, T) is Devaney chaotic if the following properties hold:

(D1) T is topological transitive, i.e., to any pair of open sets $U, V \subset X$ there is $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \phi$;

(D2) The periodic points are dense in X;

(D3) T has sensitivity, i.e., there is $\epsilon > 0$ such that for any $x \in X$ and $\delta > 0$, there is $y \in B_{\varrho}(x, \delta)$ and $n \in \mathbb{N}$ such that $\varrho(T^n x, T^n y) \geq \epsilon$.

We deduce the following equivalent characterizations from Definition 1:

Proposition 1. *Assuming that (X, T) is a dynamical system, $x \in X$, the system (X, T) is L-stable at x if only and if for any $\epsilon > 0$, there is $\delta > 0$ such that $\varrho(T^n x, T^n y) \leq \epsilon$, for each $y \in B_{\varrho}(x, \delta)$ and for each $n \in \mathbb{N}$.*

Therefore, the following three equivalent characterizations naturally hold:

Proposition 2. *Assume that (X, T) is a dynamical system, $x \in X$, the following three propositions are equivalent:*

(1) System (X, T) is not L-stable at;

(2) There is $\epsilon > 0$ such that for any $\delta > 0$, there is $y \in B_{\varrho}(x, \delta)$ and $n \in \mathbb{N}$ such that $\varrho(T^n x, T^n y) > \epsilon$;

(3) There is $\epsilon > 0$ such that for any $\delta > 0$, there is $y \in B_{\varrho}(x, \delta)$ and $n \in \mathbb{N}$ such that $\varrho(T^n x, T^n y) \geq \epsilon$.

From Definition 2, we can see that sensitivity is one of three conditions of Devaney chaos. In other words, a system (X, T) is non-Devaney chaotic if it is nonsensitive. Sensitivity is a global concept relative to a system (X, T). Thus, for any point $x \in X$, we can discuss its localization at a point x as follows:

Definition 3. A system (X, T) is sensitive at a point $x \in X$ if there is $\epsilon > 0$ such that for any $\delta > 0$, there is $y \in B_{\varrho}(x, \delta)$ and $n \in \mathbb{N}$ such that $\varrho(T^n x, T^n y) \geq \epsilon$.

To distinguish the two sensitivities, we define (D3) of Definition 2 as global sensitivity and Definition 3 as local sensitivity at a point. In addition, surprisingly, we find that Definition 3 is identical to condition (3) of Proposition 2. Therefore, the following relationships are obviously true:

Proposition 3. *Assuming that (X, T) is a system, $x \in X$, then (X, T) is L-stable at $x \Rightarrow$ the system (X, T) is nonsensitive at $x \Rightarrow (X, T)$ is nonglobally sensitive $\Rightarrow$ the system (X, T) is non-Devaney chaos.*

Proposition 3 shows that L-stability at a point $x \in X$ is a sufficient condition that a system (X, T) is nonsensitive and further non-Devaney chaotic. From Definition 2, it is clear that sensitivity at x (i.e., £nonstability at x) can be explained intuitively when $x \in X$, as follows:

If there is a positive number ϵ, for an arbitrarily sufficient small neighborhood $B_{\varrho}(x, \delta)$ centered on x with radial δ, there will be at least one point $y \in B_{\varrho}(x, \delta)$ and one moment $n \in \mathbb{N}$ such that the distance $\varrho(T^n x, T^n y)$ between both $T^n x$ and $T^n y$ at moment n becomes equal or greater to the positive number ϵ. We can see that this is a precise mathematical description of an explosion phenomenon when ϵ is considered as a sufficiently large real number. In other words, L-stability at point x implies that any sufficiently small neighborhood at x is safe relative to sensitivity and Devaney chaos because it is nonsensitive at x, i.e., an explosion will never happen at x. Therefore, system (X, T) is nonglobally sensitive, as well as non-Devaney chaotic.

3. L-Stability at a Point

In this section, we first prove the composite theorems and the product theorems of L-stable points in dynamic systems. Then, using these theorems, we further show three equivalent characterizations of L-stable at a point of Cournot duopoly mapping.

To achieve this, we first prove the following equivalent characterizations:

Theorem 1. *If (X,T) is a dynamical system, $x \in X$, the following assertions are equivalent:*

(1) The system (X,T) is L-stable at x;

(2) For any $\epsilon > 0$, there is $\delta > 0$ such that for each $y \in B_\varrho(x,\delta)$ and for each $n \in \mathbb{N}$, $\varrho(T^n x, T^n y) < \epsilon$;

(3) For any $\epsilon > 0$, there is $\delta > 0$ such that $T^n(B_\varrho(x,\delta)) \subset B_\varrho(T^n x, \epsilon)$ for each $n \in \mathbb{N}$.

Proof. It is apparent that $(1) \Rightarrow (2) \Rightarrow (3)$. Therefore, we only need to prove $(3) \Rightarrow (1)$.

In fact, for any $\epsilon > 0$, there is $\delta > 0$ such that for any $n \in \mathbb{N}$, we have

$$T^n\left(B_\varrho(x,\delta)\right) \subset B_\varrho(T^n x, \varepsilon/2). \quad (1)$$

Then, for each $y \in B_\varrho(x,\delta)$ and for any $n \in \mathbb{N}$, the following inequation holds:

$$\varrho(T^n x, T^n y) < \varepsilon/2. \quad (2)$$

Eventually, we have that $\sup_{n\in\mathbb{N}} \varrho(T^n x, T^n y) \leq \epsilon/2 < \epsilon$.

