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Research Article

Convergence Analysis of an Iteration Process for a Class of Generalized Nonexpansive Mappings with Application to Fractional Differential Equations

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We consider the class of generalized α-nonexpansive mappings in a setting of Banach spaces. We prove existence of fixed point and convergence results for these mappings under the K^*-iterative process. The weak convergence is obtained with the help of Opial's property while strong convergence results are obtained under various assumptions. Finally, we construct two numerical examples and connect our K^*-iterative process with them. An application to solve a fractional differential equation (FDE) is also provided. It has been eventually shown that the K^*- iterative process of this example gives more accurate numerical results corresponding to some other iterative processes of the literature. The main outcome is new and improves some known results of the literature.

1. Introduction and Preliminaries

In recent years, theory of fixed points gained the attention of many authors [1, 2]. Whenever the ordinary analytical techniques cannot yield a solution to a differential or an integral equation, we are interested in finding the approximate value of the requested solution (see, e.g., the recent results in [3–5] and others). Before employing the appropriate iterative processes on such problems, one needs to convert it into a form of equation of fixed point. In this way a sequence is generated by the algorithm. The intended fixed point value of the equation of fixed point and the given equation's solution is the limit of the series. In case of contraction mappings, Banach fixed point theorem [6] signals the fundamental Picard iteration $x_{n+1} = \mathscr{A} x_n$. However, when the Picard iterative process for a given mapping does not converge, we employ alternative iterative procedures with different steps. One of the other iterative processes that have been studied by authors are the Mann [7], Noor [8], Ishikawa [9], and SP iteration (Phuengrattana and Suantai) [10]; S-iteration (Agarwal et al.) [11]; S^* iteration (Karahan and Ozdemie) [12]; Picard–Mann hybrid [13]; Normal-S [14]; Krasnoselskii–Mann [15]; Abbas [16]; Picard-S [17]; and Thakur [18].

On the other hand, Ullah and Arshad [19] presented a new iterative process for generalized nonexpansive mappings and call it as a K^* iterative process. The K^*-iterative process reads as follows:

$$\begin{cases} x_1 \in \mathscr{C}, \\ z_n = (1-\beta_n)x_n + \beta_n \mathscr{A} x_n), \\ y_n = \mathscr{A}\left((1-\alpha_n)z_n + \alpha_n \mathscr{A} z_n\right), \\ x_{n+1} = \mathscr{A} y_n, n \geq 0, \end{cases} \quad (1)$$

where $\alpha_n, \beta_n \in (0,1)$.

They demonstrated that among the many other iterative processes, the K^* iteration (1) gives very high accurate results in very less steps of iterations in the setting of Suzuki mappings. We improve here their results to the larger class of generalized α-nonexpansive mappings.

Definition 1. Let $\mathscr{A}: \mathscr{C} \longrightarrow \mathscr{C}$. Then, $\mathscr{A}$ is referred to as

(i) Nonexpansive if $\|\mathscr{A}a - \mathscr{A}(a')\| \le \|a - a'\|$, for every two $a, a' \in \mathscr{C}$;

(ii) Satisfying Condition (C) (or Suzuki mapping) if $1/2\|a - \mathscr{A}a\| \le \|a - a'\|$ implies $\|\mathscr{A}a - \mathscr{A}a'\| \le \|a - a'\|$, for every two $a, a' \in \mathscr{C}$;

(iii) Generalized α-nonexpansive if $1/2\|a - \mathscr{A}a\| \le \|a - a'\|$ implies $\|\mathscr{A}a - \mathscr{A}a'\| \le \alpha\|a - \mathscr{A}a'\| + \alpha\|a' - \mathscr{A}a\| + (1 - 2\alpha)\|a - a'\|$, for every two $a, a' \in \mathscr{C}$ and $\alpha \in [0, 1)$;

(iv) Satisfying condition I [20] if there is nondecreasing f with $f(0) = 0$ and $f(r) > 0$ at $r > 0$ and $\|a - \mathscr{A}(a)\| \ge f(d(a, D_{\mathscr{A}}))$ for all $a \in \mathscr{C}$.

