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Research Article

Stability Analysis of Second-Order Linear PDEs on Time Scales

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This paper presents the stability analysis of a class of second-order linear partial differential equations (PDEs) on time scales with diffusion operator and first-order partial derivative. According to the time scale theory, the Lyapunov functional method, and some inequality techniques, sufficient conditions for exponential stability are strictly obtained, and the results are generalized for that where both the discrete-time and continuous-time cases are considered jointly. In addition, the theoretical results are applied to exponential synchronization of reaction-diffusion neural networks (RDNNs). Simulation examples are given to verify the feasibility of our results.

1. Introduction

Many real systems can be modeled as partial differential equations (PDEs), which generally express differential relationships among multiple objective things. The PDEs have shown practical applications in many fields including the electricity, light, and heat. Based on physics principles and laws, the PDEs can be roughly divided into three categories: hyperbolic, parabolic, and elliptic PDEs. Studies on various PDEs have gained wide attention among researchers in recent years [1–5].

Stability is a fundamental problem of equations or systems, and the stability of PDEs has been extensively studied in literature. For example, Taniguchi investigated the exponential stability of stochastic delayed PDEs in [6]. Practical exponential stability of nonlinear monotonic stochastic PDEs was discussed in [7]. Sufficient conditions for exponential stability of the second-order stochastic functional PDEs were obtained in [8]. Gahlawat and Valmorbida performed the Lyapunov analysis to determine the exponential stability of linear PDEs with one spatial dimension in [9]. With the aid of finite difference technique, Esmailzadeh et al. presented a numerical solution scheme to solve hyperbolic PDEs in [10]. In addition, as special parabolic PDEs, the stability of reaction-diffusion systems or reaction-diffusion neural networks (RDNNs) has also led to many achievements [11–13]. However, the above PDEs under consideration are all independent continuous-time and discrete-time cases. It is meaningful to introduce time scales to unify the continuous-time and discrete-time cases.

The time scales calculus theory was pioneered by Hilger [14], then improved and refined by Bohner and Peterson [15, 16]. Time scales theory unifies difference equation and differential equation so that they can be studied under a same framework. Recent related studies have shown that time scales are not only a theoretical territory of mathematics, but also an effective tool to handle many practical matters [17–20]. Although extensive research has been conducted on the stability analysis of dynamic equations on time scales, most of them were limited to ordinary differential equations, and few studies on the stability of PDEs on time scales were made [21–23]. To overcome this shortcoming, this study examines the stability of the second-order linear PDEs on time scales based on the Lyapunov functional method. In addition, there were many achievements on synchronization problems of RDNNs [24, 25]. We also attempt to deal with the exponential stability of RDNNs on time scales.

Inspired by the previous works, this paper adopts the Lyapunov functional method to study the global exponential stability of the second-order linear PDEs on time scales. The main contributions can be summarized as follows:

(1) Compared to the existing works [21–23], the PDEs under consideration is more general for that where both the diffusion operator and second-order partial differential terms are taking into account.

(2) Based on the Lyapunov functional and inequality methods, the sufficient conditions of stability are presented, and the results are generalized that they can unify the continuous-time and discrete-time cases. In addition, we extend the results of exponential stability to the application of RDNNs.

The rest of this paper is organized as follows. In Section 2, the relevant definitions and hypothetical lemmas are introduced. In Section 3, a model of second-order linear PDEs on time scales is defined, and its stability is analyzed. Section 4 gives an application on synchronization of RDNNs on time scales. The results are verified in two examples in Section 5. Conclusions are drawn in Section 6.

Notations: throughout the paper, $\mathbb{T} \in \mathbb{R}$ is a time scale and $\mathbb{T} \cap [0, +\infty) = [0, +\infty)_{\mathbb{T}}$; $\Omega = \{x = (x_1, x_2, \ldots, x_m)^T \mid |x_k| < l_k, k = 1, 2, \ldots, m\}$ is an open bounded domain in $\mathbb{R}^m$ with a smooth boundary $\partial\Omega$; $C_{rd}(\mathbb{T} \times \Omega, \mathbb{R}^n)$ is a set consisting of vector functions $u(t, x)$, which is rd-continuous with respect to $t \in \mathbb{T}$ and $x \in \Omega$; $C_t^{\mathbb{T}} = \{\zeta(t, \cdot): \zeta \in C(\Omega, \mathbb{R}^n)\}$is a Banach space with the norm of $\|\zeta(t, \cdot)\| = (\sum_{i=1}^n \|\zeta_i(t, \cdot)\|\|_2^2)^{(1/2)}$, where $\zeta(t, x) = (\zeta_1(t, x), \zeta_2(t, x), \ldots, \zeta_n(t, x))^T$ and $\|\zeta_i(t, \cdot)\|_2 = (\int_\Omega |\zeta_i(t, x)|^2 \mathrm{d}x)^{(1/2)}$; $\|\zeta_0\| = (\sum_{i=1}^n \|\zeta_i^0\|_2^2)^{(1/2)}$, where $\|\zeta_i^0\|_2 = (\int_\Omega \mid \zeta_i^0(x) \mid {}^2 \mathrm{d}x)^{(1/2)}$.