According to Theorem 1, we naturally obtain the following composite theorem: □

Theorem 2. *Assume that (X,T) is a compact dynamic system, $x \in X$, then (X,T) is L-stable at x if and only if (X,T^2) is L-stable at x.*

Proof. $(\Rightarrow)$ For an arbitrary $\epsilon > 0$, since (X,T) is L-stable at x, there is $\delta > 0$ such that for any $y \in B_\varrho(x,\delta)$ and for each $n \in \mathbb{N}$, we have $\varrho(T^n x, T^n y) < \varepsilon/2$, thus $\sup_{n\in\mathbb{N}} \varrho(T^n x, T^n y) \leq \varepsilon/2 < \varepsilon$. In addition, because $\sup_{n\in\mathbb{N}} \varrho(T^{2n} x, T^{2n} y) \leq \sup_{n\in\mathbb{N}} \varrho(T^n x, T^n y)$, then the inequation

$$\sup_{n\in\mathbb{N}} \varrho\left(T^{2n} x, T^{2n} y\right) < \epsilon, \quad (3)$$

holds. Therefore, (X,T^2) is L-stable at x. $(\Leftarrow)$ For an arbitrary $\epsilon > 0$, since $T\colon X \longrightarrow X$ is uniformly continuous, there is $\eta > 0$, such that when $\varrho(x,y) < \eta$ for each pair $x, y \in X$, we have

$$\varrho(Tx, Ty) < \varepsilon/2. \quad (4)$$

Since point x is an L-stable point in (X,T^2), for $\eta > 0$, there is $\delta \in (0,\eta)$, such that for each $y \in B_\varrho(x,\delta)$ and each $n \in N$, the following inequations are satisfied:

$$\varrho\left(T^{2n} x, T^{2n} y\right) \leq \sup_{n\in\mathbb{N}} \varrho\left(T^{2n} x, T^{2n} y\right) < \eta < \varepsilon/2. \quad (5)$$

Then, for each $n \in \mathbb{N}$

$$\varrho\left(T\left(T^{2n}\right)x, T\left(T^{2n}\right)y\right) = \varrho\left(T^{2n+1} x, T^{2n+1} y\right) < \epsilon/2. \quad (6)$$

Thus, we have

$$\sup_{n\in\mathbb{N}} \varrho(T^n x, T^n y) \leq \varepsilon/2 < \varepsilon. \quad (7)$$

That is, (X,T) is L-stable at the point x.

The following corollary naturally follows from Theorem 2: □

Corollary 1. *Assume that (X,T) is a compact dynamic system, $x \in X$, then (X,T) is L-stable at x if and only if the system (X,T^n) is L-stable at x for any $n \in \mathbb{N}$.*

For a nature number $k \geq 1$, let $\{(X_i, T_i)\}_{i=1}^k$ be k compact dynamic systems, Cartesian set $X = \prod_{i=1}^k X_i$, and product mapping $T = \prod_{i=1}^k T_i$. Then, for $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k) \in X$, (X,T) is a product system with the metric $\varrho(x,y) = \max_{1\leq i\leq k} \varrho_i(x_i, y_i)$, where ϱ_i is a metric on X_i. Therefore, we have Theorem 3.

Theorem 3. *Product system (X,T) is L-stable at point $x = (x_1, \ldots, x_k) \in X$, if and only if for $i\,(1 \leq i \leq k)$, each (X_i, T_i) is L-stable at point x_i.*

Proof. $(\Rightarrow)$ Arbitrarily taking a natural number $i_0\,(1 \leq i_0 \leq k)$ and a real number $\epsilon > 0$, since (X,T) is L-stable at $x = (x_1, \ldots, x_k) \in X$, there is $\delta > 0$ such that for any $y = (y_1, \ldots, y_k) \in B_\varrho(x,\delta)$ and any natural number $n \in N$, the following inequation holds:

$$\varrho(T^n x, T^n y) < \epsilon. \quad (8)$$

For an arbitrary $y_{i_0} \in B_{\varrho_{i_0}}(x_{i_0}, y_{i_0})$, we take $y_i = x_i$ when $i \neq i_0$, and have $y = (y_1, \ldots, y_k)$; then,

$$\varrho_{i_0}\left(x_{i_0}, y_{i_0}\right) = \max_{1\leq i\leq k} \varrho_i(x_i, y_i) = \varrho(x,y) < \delta. \quad (9)$$

Therefore, we have

$$\varrho_{i_0}\left(T_{i_0}^n x_{i_0}, T_{i_0}^n y_{i_0}\right) \leq \max_{1\leq i\leq k} \varrho_i\left(T_i^n x_i, T_i^n y_i\right) = \varrho(T^n x, T^n y) < \epsilon. \quad (10)$$

That is, (X_{i_0}, T_{i_0}) is L-stable at the point $x_{i_0} \in X_{i_0}$. $(\Leftarrow)$ For any $\epsilon > 0$, since each (X_i, T_i) is L-stable at x_i, there is $\delta_i > 0$ such that for any $y_i \in B_{\varrho_i}(x_i, \delta_i)$ and for any $n \in N$, the following inequation holds:

$$\varrho_i\left(T_i^n x_i, T_i^n y_i\right) < \epsilon. \quad (11)$$

Let $\delta = \max_{1\leq i\leq k} \delta_i$; then for $y \in B_\varrho(x,\delta)$ and any $i\,(1 \leq i \leq k)$, we have the following equation:

$$\varrho_i(x_i, y_i) \leq \max_{1\leq i\leq k} \varrho_i(x_i, y_i) = \varrho(x,y) < \delta \leq \delta_i, \quad (12)$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$.

Subsequently, for any $n \in N$, we have

$$\varrho(T^n x, T^n y) = \max_{1\leq i\leq k} \varrho_i\left(T_i^n x_i, T_i^n y_i\right) < \epsilon. \quad (13)$$

Therefore, system (X,T) is L-stable at point x.