First time, a fixed point existence result for nonexpansive mappings established by Gohde [21] and Browder [22] in a setting of uniform convex Banach space (UCBS), and in the same year, Kirk [23] obtained the same result in a setting of reflexive Banach space. In [24], Suzuki suggested a very interesting generalized of nonexpansive mappings and any mapping of this class if named as a mapping satisfying condition (C) (or Suzuki mapping). He established several convergence and existence results for these mappings in different Banach space settings. Since mappings with condition (C) are more general than the concept of nonexpansive mappings. Thus, Pant and Shukla [25] generalized the Suzuki mappings by introducing the class of generalized α-nonexpansive mappings. They proved that every Suzuki mapping is generalized α-nonexpansive but the converse is not valid in general; that is, they proved that the class of generalized α-nonexpansive mappings properly includes the class of Suzuki mappings. Moreover, they used the Agarwal iteration [11] for establishing the main convergence results. The purpose of this work is to obtain the strong and weak convergence for the K^*-iterative processes for generalized α-nonexpansive mappings. In this way, we extend some main results of Pant and Shukla [25], Ullah and Arshad, and many others.

Definition 2 (see [26, 27]). Let $\mathscr{C}$ be any nonvoid closed convex subset of a UCBS $\mathscr{H}$. $\{a_n\} \subseteq \mathscr{H}$ is bounded. If $h \in \mathscr{H}$ is any fixed element then we set the following:

(b1) For a bounded sequence $\{a_n\}$ at point h, $r(h, \{a_n\})\colon = \limsup_{n \longrightarrow \infty}\|h - a_n\|$ is termed as asymptotic radius;

(b2) For a bounded sequence $\{a_n\}$ with the connection of $\mathscr{C}$, $r(\mathscr{C}, \{a_n\}) = \inf\{r(h, \{a_n\})\colon h \in \mathscr{C}\}$ is termed as asymptotic radius;

(b3) For a bounded sequence $\{a_n\}$ with the connection of $\mathscr{C}$, $A(\mathscr{C}, \{a_n\}) = \{h \in \mathscr{C}\colon r(h, \{a_n\}) = r(\mathscr{C}, \{a_n\})\}$ is termed as asymptotic center.

Definition 3 (see [28]). A Banach space $\mathscr{H}$ space is said to satisfy the Opial's condition in the case of any weakly convergent sequence $\{a_n\} \subseteq \mathscr{H}$ whose weak limit is $h \in \mathscr{H}$, one is able to obtain the following:

$$\limsup_{n \longrightarrow \infty}\|a_n - h\| < \limsup_{n \longrightarrow \infty}\left\|a_n - h'\right\| \text{for each } h' \in \mathscr{C} - \{h\}. \tag{2}$$

In [25], the authors obtained some characterizations for the class of generalized α-nonexpansive mappings. We write all these characterizations in the following proposition:

Proposition 1. *Assume that $\mathscr{A}$ is a self-map on any nonempty subset $\mathscr{C}$ of a Banach space. Then,*

(p1) If $\mathscr{A}$ satisfy condition (C), then $\mathscr{A}$ is essentially generalized α-nonexpansive.

(p2) If $\mathscr{A}$ is generalized α-nonexpansive having a nonempty fixed point set, then $\|\mathscr{A}(a) - a^*\| \le \|a - a^*\|$ for $a \in \mathscr{C}$ and a^* in $D_{\mathscr{A}}$.

(p3) If $\mathscr{A}$ is generalized α-nonexpansive, then $D_{\mathscr{A}}$is closed. Moreover, if the given space $\mathscr{H}$ is strictly convex, $\mathscr{C}$ is convex, then the set $D_{\mathscr{A}}$ is also convex.

(p4) If $\mathscr{A}$ is generalized α-nonexpansive, then for all pair of elements $a, a' \in \mathscr{C}$, one has

$$\left\|a - \mathscr{A}a'\right\| \le \left(\frac{3 + \alpha}{1 - \alpha}\right)\|a - \mathscr{A}a\| + \left\|a - a'\right\|;. \tag{3}$$

(p5) If the space $\mathscr{H}$ is endowed with the Opial condition, $\mathscr{A}$ is generalized α-nonexpansive, $\{s_n\}$ is any weakly convergent sequence to r with the property $\lim_{n \longrightarrow \infty}\|\mathscr{A}s_n - s_n\| = 0$, then $r \in D_{\mathscr{A}}$.