2. Preliminaries

Before giving the stability analysis, it is necessary to introduce some basic concepts and lemmas of time scales theory [15, 16].

Definition 1. [15] For $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho: \mathbb{T} \longrightarrow \mathbb{T}$ are, respectively, defined by $\sigma(t) = \inf\{\tau \in \mathbb{T}: \tau > t\}$ and $\rho(t) = \sup\{\tau \in \mathbb{T}: \tau < t\}$. If $\sigma(t) > t$, then t is called right-scattered, and t is called left-scattered if $\rho(t) < t$. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense;t is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. The graininess function $\mu: \mathbb{T} \longrightarrow [0, +\infty)$ is defined by $\mu(t) = \sigma(t) - t$. Set $\mathbb{T}^k$ is obtained from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \backslash m$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2. [15] Let $f: \mathbb{T} \longrightarrow \mathbb{R}$ and define $f^\sigma: \mathbb{T} \longrightarrow \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, namely, $f^\sigma = f^\circ\sigma$.

Definition 3. [15] Let $f: \mathbb{T} \longrightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$, then $f^\triangle(t)$ can be defined as a number with a property that for any $\varepsilon > 0$, and there exists $\delta > 0$, for all $s \in U = (t - \delta, t + \delta)_{\mathbb{T}}$, such that:

$$\left|(f(\sigma(t)) - f(s)) - f^\triangle(t)(\sigma(t) - s)\right| \leq \varepsilon|\sigma(t) - s|, \quad (1)$$

then, $f^\triangle(t)$ is called a delta derivative of f on $\mathbb{T}^k$.$f: \mathbb{T} \longrightarrow \mathbb{R}$ is called rd-continuous if it is continuous at right dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. A set of all rd-continuous functions f is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.$f$ is called regulated if it is rd-continuous and its right-side limits exist at all right-side points in $\mathbb{T}$ and its left-side limits exist at all left-side points in $\mathbb{T}$.

Definition 4. [15] If $F^\triangle(t) = f(t), t \in \mathbb{T}^k$, then for any $a, b \in \mathbb{T}$, the integral is defined as follows:

$$\int_a^b f(t)\Delta t = F(b) - F(b). \quad (2)$$

Definition 5. [16] If $p \in \mathbb{R}(\mathbb{T}, \mathbb{R})$, then the exponential function can be defined by $e_p(t, s) = \exp(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta\tau)$ for all $s, t \in \mathbb{T}$. The cylinder transformation $\xi_h(z)$ is given by the following expression:

$$\xi_h(z) = \begin{cases} \dfrac{1}{h}\mathrm{Log}(1 + zh), & h > 0, \\ z, & h = 0. \end{cases} \quad (3)$$

Lemma 1. [16] *Assume that Ω is a cube $|x_i| < l_i$ $(i = 1, 2, \ldots, n)$ and $h(x) \in C(\Omega)$ is a real-valued function, which vanish on the boundary $\partial\Omega$, then it holds:*

$$\int_\Omega h^2(x)\mathrm{d}x \leq l_i^2 \int_\Omega \left(\frac{\partial h}{\partial x_i}\right)^2 \mathrm{d}x. \quad (4)$$

Lemma 2. [16] *If $f, g: \mathbb{T} \longrightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$, then it holds:*

$$\begin{aligned} (fg)^\triangle(t) &= f^\triangle(t)g(t) + f^\sigma(t)g^\triangle(t) \\ &= f(t)g^\triangle(t) + f^\triangle(t)g^\sigma(t). \end{aligned} \quad (5)$$

Lemma 3. [16] *If $f: \mathbb{T} \longrightarrow \mathbb{R}$ is differentiable at $t \in \mathbb{T}^k$, then it holds:*

$$f^\sigma(t) = f(t) + \mu(t)f^\triangle(t). \quad (6)$$

Lemma 4. [16] *Let $p, q: \mathbb{T} \longrightarrow \mathbb{R}$ be two regressive functions and denote $p \oplus q = p + q + \mu pq, \ominus p = -(p/1 + \mu p), p \ominus q = p \oplus p(\ominus q)$; then the following expressions hold:*