When $k = 2$ in this theorem, the following corollary is obviously true: □

Corollary 2. *Product system $(X_1 \times X_2, T_1 \times T_2)$ is L-stable at point $x = (x_1, x_2) \in X_1 \times X_2$, if and only if each (X_i, T_i) is L-stable at point x_i for $i \in \{1, 2\}$.*

Assuming that $f\colon Y \longrightarrow X$ and $g\colon X \longrightarrow Y$ are two continuous mappings, where X and Y are two compact metric spaces, the product mapping $\Phi\colon X \times Y \longrightarrow X \times Y$ is said to be a Cournot duopoly mapping, if for any pair $(x, y) \in X \times Y$, $\Phi(x, y) = (f(y), g(x))$. The system $(X \times Y, \Phi)$ is defined as the Cournot duopoly game system, and $\Phi\colon X \times Y \longrightarrow X \times Y$ is defined as Cournot duopoly mapping where the metric ϱ on $X \times Y$ satisfies the following: for $(x_1, y_1), (x_2, y_2) \in X \times Y$,

$$\varrho((x_1, y_1), (x_2, y_2)) = \max\{\varrho_X(x_1, x_2), \varrho_Y(y_1, y_2)\}. \tag{14}$$

ϱ_X and ϱ_Y are the metrics on X and Y, respectively. We have the subsequent theorem:

Theorem 4. *Let $\Phi\colon X \times Y \longrightarrow X \times Y$ to be the Cournot duopoly mapping, $(x^*, y^*) \in X \times Y$; then, the following items are equivalent:*

(1) Φ is L-stable at the point (x^*, y^*);

(2) Φ^2 is L-stable at the point (x^*, y^*);

(3) $f^\circ g$ and $g^\circ f$ are L-stable at both x^* and y^*, respectively.

Proof. Theorem 2 implies the equivalence of (1) and (2). Thus, we only need to prove the equivalence of (2) and (3):

In fact, for any pair $(x, y) \in X \times Y$, we have the following quation:

$$\begin{aligned}\Phi^2(x, y) &= \Phi(f(y), g(x)) \\ &= (f^\circ g(x), g^\circ f(y)) \\ &= ((f^\circ g) \times (g^\circ f))(x, y),\end{aligned} \tag{15}$$

That is, $\Phi^2 = (f^\circ g) \times (g^\circ f)$. Then, according to Corollary 2, the equivalence of (2) and (3) is shown to be true. □

4. L-Stability at Equilibrium in the Cournot Duopoly Game System

In this section, we discuss the L-stability at equilibrium in the Cournot duopoly game system. Assume that duopoly firms produce homogenous goods that are perfect substitutes and offer goods at discrete time periods. Therefore, duopoly firms encounter the same market demands. They both choose the adaptive expectation rule to decide the amount of goods in the next period as their response strategy. Assume that the demanded quantity is reciprocal to price p. This represents an "iso-elastic" demand function reflecting a case where consumers always spend a constant sum on the commodity, regardless of price. Inverting the demand function, we obtain

$$p = \frac{1}{x + y}, \tag{16}$$

where the total quantity in the denominator is the sum of the supplies, x and y, of two firms.

The revenues of the two firms equal price times quantity, that is, $px = x/(x + y)$ and $py = y/(x + y)$. We assume that the firms operate under constant unit costs, denoted as a and b. Their total costs are accordingly ax and by and their profits become

$$\begin{aligned}\pi_1(x, y) &= \frac{x}{x + y} - ax, \\ \pi_2(x, y) &= \frac{y}{x + y} - by,\end{aligned} \tag{17}$$

respectively. To maximize the profits of the two firms, we have to set the partial derivative of the previous with respect to x and y to zero, that is, let $\partial\pi_1/\partial x = 0$ and $\partial\pi_2/\partial y = 0$. Then, we obtain the two reaction functions as follows:

$$\begin{cases} x(y) = \sqrt{\dfrac{y}{a}} - y \\ & , (x, y) \in \left[0, \dfrac{1}{b}\right] \times \left[0, \dfrac{1}{a}\right]. \\ y(x) = \sqrt{\dfrac{x}{b}} - x \end{cases} \tag{18}$$

Let $\Phi(x, y) = (f(y), g(x))$, where $g(x) = \sqrt{x/b} - x$ for any $x \in [0, 1/b]$ and $f(y) = \sqrt{y/a} - y$ for any $y \in [0, 1/a]$. We have

$$\Phi\colon \left[0, \frac{1}{b}\right] \times \left[0, \frac{1}{a}\right] \longrightarrow \left[0, \frac{1}{b}\right] \times \left[0, \frac{1}{a}\right]. \tag{19}$$

It is a continuous self-mapping on the compact metric space $[0, 1/b] \times [0, 1/a]$. $\Phi^2 = (f^\circ g) \times (g^\circ f)$.

According to Theorem 4, we first state the following lemmas to prove that point $(b/(a + b)^2, a/(a + b)^2)$ is an L-stable point of Φ.

Lemma 1. *Let $g(x) = \sqrt{x/b} - x$, for any $x \in [0, 1/b]$ and $x^* = b/(a + b)^2$. Then,*

(1) $|g'(x)| \le 1$, if and only if $x \in [1/16b, 1/b]$, where $g'(x)$ is the derivative of $g(x)$ to x;

(2) $x^* \in (1/16b, 1/b)$ if and only if $3b > a > 0$.

Proof. The following two inferences are true:

(1) $|g'(x)| = |1/2\sqrt{bx} - 1| \le 1 \rightleftharpoons 1/\sqrt{bx} \le 4 \rightleftharpoons 1/16b \le x \rightleftharpoons 1/16b \le x \le 1/b$,and

(2) $1/16b < x^* = b/(a + b)^2 < 1/b \rightleftharpoons 1/16b^2 < 1/(a + b)^2 < 1/b^2 \rightleftharpoons 4b > a + b > b \rightleftharpoons 3b > a > 0$. □

Lemma 2. *Assume that $g(x) = \sqrt{x/b} - x$ for $x \in [0, 1/b]$, $f(y) = \sqrt{y/a} - y$ for $y \in [0, 1/a]$, and $3a > b$, then,* *(1)* $|f'(g(x))| \le 1$ *if and only if* $(1/2\sqrt{b} - 1/2\sqrt{1/b - 1/4a})^2 \le x \le (1/2\sqrt{b} + 1/2\sqrt{1/b - 1/4a})^2$, *and*

(2) $(1/2\sqrt{b} - 1/2\sqrt{1/b - 1/4a})^2 < x^* < (1/2\sqrt{b} + 1/2\sqrt{1/b - 1/4a})^2$, *where* $x^* = b/(a+b)^2$.

Proof. Using a similar method as Lemma 1 (1), it is obvious that $|f'(y)| = |1/2\sqrt{ay} - 1| \le 1$, if and only if $1/16a \le y \le 1/a$. Then, $|f'(g(x))| \le 1$, if and only if $1/16a \le g(x) = \sqrt{x/b} - x \le 1/a$, and a series of equivalence relations follows:

$$\begin{aligned}\frac{1}{16a} \le \sqrt{\frac{x}{b}} - x &\rightleftharpoons x - \frac{1}{\sqrt{b}}\sqrt{x} + \frac{1}{16a} \le 0 \rightleftharpoons \left(\sqrt{x} - \frac{1}{2\sqrt{b}}\right)^2 \le \frac{1}{4b} - \frac{1}{16a} \rightleftharpoons \frac{1}{2\sqrt{b}} - \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}} \\ &\le \sqrt{x} \le \frac{1}{2\sqrt{b}} + \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}} \rightleftharpoons \left(\frac{1}{2\sqrt{b}} - \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}}\right)^2 \le x \le \left(\frac{1}{2\sqrt{b}} + \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}}\right)^2.\end{aligned} \tag{20}$$

Therefore, (1) is true.

In addition, the following equivalence relations hold:

$$\begin{aligned}3a > b \rightleftharpoons 4a > (a+b) > a &\rightleftharpoons \frac{1}{16a} < \frac{a}{(a+b)^2} = y^* = \sqrt{\frac{x^*}{b}} - x^* \le \frac{1}{a} \rightleftharpoons x^* - \sqrt{\frac{x^*}{b}} + \frac{1}{16a} < 0 \\ \rightleftharpoons \left|\sqrt{x^*} - \frac{1}{2\sqrt{b}}\right|^2 < \frac{1}{4b} - \frac{1}{16a} &\rightleftharpoons \left(\frac{1}{2\sqrt{b}} - \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}}\right)^2 < x^* = \frac{b}{(a+b)^2} < \left(\frac{1}{2\sqrt{b}} + \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}}\right)^2.\end{aligned} \tag{21}$$

Therefore, (2) is also true.

Let $UCF = \{a, b \mid 3a > b, 3b > a\}$ which is the unit cost field of Φ, UCF is an unbounded triangle open area on the real plane as shown in Figure 1.

For each $(a, b) \in UCF$, $3a > b$ and $3b > a$. According to Lemma 1 (2) and Lemma 2 (2), we define $x^* = b/(a+b)^2$, the subsequent two inequalities hold:

(i) $1/16b < x^* \le 1/b$ and

(ii) $(1/2\sqrt{b} - 1/2\sqrt{1/b - 1/4a})^2 < x^* < (1/2\sqrt{b} + 1/2\sqrt{1/b - 1/4a})^2$.

Let

$$\lambda_1 = \min\left\{x^* - \frac{1}{16b}, \left(\frac{1}{2\sqrt{b}} + \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}}\right)^2 - x^*, x^* - \left(\frac{1}{2\sqrt{b}} - \frac{1}{2}\sqrt{\frac{1}{b} - \frac{1}{4a}}\right)^2\right\}. \tag{22}$$

It is easy to see that $\lambda_1 > 0$ from Lemma 1 (2) and Lemma 2 (2). The following conclusion is true when $(a, b) \in UCF$: □

Lemma 3. $|(f^\circ g'(x)| \le 1$ *always holds when* $|x - x^*| \le \lambda_1$ *for* $x \in [0, 1/b]$.

Proof. The derivative of the composite function $f^\circ g(x)$ with respect to x is as follows:

$$\left(f^\circ g'(x) = f'(g(x))g'(x). \tag{23}$$

Since $|x - x^*| \le \lambda_1$, the following two inequalities hold:

(a) $1/16b - x^* \le -\lambda_1 \le x - x^*$ and

(b) $(1/2\sqrt{b} - 1/2\sqrt{1/b - 1/4a})^2 - x^* \le -\lambda_1 \le x - x^* \le \lambda_1 \le (1/2\sqrt{b} + 1/2\sqrt{1/b - 1/4a})^2 - x^*$.

Then, we have.

(c) $1/16b \le x \le 1/b$ and

(d) $(1/2\sqrt{b} - 1/2\sqrt{1/b - 1/4a})^2 \le x \le (1/2\sqrt{b} + 1/2\sqrt{1/b - 1/4a})^2$

According to Lemma 1 (1) and Lemma 2 (1), the two following inequalities hold:

(e) $|f'(g(x))| \le 1$ and

(f) $|g'(x)| \le 1$.

Then,

$$\left|\left(f^\circ g'(x)\right)\right| = \left|f'(g(x))\right| \cdot \left|g'(x)\right| \le 1. \tag{24}$$

□

Lemma 4. *Point* $x^* = b/(a+b)^2$ *is an L-stable point in the system* $(X, f^\circ g)$ *, where* $(a, b) \in UCF$.