(i) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$

(ii) $(1/e_p(t, s)) = e_{\ominus p}(t, s)$

(iii) $e_p(t, s) = (1/e_p(t, s)) = e_{\ominus p}(s, t)$

(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$

(v) $[e_p(t, s)]^\triangle = p(t)e_p(t, s)$

(vi) $(d[e_p(t, s)]/\mathrm{d}z) = (\int_s^t (1/1 + \mu(\tau)z)\Delta\tau)e_z(t, s)$

Definition 6. [16] The equilibrium solution $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of PDEs (10) is called global exponential stability if there exist positive constants $\alpha > 0 and M \geq 1$ such that:

$$\left\| u(t,\cdot) - u^* \right\| \leq M e_{\ominus\alpha}(t,0), t \in \mathbb{T}. \tag{7}$$

Definition 7. [16] For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, for $V \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, define $D^+V^{\triangle}(t, x(t))$ as a mean such that, for $\varepsilon > 0$, and there exists a right neighborhood $N_\varepsilon \cap N$ of t such that:

$$\begin{aligned} &\frac{1}{\mu(t,s)} [V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) \\ &- \mu(s,t) f(t, x(t))] < D^+V^{\triangle}(t, x(t)) + \varepsilon. \end{aligned} \tag{8}$$

For each $s \in N_\varepsilon, s > t$, where $\mu(t,s) = \sigma(t) - s$, if t is right-scattered and $V(t, x(t))$ is continuous at t, it holds:

$$D^+V^{\triangle}(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(\sigma(t)))}{\sigma(t) - t}. \tag{9}$$

3. Main Results

In this work, discrete and continuous domains are considered directly and jointly using the equations on time scales, and the second-order linear PDEs on the time scales under investigation is given as follows:

$$\begin{cases} u_i^{\triangle}(t,x) = \sum_{j=1}^{m} a_{ij} \frac{\partial^2 u_i(t,x)}{\partial x_j^2} + \sum_{j=1}^{m} b_j \frac{\partial u_i(t,x)}{\partial x_j} + c u_i(t,x) + f_i(u_i(t,x)), \\ \frac{\partial u_i(t,x)}{\partial x} = 0, \ (t,x) \in [0, +\infty_{\mathbb{T}} \times \partial\Omega, \\ u_i(0,x) = u_i^0(x), x \in \Omega, \end{cases} \tag{10}$$

where $i = 1, 2, \ldots, n$; $t \in \mathbb{T}$ is the time variable; $x \in \Omega$ is the spacial variable; $u_i(t,x)$ is the state variable; a_{ij}, b_j, and c are constant coefficients; $f_i(\cdot)$ is the nonlinear function; $u_i^0(x)$ is the initial value function.

Assumption 1. [16] Assume that $f_i(\cdot)$ is Lipschitz continuous, and there exists a constant $F_i > 0$ such that:

$$\left| f_i(\xi) - f_i(\eta) \right| \leq F_i \left| \xi - \eta \right|. \tag{11}$$

Theorem 1. *Under the Assumption 1, the PDEs on time scales (10) is global exponential stability if the following condition is satisfied as follows:*

$$-2 \sum_{j=1}^{m} \frac{a_{\mathrm{ij}}}{l_j^2} + \sum_{j=1}^{m} b_j + 2c + 2F_i \leq 0. \tag{12}$$

Proof. Let $\zeta_i(x,t) \triangleq u_i(x,t) - u_i^*$, and the error dynamics equation is obtained as follows:

$$\begin{cases} \zeta_i^{\triangle}(t,x) = \sum_{j=1}^{m} a_{ij} \frac{\partial^2 \zeta_i(t,x)}{\partial x_j^2} + \sum_{j=1}^{m} b_j \frac{\partial \zeta_i(t,x)}{\partial x_j} + c\zeta_i(t,x) + f_i(u_i(t,x)) - f_i(u_i^*), \\ \frac{\partial \zeta_i(t,x)}{\partial x} = 0, \ (t,x) \in [0, +\infty)_{\mathbb{T}} \times \partial\Omega, \\ \zeta_i(x,0) = \zeta_i^0(x).\mathrm{ed}, \end{cases} \tag{13}$$

where $\zeta_i^0(x) = u_i^0(x) - u_i^*$. Calculating the delta derivation of $\|\zeta_i(t,\cdot)\|\|_2^2$ yields to