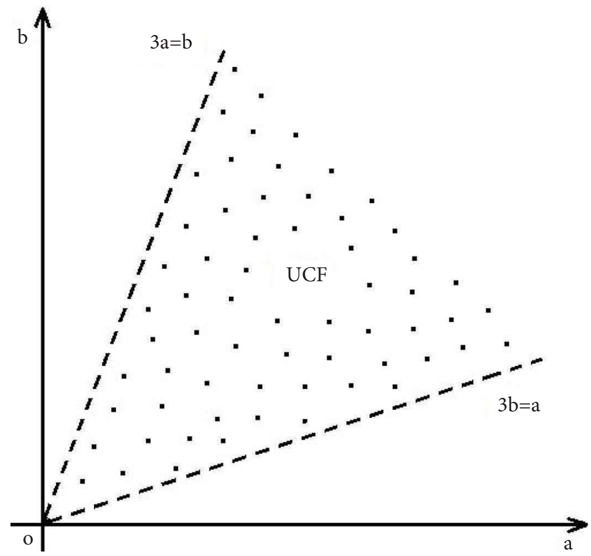

Figure 1: Unit cost field.

Proof. When $|x - x^*| \le \lambda_1$, for $x \in [0, 1/b]$, according to the differential mean value theorem, there is a real number ξ between both x and x^*, such that

$$\left(f^\circ g\right)(x) - (f^\circ|g)(x^*) = (f^\circ g^{'}(\xi)(x - x^*) = f^{'}\left(g|(\xi))g^{'}(\xi)(x - x^*\right), \tag{25}$$

and subsequently,

$$\left|(f^\circ g)(x) - (f^\circ g)(x^*)\right| = \left|f^{'}(g(\xi))g^{'}(\xi)(x - x^*\right| \le \left|x - x^*\right|. \tag{26}$$

Since $f^\circ g$ is a uniformly continuous function, for each natural number $n \ge 1$, the following recursive formula holds:

$$\left|(f^\circ g)^n(x) - (f^\circ g)^n(x^*)\right| \le \left|(f^\circ g)^{(n-1)}(x) - (f^\circ g)^{(n-1)}(x^*)\right| \le \ldots \le \left|(f^\circ g)(x) - (f^\circ g)(x^*)\right| \le \left|x - x^*\right|, \tag{27}$$

and $(f^\circ g)^n(x^*) = x^*$ (since x^* is a fixed point of $f^\circ g$). Therefore, for any $\epsilon \in (0, \lambda_1)$, $\delta = \epsilon > 0$, when for each $x \in [0, 1/b]$, $|x - x^*| < \delta$,

$$\left|(f^\circ g)^n(x) - x^*\right| = \left|(f^\circ g)^n(x) - (f^\circ g)^n(x^*)\right| \le \left|x - x^*\right| < \varepsilon (= \delta), \text{for} n = 0, 1, 2, \ldots, \tag{28}$$

holds. Therefore, system $(X, f^\circ g)$ is L-stable at point x^*.

In the previous Lemmas, if we swap the positions of x and y, a and b, as well as f and g, and let

$$\lambda_2 = \min\left\{ y^* - \frac{1}{16a}, \left(\frac{1}{2\sqrt{a}} + \frac{1}{2}\sqrt{\frac{1}{a} - \frac{1}{4b}}\right)^2 - y^*, y^* - \left(\frac{1}{2\sqrt{a}} - \frac{1}{2}\sqrt{\frac{1}{a} - \frac{1}{4b}}\right)^2 \right\}, \tag{29}$$

where $y^* = a/(a + b)^2$.

Then, the following corollaries hold: □

Corollary 3. *Let $f(y) = \sqrt{y/a} - y$, $y \in [0, 1/a]$, we have*

(1) $|f'(y)| \le 1$ *if and only if* $y \in [1/16a, 1/a]$ *where* $f'(y)$ *is the derivative of* $f(y)$ *to* y;

(2) $y^* = a/(a+b)^2 \in (1/16a, 1/a)$ *if and only if* $3a > b > 0$.

Corollary 4. *Assume that* $f(y) = \sqrt{y/a} - y$ *for any* $y \in [0, 1/a]$, $g(x) = \sqrt{x/b} - x$ *for any* $x \in [0, 1/b]$, *and* $3b > a$, *then we have*

(1) $|g'(f(y))| \le 1$ *if and only if* $(1/2\sqrt{a} - 1/2\sqrt{1/a - 1/4b})^2 \le y \le (1/2\sqrt{a} + 1/2\sqrt{1/a - 1/4b})^2$;

(2) $(1/2\sqrt{a} - 1/2\sqrt{1/a - 1/4b})^2 < y^* < (1/2\sqrt{a} + 1/2\sqrt{1/a - 1/4b})^2$, *where* $y^* = a/(a+b)^2$.

Corollary 5. $|(g^\circ f'(y)| \le 1$ *always holds when* $|y - y^*| \le \lambda_2$ *for* $y \in [0, 1/a]$.

Corollary 6. *The point* $y^* = a/(a+b)^2$ *is an L-stable point of the system* $(Y, g^\circ f)$ *where* $(a, b) \in UCF$.

Let the Cartesian set $\mathrm{LSD} = (x^* - \lambda_1, x^* + \lambda_1) \times (y^* - \lambda_2, y^* + \lambda_2)$, where $(x^*, y^*) = (b/(a+b)^2, a/(a+b)^2)$; we name LSD an L-stable domain of Φ. Then, we have

Theorem 5. Φ *(LSD)* $\subset$ *LSD, that is, LSD is an invariant set of the Cournot duopoly mapping* Φ.

Proof. Following the definitions of λ_1 and λ_2, for each $(x, y) \in \mathrm{LSD}$, we have

(1) $1/16b \le x^* - \lambda_1 < x \le 1/b$, and

(2) $1/16a \le y^* - \lambda_2 < y \le 1/a$.