$$\begin{aligned}\left(\|\zeta_i(t,\cdot)\|_2^2\right)^{\triangle} &= \int_\Omega \left(\zeta_i(t,x)^2\right)^{\triangle} \mathrm{d}x \\ &= 2\int_\Omega \zeta_i(t,x)\zeta_i^{\triangle}(t,x)\mathrm{d}x + \mu(t)\int_\Omega \left(\zeta_i^{\triangle}(t,x)\right)^2 \mathrm{d}x \\ &= 2\int_\Omega \zeta_i(t,x)\sum_{j=1}^m a_{\mathrm{ij}}\frac{\partial^2\zeta_i(t,x)}{\partial x_j^2}\mathrm{d}x + 2\int_\Omega \zeta_i(t,x)\sum_{j=1}^m b_j\frac{\partial\zeta_i(t,x)}{\partial x_j}\mathrm{d}x \\ &\quad + 2c\int_\Omega (\zeta_i(t,x))^2\mathrm{d}x + 2\int_\Omega \zeta_i(t,x)\left(f_i(u_i(t,x)) - f_i(u_i^*)\right)\mathrm{d}x \\ &\quad + \mu(t)\left(\|\zeta_i(t,\cdot)\|\|_2^2\right)^{\triangle}.\end{aligned} \tag{14}$$

In view of integration by parts and Lemma 1, one has

$$\begin{aligned}\int_\Omega \zeta_i(t,x)\sum_{j=1}^m a_{\mathrm{ij}}\frac{\partial^2\zeta_i(t,x)}{\partial x_j^2}\mathrm{d}x &= -\int_\Omega \sum_{j=1}^m a_{\mathrm{ij}}\left(\frac{\partial\zeta_i(t,x)}{\partial x_j}\right)^2\mathrm{d}x \\ &\le -\sum_{j=1}^m \frac{a_{ij}}{l_j^2}\int_\Omega (\zeta_i(t,x))^2\mathrm{d}x \\ &= -\sum_{j=1}^m \frac{a_{ij}}{l_j^2}\|\zeta_i(t,\cdot)\|_2^2,\end{aligned} \tag{15}$$

$$\begin{aligned}&\int_\Omega \sum_{j=1}^m \zeta_i(t,x) b_j \frac{\partial\zeta_i(t,x)}{\partial x_j}\mathrm{d}x \\ &= \frac{1}{2}\sum_{j=1}^m b_j\int_\Omega (\zeta_i(t,x))^2\mathrm{d}x \\ &= \frac{1}{2}\sum_{j=1}^m b_j\|\zeta_i(t,\cdot)\|_2^2.\end{aligned} \tag{16}$$

Considering Assumption 1, we can get

$$\begin{aligned}&\int_\Omega \zeta_i(t,x)\left(f_i(u_i(t,x)) - f_i(u_i^*)\right)\mathrm{d}x \\ &\le F_i\int_\Omega (\zeta_i(t,x))^2\mathrm{d}x = F_i\|\zeta_i(t,\cdot)\|_2^2.\end{aligned} \tag{17}$$

Based on the above inequalities (15–17), it can be obtained

$$\begin{aligned}\left(\|\zeta_i(t,x)\|_2^2\right)^{\triangle} &\le -2\sum_{j=1}^m \frac{a_{ij}}{l_j^2}\left\|\zeta_i(t,\cdot)\|\right\|_2^2 + \sum_{j=1}^m b_j\left\|\zeta_i(t,\cdot)\|\right\|_2^2 \\ &\quad + 2c\|\zeta_i(t,\cdot)\|_2^2 + 2F_i\|\zeta_i(t,\cdot)\|_2^2 + \mu(t)\left(\|\zeta_i(t,\cdot)\|_2^2\right)^{\triangle} \\ &\le \left(-2\sum_{j=1}^m \frac{a_{ij}}{l_j^2} + \sum_{j=1}^m b_j + 2c + 2F_i\right)\|\zeta_i(t,\cdot)\|\|_2^2 + \mu(t)q(t)\|\zeta_i(t,\cdot)\|\|_2^2,\end{aligned} \tag{18}$$

where $(\|\zeta_i(t,\cdot)\|_2^2)^{\triangle} = q(t)\|\zeta_i(t,\cdot)\|_2^2$, $q(t) \ge 0$, $i = 1, 2, \ldots, n$.

If formula (18) holds, there exists an appropriate positive number σ, perhaps small, such that for any $i = 1, 2, \ldots, n$:

$$-2\sum_{j=1}^m \frac{a_{\mathrm{ij}}}{l_j^2} + \sum_{j=1}^m b_j + 2c + 2F_i + \sigma \le 0. \tag{19}$$

Then, select a function defined as follows:

$$q_i(z_i) = z_i \oplus z_i - 2\sum_{j=1}^{m} \frac{a_{ij}}{l_j^2} + \sum_{j=1}^{m} b_j + 2c + 2F_i + \frac{w(z_i)\mu(t)q(t)\max\left\{e_{z_i\oplus z_i}(\sigma(t),0), e_{(w(z_i)-1)\mu(t)q(t)\|\zeta_i(t,x)\|_2^2}(t,0)\right\}}{e_{z_i\oplus z_i}(\sigma(t),0)}, \tag{20}$$

where $w(z_i) = \int_0^{z_i}(e^{z_i-s}/(z_i-s)^2\mathrm{d}s$. By equation (20), it can be obtained that $q_i(0) < -\sigma < 0$ and $q_i(z_i)$ is continuous. Next, ensure that $q<0$ at some point. For any $z_i \in [0, +\infty)$, when $z_i \longrightarrow +\infty$, $q_i(z_i) \longrightarrow +\infty$, there exists a constant $\varepsilon_i \in (0, +\infty)$ such that $q_i(\varepsilon_i^*) = 0$ and $q_i(\varepsilon) < 0$ for $\varepsilon_i \in (0, \varepsilon_i^*) \cap (0,1)$. Setting $\varepsilon = \min_{1\le i\le n}\{\varepsilon_i\}$, $0<\varepsilon<1$, one can get

$$q_i(\varepsilon) = \varepsilon\oplus\varepsilon - \sum_{j=1}^{m} \frac{a_{ij}}{l_j^2} + \sum_{j=1}^{m} b_j + 2c + 2F_i + \frac{w(\varepsilon)\mu(t)q(t)\max\left\{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0), e_{(w(\varepsilon)-1)\mu(t)q(t)\|\zeta_i(t,\cdot)\|_2^2}(t,0)\right\}}{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)} \le 0. \tag{21}$$

Taking the Lyapunov function as follows:

$$V(t,\zeta(t)) = \sum_{i=1}^{n}\left\{e_{\varepsilon\oplus\varepsilon}(t,0)\|\zeta_i(t,\cdot)\|_2^2 + e_{(w(\varepsilon)-1)\mu(t)q(t)\|\zeta_i(t,\cdot)\|_2^2}(t,0)\right\}. \tag{22}$$

It should be noted that $(d/ds)[e_\zeta(t,x)] = (\int_s^t(1/1+\mu(\tau)\zeta)\mathrm{d}\tau)e_\zeta(t,x) > 0$. Then, the delta derivatives of $V(t,\zeta(t))$ can be expressed as follows:

$$\begin{aligned}
&D^+V^{\triangle}(t,\zeta(t)) \\
&= \sum_{i=1}^{n}\Big\{(\varepsilon\oplus\varepsilon)e_{\varepsilon\oplus\varepsilon}(t,0)\|\zeta_i(t,\cdot)\|_2^2 + e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\left(\|\zeta_i(t,\cdot)\|_2^2\right)^{\triangle} \\
&\quad + (w(\varepsilon)-1)\mu(t)q(t)\|\zeta_i(t,\cdot)\|_2^2 e_{(w(\varepsilon)-1)\mu(t)q(t)\|\zeta_i(t,\cdot)\|_2^2}(t,0)\Big\} \\
&\le \sum_{i=1}^{n}\Big\{(\varepsilon\oplus\varepsilon)e_{\varepsilon\oplus\varepsilon}(t,0)\|\zeta_i(t,\cdot)\|_2^2 + e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\Big(-2\sum_{j=1}^{m}\frac{a_{ij}}{l_j^2}\|\zeta_i(t,\cdot)\|_2^2 \\
&\quad + \sum_{j=1}^{m} b_j\|\zeta_i(t,\cdot)\|_2^2 + 2c\|\zeta_i(t,\cdot)\|_2^2 + 2c + 2F_i\|\zeta_i(t,\cdot)\|_2^2 + 2c + \mu(t)\left(\|\zeta_i(t,\cdot)\|_2^2\right)^{\triangle}\Big) \\
&\quad + (w(\varepsilon)-1)\mu(t)q(t)\|\zeta_i(t,\cdot)\|_2^2 e_{(w(\varepsilon)-1)\mu(t)q(t)\|\zeta_i(t,\cdot)\|_2^2}(t,0)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\sum_{i=1}^{n}\left\{(\varepsilon\oplus\varepsilon)\left\|\zeta_i(t,0)\right\|_2^2+\left(-2\sum_{j=1}^{m}\left(a_{ij}/l_j\right)^2+\sum_{j=1}^{m}b_j+2c+2F_i\right)\left\|\zeta_i(t,\cdot)\right\|_2^2\right\}\\
&\quad+\frac{w(\varepsilon)\mu(t)q(t)\left\|\zeta_i(t,\cdot)\right\|_2^2\max\left\{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0),e_{(w(\varepsilon)-1)\mu(t)q(t)\left\|\zeta_i(t,x)\right\|_2^2}(t,0)\right\}}{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)}\\
&\leq e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)\sum_{i=1}^{n}\left\|\zeta_i(t,0)\right\|_2^2\left\{\varepsilon\oplus\varepsilon+\left(-2\sum_{j=1}^{m}\left(a_{ij}/l_j\right)^2+\sum_{j=1}^{m}b_j+2c+2F_i\right)+\mu(t)q(t)\right.\\
&\quad\left.+\frac{w(\varepsilon)\mu(t)q(t)\max\left\{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0),e_{(w(\varepsilon)-1)\mu(t)q(t)\left\|\zeta_i(t,x)\right\|_2^2}(t,0)\right\}}{e_{\varepsilon\oplus\varepsilon}(\sigma(t),0)}\right\}\\
&\leq 0.
\end{aligned}
\tag{23}
$$