By Lemma 1 (1) and Corollary 3 (1), the following inequations hold:

$$\begin{aligned} &\left|f'(y)\right| \le 1, \\ &\left|g'(x)\right| \le 1. \end{aligned} \tag{30}$$

For each $(x, y) \in \mathrm{LSD}$, since g and f are two differentiable functions, there is a real number ξ between both x and x^*, such that

$$\varrho_Y(g(x), g(x^*)) = \left|g(x) - g(x^*)\right| = \left|g'(\xi)\right| \cdot \left|x - x^*\right| \le \left|x - x^*\right| = \varrho_X(x, x^*), \tag{31}$$

and there is a real number η between both y and y^*, such that

$$\varrho_X(f(y), f(y^*)) = \left|f(y) - f(y^*)\right| = \left|f'(\eta)\right| \cdot \left|y - y^*\right| \le \left|y - y^*\right| = \varrho_Y(y, y^*). \tag{32}$$

Then, we have

$$\begin{aligned} \varrho(\Phi(x, y), (x^*, y^*)) &= \varrho(\Phi(x, y), \Phi(x^*, y^*)) = \max\{\varrho_X(f(y), f(y^*)), \varrho_Y(g(x), g(x^*)\} \\ &\le \max\{\varrho_X(x, x^*), \varrho_Y(y, y^*)\} = \varrho((x, y), (x^*, y^*)). \end{aligned} \tag{33}$$

Therefore, $\Phi(x, y) \in \mathrm{LSD}$. □

Theorem 6. *For any pair* $(a, b) \in UCF$, *the Cournot equilibrium point* $(b/(a+b)^2, a/(a+b)^2)$ *of the mapping* Φ *is an L-stable point of the system* $(X \times Y, \Phi)$, *where* $X = [0, 1/b]$ *and* $Y = [0, 1/a]$.

Proof. By Lemma 4 and Corollary 6, point $x^* = b/(a+b)^2$ and point $y^* = a/(a+b)^2$ are L-stable points of $(X, f^\circ g)$ and $(Y, g^\circ f)$, respectively. By Theorem 4 and the formula $\Phi^2 = (f^\circ g) \times (g^\circ f)$, the pair (x^*, y^*) is an L-stable point of the system $(X \times Y, \Phi)$. □

Corollary 7. *When* $(a, b) \in UCF$, *the Cournot duopoly game system* $(X \times Y, \Phi)$ *where* $X \times Y = [0, 1/b] \times [0, 1/a]$, *is safe (nonsensitive, non-Devaney chaos).*

Will the Cournot duopoly game system $(X \times Y, \Phi)$ still be safe when $(a, b) \notin$ UCF? We answer this question through a numerical simulation in the next section.

5. Numerical Simulation

In this section, by conducting numerical simulations, we intuitively show the correctness of Theorem 2 (or Corollary

5). During this process, we also suitably demonstrate that the final question posted in the previous section is not necessarily true.

We consider the iterative Cournot Duopoly reaction functions (18) as follows:

$$\begin{cases} x_n = \sqrt{\dfrac{y_{n-1}}{a}} - y_{n-1} \\ y_n = \sqrt{\dfrac{x_{n-1}}{b}} - x_{n-1} \end{cases}, n = 1, 2, \ldots. \quad (34)$$

5.1. The Situation When $(a, b) \in UCF$. Without loss of generality, we take $a = 0.15$ and $b = 0.16$, then, $(a, b) \in \mathrm{UCF}$ and the duopoly mapping Φ has the iterative form: its fixed point (equilibrium point) is as follows:

$$\begin{cases} x^* = \dfrac{b}{(a+b)^2} = 1.6649 \\ y^* = \dfrac{a}{(a+b)^2} = 1.5609 \end{cases}, \text{that is, } (x^*, y^*) = (1.6649, 1.5609). \quad (35)$$

By Lemma 1 (2) and Corollary 3 (2), we have $x^* \in (1/16b, 1/b)$ and $y^* \in (1/16a, 1/a)$ since $3b > a > 0$ and $3a > b > 0$. Subsequently, $\lambda_i > 0$, for $i = 1, 2$, there is an L-stable domain (an open rectangle), as follows:

$$\mathrm{LSD} = (x^* - \lambda_1, x^* + \lambda_1) \times (y^* - \lambda_2, y^* + \lambda_2) = (0.3677, 2.9527) \times (0.4168, 2.5128). \quad (36)$$

Now, we present the numerical simulation results of two situations when both $(x_0, y_0) \in \mathrm{LSD}$ and $(x_0, y_0) \notin \mathrm{LSD}$.

(1) When $(x_0, y_0) \in$ LSDreports five track examples of the duopoly iteration when the initial point $(x_0, y_0) \in \mathrm{LSD}$

For each $(x_0, y_0) \in \mathrm{LSD}$, the following relationship always hold:

$$\Phi^n(x_0, y_0) = (x_n, y_n) \in LSD, \text{for } n = 0, 1, 2, \ldots. \quad (37)$$

Thus, Theorem 5 is verified by numerical simulation.

From Table 1, we can also find that Theorem 6 is correct since (x_{n-1}, y_{n-1}) and (x^*, y^*) become increasingly closer.

(2) When $(x_0, y_0) \notin \mathrm{LSD}$

Since the initial point $(x_0, y_0) \notin \mathrm{LSD}$, we consider

$$\begin{aligned} (x_0, y_0) &\in \left[0, \frac{1}{b}\right] \times \left[0, \frac{1}{a}\right] - LSD \\ &= \left\{\left[0, \frac{1}{b}\right] - (x^* - \lambda_1, x^* + \lambda_1)\right\} \times \left\{\left(\left[0, \frac{1}{a}\right] - (y^* - \lambda_2, y^* + \lambda_2)\right)\right\} \\ &= \{[0, 0.3677] \cup [2.9527, 6.2500]\} \times \{([0, 0.4168] \cup [2.5128, 6.6667]\} \\ &= \{[0, 0.3677] \times [0, 0.4168]\} \cup \{[0, 0.3677] \times [2.5128, 6.6667]\} \\ &\cup \{[2.9527, 6.2500] \times [0, 0.4168]\} \cup \{[2.9527, 6.2500] \times [2.5128, 6.6667]\}. \end{aligned} \quad (38)$$

Presents five track examples of the duopoly iteration in these conditions.