Simplifying the above formula gives as follows:

$$
\begin{aligned}
e_{\varepsilon\oplus\varepsilon}(t,0)\left\|\zeta_i(0,\cdot)\right\|_2^2 &= e_{\varepsilon\oplus\varepsilon}(t,0)\sum_{i=1}^{n}\left\|\zeta_i(0,\cdot)\right\|_2^2\leq V(t,\zeta(t))\leq V(0,\zeta(0))\\
&=\sum_{i=1}^{n}\left\{\left\|\zeta_i(0,\cdot)\right\|_2^2+1\right\}\\
&=\sum_{i=1}^{n}\left\{\left\|\zeta_i^0\right\|_2^2+1\right\}\\
&=\left\|\zeta^0\right\|^2+n.
\end{aligned}
\tag{24}
$$

In view of Lemma 4, it can be obtained that

$$
\|\zeta(t,\cdot)\|_2^2\leq\frac{1}{e_{\varepsilon\oplus\varepsilon}(t,0)}\left(\left\|\zeta^0\right\|^2+n\right),\tag{25}
$$

which implies

$$
\|\zeta(t,\cdot)\|_2\leq Me_{\ominus\varepsilon}(t,0),\tag{26}
$$

where $M=\sqrt{\|\zeta^0(x)\|^2+n}$. According to Definition 6, the PDEs on time scales (10) is global exponential stability under the condition (12). □

Remark 1. At present, few research studies have been conducted to distinguish the continuous-time and discrete-time cases of PDEs. However, the time domain of PDEs is not always strictly continuous-time or discrete-time in practice. In this paper, the time scale theory is utilized to unify the continuous-time or discrete-time cases.

Remark 2. If $\mathbb{T}=\mathbb{R}$, we have $\mu(t)=0$, and equation (10) can be expressed as a continuous-time PDEs:

$$
\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t} &= \sum_{j=1}^{m}a_{\mathrm{ij}}\frac{\partial^2 u_i(t,x)}{\partial x_j^2}\\
&\quad+\sum_{j=1}^{m}b_j\frac{\partial u_i(t,x)}{\partial x_j}+\mathrm{cu}_i(t,x)+f_i\left(u_i(t,x)\right).
\end{aligned}
\tag{27}
$$

If $\mathbb{T}=\mathbb{N}$, then $\mu(t)=1$, and equation (10) can be given as a discrete-time PDEs:

$$
\begin{aligned}
u_i(t+1,x) &= \sum_{j=1}^{m}a_{\mathrm{ij}}\frac{\partial^2 u_i(t,x)}{\partial x_j^2}+\sum_{j=1}^{m}b_j\frac{\partial u_i(t,x)}{\partial x_j}\\
&\quad+\mathrm{cu}_i(t,x)+f_i\left(u_i(t,x)\right).
\end{aligned}
\tag{28}
$$

Remark 3. By choosing appropriate equation parameter, the equation considered in this paper can be the hyperbolic, the elliptic and the parabolic equations on time scales, the stability conditions can be easily obtained. For example, if

$a_{ij} = 1, b_{ij} = 0, c = 0$, the equation is a semilinear parabolic equation on time scales:

$$u_i^{\triangle}(t,x) = \sum_{j=1}^{m} a_{\mathrm{ij}} \frac{\partial^2 u_i(t,x)}{\partial x_j^2} + f_i(u_i(t,x)), \tag{29}$$

then sufficient condition for exponential stability of the above equation is given as follows:

$$-\sum_{j=1}^{m} \frac{1}{l_j^2} + 2F_i \leq 0. \tag{30}$$

4. Application

The RDNNs on time scales is a special case of the second-order linear PDEs on time scales. In this section, the exponential stability problem of driven-response RDNNs on time scales is considered. Consider the following RDNNs model as follows:

$$\begin{aligned} w_i^{\triangle}(t,x) = {} & \sum_{k=1}^{m} d_{\mathrm{ik}} \frac{\partial^2 w_i(t,x)}{\partial x_k^2} - a_i w_i(t,x) \\ & + \sum_{j=1}^{n} b_{\mathrm{ij}} f_j\big(w_j(t,x)\big) + I_i, \end{aligned} \tag{31}$$

where $i = 1,2,\ldots,n$, and n is the number of neurons. $x = (x_1, x_2, \ldots, x_m)$ is the space variable. $d_{\mathrm{ik}} \geq 0$ is the diffusion parameter. a_i is the self-feedback connection weight. b_{ij} is the connection weight coefficient.

Take the system (31) as the drive system, then the corresponding response system can be given as follows:

$$\begin{aligned} v_i^{\triangle}(t,x) = {} & \sum_{k=1}^{m} d_{\mathrm{ik}} \frac{\partial^2 v_i(t,x)}{\partial x_k^2} - a_i v_i(t,x) \\ & + \sum_{j=1}^{n} b_{\mathrm{ij}} f_j\big(v_j(t,x)\big) + I_i. \end{aligned} \tag{32}$$

Define the error $e_i(t,x) = w_i(t,x) - v_i(t,x)$, and the error equation is obtained as follows:

$$\begin{aligned} e_i^{\triangle}(t,x) = {} & \sum_{k=1}^{m} d_{\mathrm{ik}} \frac{\partial^2 e_i(t,x)}{\partial x_k^2} - a_i e_i(t,x) \\ & + \sum_{j=1}^{n} b_{\mathrm{ij}} f_j\big(e_j(t,x)\big). \end{aligned} \tag{33}$$

We can easily obtain the following proposition.

Proposition 1. *The RDNNs on time scales (31) is exponential stability if the following sufficient condition is satisfied:*

$$-2\sum_{k=1}^{m} \frac{a_{ij}}{l_j^2} - 2a_i + 2\sum_{j=1}^{n} b_{ij} F_i \leq 0. \tag{34}$$

5. Illustrative Examples

In this section, two examples are given to verify the validity of our results.

Example 1. Consider a PDE on time scales defined as follows:

$$\begin{aligned} u_i^{\triangle}(t,x) = {} & \sum_{j=1}^{2} a_{\mathrm{ij}} \frac{\partial^2 u_i(t,x)}{\partial x_j^2} + \sum_{j=1}^{2} b_j \frac{\partial u_i(t,x)}{\partial x_j} \\ & + \mathrm{cu}_i(t,x) + f_i(u_i(t,x)), \end{aligned} \tag{35}$$

where $f_i(u_i) = \tanh(u_i) = (e_i^u - e^{-u_i}/e_i^u + e^{-u_i})$, and $f(u_i)$ satisfies Lipschitz condition, and $F_i = 1, i = 1,2$. Let $c = 0.5, L = 2, l_1 = 1, l_2 = 1$, and we consider the system with the following parameters:

$$\begin{aligned} A &= \begin{bmatrix} 0.1 & -0.3 \\ 0.3 & 0.1 \end{bmatrix}, \\ B &= \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \end{aligned} \tag{36}$$

and the initial conditions:

$$u_i^0 = \begin{bmatrix} \sin 2x + 2\sin(3t)^2 + 1.5 \\ 3\cos 3x + 3\sin 2t + 1.1 \end{bmatrix}. \tag{37}$$

Let $\mathbb{T} = \cup_{k=0}^{\infty} [0.1k, 0.1k + 0.07], k\epsilon Z$; we can have

$$\mu(t) = \begin{cases} 0, & t \in [0.1k, 0.1k{+}0.07), \\ 0.03, & t = 0.1k + 0.07.\mathrm{ed}. \end{cases} \tag{38}$$

By calculating the inequality (12), we can have

$$\begin{aligned} -2\sum_{j=1}^{2} \frac{a_{1j}}{l_j^2} + \sum_{j=1}^{2} b_j + 2c + 2F_1 &= 6.4, \\ -2\sum_{j=1}^{2} \frac{a_{2j}}{l_j^2} + \sum_{j=1}^{2} b_j + 2c + 2F_2 &= 5.2. \end{aligned} \tag{39}$$

In this case, system (35) does not satisfy sufficient condition (2), so the system is not exponentially stable under this numerical condition (See Figure 1).