In Table 2, Tracks (1), (3), and (4) cannot provide coverage to any point since they swing between odd and even terms. However, Tracks (2) and (5) provide coverage to a fixed point (the equilibrium point) (1.6649 and 1.5609). These results suggest that Theorem 5 and Theorem 6 are not necessarily true if the initial point $(x_0, y_0) \notin LSD$.

Table 1: Numerical simulation results if $(a, b) \in$ UCF and $(x_0, y_0) \in$ LSD.

Track	Track (1)	Track (2)	Track (3)	Track (4)	Track (5)
(x_0, y_0)	(2.9500, 0.4100)	(1.9500, 1.4100)	(1.4500, 1.9100)	(0.9500, 2.1100)	(0.3778, 0.5127)
(x_1, y_1)	(1.2533, 1.3573)	(1.6559, 1.5411)	(1.6584, 1.5604)	(1.6406, 1.48867)	(0.3778, 0.5127)
(x_2, y_2)	(1.1984, 1.5455)	(1.6642, 1.5611)	(1.6649, 1.5611)	(1.6615, 1.5615)	(3.0557, 1.1588)
(x_3, y_3)	(1.5099, 1.5384)	(1.6649, 1.5609)	(1.6649, 1.5609)	(1.6650, 1.5610)	(1.6207, 1.3144)
(x_4, y_4)	(1.6641, 1.5620)	.	.	(1.6649, 1.5609)	(1.6458, 1.5620)
(x_5, y_5)	(1.6650, 1.5609)	.	.	.	(1.6550, 1.5614)
(x_6, y_6)	(1.6649, 1.5609)	.	.	.	(1.6649, 1.5609)
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
(x_n, y_n)	(1.6649, 1.5609)	(1.6649, 1.5609)	(1.6649, 1.5609)	(1.6649, 1.5609)	(1.6649, 1.5609)
↓	↓	↓	↓	↓	↓
(x^*, y^*)	(1.6649, 1.5609)	(1.6649, 1.5609)	(1.6649, 1.5609)	(1.6649, 1.5609)	(1.6649, 1.5609)
		$\lim_{n\longrightarrow\infty}(x_n, y_n) = (x^*, y^*) = (1.6649, 1.5609)$			

Table 2: Numerical simulation results if $(a, b) \in$ UCF but $(x_0, y_0) \notin$ LSD.

Track	Track (1)	Track (2)	Track (3)	Track (4)	Track (5)
(x_0, y_0)	(0.0000, 0.4168)	(0.3677, 2.5128)	(2.9527, 0.0000)	(6.2500, 0.4160)	(2.9527, 6.666)
(x_1, y_1)	(1.2501, 0.0000)	(1.5801, 1.1483)	(0.0000, 1.3432)	(1.2501, 0.0000)	($3.3333e-05$, 1.3432)
(x_2, y_2)	(0.0000, 1.5451)	(1.6185, 1.5625)	(1.6492, 0.0000)	(0.0000, 1.5451)	(1.6492, 0.0144)
(x_3, y_3)	(1.6650, 0.0000)	(1.6650, 1.5620)	(0.0000, 1.5613)	(1.6650, 0.0000)	(0.2954, 1.5613)
(x_4, y_4)	(0.0000, 1.5609)	(1.6650, 1.5609)	(1.6649, 0.0000)	(0.0000, 1.5609)	(1.6649, 1.0634)
(x_5, y_5)	(1.6649, 0.0000)	(1.6649, 1.5609)	(0.0000, 1.5609)	(1.6649, 0.0000)	(1.5609, 1.6649)
(x_6, y_6)	(0.0000, 1.5609)	.	(1.6649, 0.0000)	(0.0000, 1.5609)	(1.6649, 1.5609)
(x_7, y_7)	(1.6649, 0.0000)	.	(0.0000, 1.5609)	(1.6649, 0.0000)	(1.6650, 1.5609)
(x_8, y_8)	(0.0000, 1.5609)	.	(1.6649, 0.0000)	(0.0000, 1.5609)	(1.5609, 1.6649)
(x_9, y_9)	(1.6649, 0.0000)	.	(0.0000, 1.5609)	(1.6649, 0.0000)	(1.6649, 1.5609)
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
(x_n, y_n)	.	(1.6649, 1.5609)	.	.	(1.6649, 1.5609)
.	.	↓	.	.	↓
.	final2-period	(1.6649, 1.5609)	final2-period	final2-period	(1.6649, 1.5609)

A track $\{T^n(x_0)\}_{n\geq 0}$ is called the final k-periodic if there is a natural number $m \geq 1$ such that $\{T^n(x_0)\}_{n\geq m}$ is a k-period track where $k = 2$ in Table 2.

Table 3: Numerical simulation results if (a, b) g UCF.

	Track (1)	Track (2)	Track (3)	Track (4)	Track (5)
(x_0, y_0)	$(x^* + 1,\ y^* - 1)$	$(x^* + 0.1,\ y^* - 0.1)$	$(x^* + 0.1^2,\ y^* - 0.1^2)$	$(x^* + 0.1^3,\ y^* - 0.1^3)$	$(x^* + 0.1^4,\ y^* - 0.1^4)$
(x_1, y_1)	(1.2138, 6.6068)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334,4.8573)
(x_2, y_2)	(0.0299, 5.6538)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)
(x_3, y_3)	(0.4856, 1.0471)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_4, y_4)	(1.5950, 3.8581)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
(x_5, y_5)	(1.2135, 6.2775)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334, 4.8573)
(x_6, y_6)	(0.1917, 5.6531)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)
(x_7, y_7)	(0.4859, 2.5374)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_8, y_8)	(1.5755, 3.8592)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
(x_9, y_9)	(1.2131, 6.2487)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334, 4.8573)
(x_{10}, y_{10})	(0.2056, 5.6525)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)
(x_{11}, y_{11})	(0.4862, 1.6208)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_{12}, y_{12})	(1.5591, 3.8602)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
(x_{13}, y_{13})	(1.2127, 6.2243)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334, 4.8573)
(x_{14}, y_{14})	(0.2174, 5.6518)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)
(x_{15}, y_{15})	(0.4865, 2.6889)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_{16}, y_{16})	(1.5450, 3.8613)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
(x_{17}, y_{17})	(1.2124, 6.2031)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334, 4.8573)
(x_{18}, y_{18})	(0.2276, 5.6512)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)

Table 3: Continued.

	Track (1)	Track (2)	Track (3)	Track (4)	Track (5)
(x_{19}, y_{19})	(0.4868, 2.7463)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_{20}, y_{20})	(1.5362, 3.8623)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
(x_{21}, y_{21})	(1.2129, 6.1843)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334, 4.8573)
(x_{22}, y_{22})	(0.2367, 5.6505)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)
(x_{23}, y_{23})	(0.4871, 2.7958)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_{24}, y_{24})	(1.5215, 3.8634)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
(x_{25}, y_{25})	(1.2116, 6.1674)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334, 4.8573)
(x_{26}, y_{26})	(0.2448, 5.6498)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)
(x_{27}, y_{27})	(0.4871, 2.8393)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_{28}, y_{28})	(1.5114, 3.8644)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
(x_{29}, y_{29})	(1.2133, 6.1520)	(0.8745, 5.0888)	(0.8375, 4.8811)	(0.8337, 4.8594)	(0.8334, 4.8573)
(x_{30}, y_{30})	(0.2522, 5.6492)	(0.7358, 4.9546)	(0.8234, 4.8670)	(0.8323, 4.8580)	(0.8332, 4.8571)
(x_{31}, y_{31})	(0.4877, 2.8781)	(0.7926, 4.6111)	(0.9292, 4.8328)	(0.8329, 4.8546)	(0.8333, 4.8568)
(x_{32}, y_{32})	(1.5022, 3.8654)	(0.9333, 4.7570)	(0.8433, 4.8470)	(0.8343, 4.8560)	(0.8334, 4.8569)
⋮	⋮	⋮	⋮	⋮	⋮
State	Disordered	4-period	4-period	4-period	4-period

5.2. The Situation When $(a,b) \notin$ UCF. Without loss of generality, we consider $a = 0.15$ and $b = (3 - 2\sqrt{2})a = 0.0259$, then $(a,b) \in$ UCF since $3a > b$. This implies $y^* \notin (1/16a, 1/a]$ by Corollary 1 (2). That is, $y^* \leq 1/16a$. Then, $\lambda_2 \leq y^* - 1/16a \leq 0$. That is, $\lambda_2 > 0$. This suggests that there is no L-stable domain (LSD) for duopoly mapping Φ. Therefore, we cannot discuss in two scenarios as in Section 5.1. However, we have the fixed point (the equilibrium point) of duopoly mapping Φ as follows:

$$\begin{cases} x^* = \dfrac{b}{(a+b)^2} = 0.8333, \\ \\ y^* = \dfrac{a}{(a+b)^2} = 4.8570. \end{cases} \tag{39}$$

To understand the dynamic behavior of Φ nears its equilibrium point (x^*, y^*), we choose a series of the initial points of the tracks of Φ as the follows:

$$(x_0, y_0) = \left(x_0^{(k)}, y_0^{(k)}\right) = \left(x^* + 0.1^k, y^* - 0.1^k\right), k = 1, 2, \ldots. \tag{40}$$

It is apparent that $(x_0, y_0) = (x_0^{(k)}, y_0^{(k)})$ is sufficiently close to the equilibrium point (x^*, y^*), when the natural number k is sufficiently large.

Table 3 shows the evolution of the duopoly game as k goes from 1 to 5.

The results of Table 3 suggest that duopoly game track (1) is disordered, and each track from Track (2) to Track (5) has 4 periods. In other words, L-stability (or nonsensitivity) is not guaranteed at the equilibrium point (x^*, y^*) of the Cournot duopoly system $([0, 1/b] \times [0, 1/a], \Phi)$ when $(a,b) \notin$ UCF.

6. Conclusion

This paper is the first to study the nature and characterization of L-stability at a point. First, we explain the relationship between L-stability at a point, nonsensitivity, and non-Devaney chaos. We find that a dynamical system at a point must be nonsensitive, as well as non-Devaney chaotic if the system is L-stable at this point. This paper is also the first to apply this method to the Cournot duopoly game. Employing the method of topological dynamic systems, three equivalent characterizations of general Cournot duopoly mapping (Theorem 4) are derived. Then, we discuss a specific Cournot duopoly game system $(X \times Y, \Phi)$ where $X = [0, 1/b]$ and $Y = [0, 1/a]$. By using calculus, we find a unit cost field UCF and an L-stable domain LSD of this specific Cournot duopoly mapping Φ, and by utilizing Theorem 4, we show that this specific Cournot duopoly game system is L-stable at its Cournot equilibrium point $(x^*, y^*) = (b/(a+b)^2, a/(a+b)^2)$ when the unit cost pair UCF (see Theorem 6). The robustness of Theorem 6 is further verified by conducting numerical simulations. In addition, this paper may elicit research of relative security in economics and other sciences since the security of systems is also an important aspect of these sciences.

Data Availability

The data generated during and/or analysed during the current study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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