Example 2. Consider a PDE with the following parameters, and the other parameters are the same as in Example 1:

$$\begin{aligned} A &= \begin{bmatrix} 0.5 & 1.5 \\ 1.3 & 0.7 \end{bmatrix}, \\ B &= \begin{bmatrix} 1.3 \\ 0.4 \end{bmatrix}, \\ c &= 0.1, \end{aligned} \tag{40}$$

and the initial conditions:

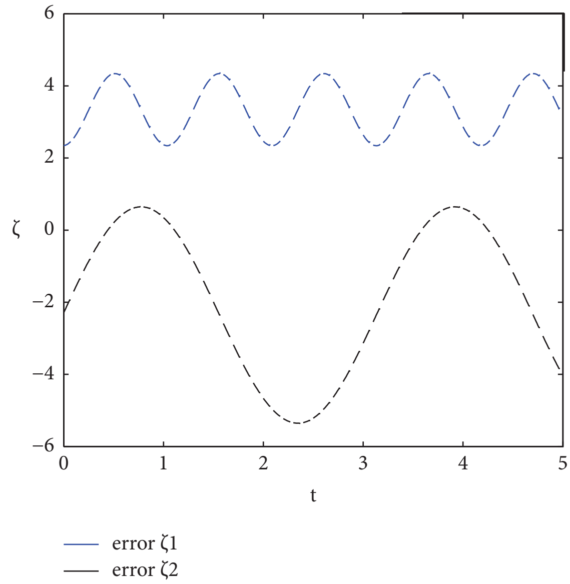

Figure 1: Error ζ of the unstable PDE system (35).

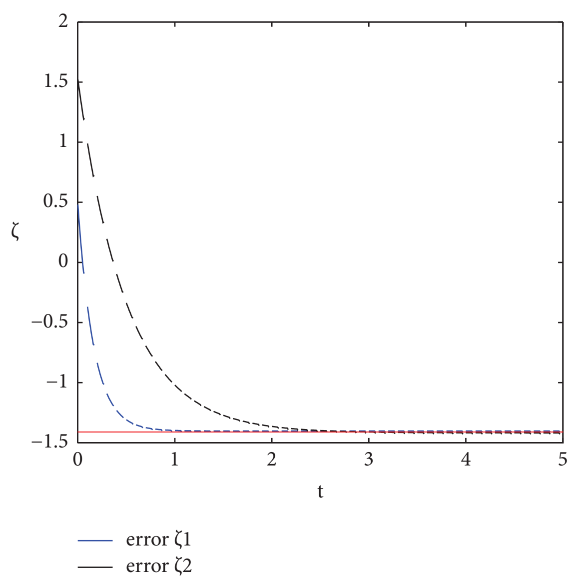

Figure 2: Error ζ of the stable PDE system (35).

$$u_i^0 = \begin{bmatrix} -2e^{\left(-\dfrac{0.2x}{L}\right)} + 2e^{(-3t)^2} + 0.5 \\ \\ -1.2e^{\left(-\dfrac{0.3x}{L}\right)} + 3e^{(-2t)} + 0.1 \end{bmatrix}. \quad (41)$$

Let $\mathbb{T} = \cup_{k=0}^{\infty}[0.1k, 0.1k + 0.07], k\epsilon Z$. By calculating the inequality (12), we have

$$-2\sum_{j=1}^{2}\frac{a_{1j}}{l_j^2} + \sum_{j=1}^{2} b_j + 2c + 2F_1 = -2\times(0.5+1.5) + 1.3 + 0.4 + 2\times 0.1 + 2 \le 0,$$

$$-2\sum_{j=1}^{2}\frac{a_{2j}}{l_j^2}+\sum_{j=1}^{2}b_j+2c+2F_2=-2\times(1.3+0.7)+1.3+0.4+2\times0.1+2\leq0. \quad (42)$$

Obviously, the conditions obtained are satisfied. Figure 2 shows that the equation is the global exponential stability. According to Theorem 1, the equation on time scales satisfies the global exponential stability condition.

From Figure 2, we can see that the error state of PDEs (35) is gradually reduced to the corresponding stable point. From the simulation results, it can be seen that the stable points of continuous-time and discrete-time are different, but they tend to be stable in the end.

Remark 4. The reason why equation (35) fails to converge to zero is that the numerical stability point of the equation under this condition is not zero. In the same way, when $\mathbb{T}=\cup_{k=0}^{\infty}[0.1k,0.1k+0.07]$, the stable point is -1.41.

Remark 5. Examples 1 and 2 give two simulation examples of stable and unstable PDE systems. It can be seen that the selection of system parameters and initial values determine the steady state of the system. The obtained condition of exponential stability in Theorem 1 is a sufficient condition, so the system that does not meet this condition may also be stable, which will be the goal of the PDE system in our next study.

6. Conclusion

This study considers a common second-order linear PDEs and obtains the corresponding equations on time scales. Based on the time scales theory, the Lyapunov functional method, and some inequality techniques, the exponential stability of second-order linear PDEs on time scales is analyzed, and sufficient conditions for the global exponential stability are strictly presented. In addition, the stability result is applied to the synchronization of RDNNs on time scales. Finally, two examples are given to verify the correctness of our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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