Following lemma is a well-known property of any UCBS that is needed for our main results.

Lemma 1 (see [29]). *Let $\mathscr{H}$ be any UCBS. For $0 < i \le \alpha_n \le j < 1$, $\{a_n\}, \{b_n\} \subseteq \mathscr{H}$ so that $\limsup_{n \longrightarrow \infty}\|a_n\| \le u$, $\limsup_{n \longrightarrow \infty}\|b_n\| \le u$ and $\lim_{n \longrightarrow \infty}\|\alpha_n a_n + (1 - \alpha_n)b_n\| = u$ for some $u \ge 0$. Subsequently, we have $\lim_{n \longrightarrow \infty}\|a_n - b_n\| = 0$.*

2. Main Findings

To obtain our weak and strong convergence results, we need a key lemma as follows:

Lemma 2. *If $\mathscr{A}$ is generalized α-nonexpansive self-map on a closed convex subset $\mathscr{C}$ of a Banach space $\mathscr{H}$ with $D_{\mathscr{A}} \ne \varnothing$ and $\{x_n\}$ is a sequence of iterates obtained from the K^*-iterative process (1). Subsequently,$\lim_{n \longrightarrow \infty}\|x_n - a^*\|$ exists for each $a^* \in D_{\mathscr{A}}$.*

Proof. Let us take any $a^* \in D_{\mathscr{A}}$. Using Proposition 1 (p_2), we see that

$$\begin{aligned}\|z_n - a^*\| &= \|(1-\beta_n)x_n + \beta_n \mathscr{A}x_n - a^*\| \\ &\le \|(1-\beta_n)x_n + \beta_n x_n - a^*\| \\ &\le (1-\beta_n)\|x_n - a^*\| + \beta_n \|x_n - a^*\| \\ &= \|x_n - a^*\|.\end{aligned} \tag{4}$$

This implies that

$$\begin{aligned}\|x_{x+1} - a^*\| &= \|\mathscr{A}y_n - a^*\| \\ &\le \|y_n - a^*\| \\ &= \|\mathscr{A}(1-\alpha_n)z_n + \alpha_n \mathscr{A}z_n - a^*\| \\ &\le \|(1-\alpha_n)z_n + \alpha_n \mathscr{A}z_n - a^*\| \\ &\le (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|\mathscr{A}z_n - a^*\| \\ &\le (1-\alpha_n)\|z_n - a^*\| + \alpha_n\|z_n - a^*\| \\ &\le \|z_n - a^*\| \\ &\le \|x_n - a^*\|.\end{aligned} \tag{5}$$

Hence, $\|x_{n+1} - a^*\| \le \|x_n - a^*\|$, which shows that $\{\|x_n - a^*\|\}$ is bounded and nonincreasing. This gives result $\lim_{n\longrightarrow\infty}\|x_n - a^*\|$ exists for each $a^* \in D_{\mathscr{A}}$. □

Now we discuss about necessary and sufficient conditions that must be met in order for any generalized nonexpansive mapping in a Banach space to have fixed points.

Theorem 1. *If $\mathscr{A}$ is generalized α-nonexpansive self-map on a closed convex subset $\mathscr{C}$ of a UCBS $\mathscr{H}$ and $\{x_n\}$ is a sequence of iterates obtained from the K^*-iterative process (1). Subsequently, $D_{\mathscr{A}} \neq \varnothing$ if and only if $\{x_n\}$ is bounded in $\mathscr{C}$ and $\lim_{n\longrightarrow\infty}\|\mathscr{A}x_n - x_n\| = 0$.*

Proof. Let $D_{\mathscr{A}} \neq \varnothing$ and $a^* \in D_{\mathscr{A}}$. Applying Lemma 2, we get existence of $\lim_{n\longrightarrow\infty}\|x_n - a^*\|$ and $\{x_n\}$ is bounded. Let u be that limit; thus,

$$\lim_{n\longrightarrow\infty}\|x_n - a^*\| = u. \tag{6}$$

As we have demonstrated in the proof of Lemma 2 that

$$\|z_n - a^*\| \le \|x_n - a^*\|. \tag{7}$$

This together with (6) gives

$$\limsup_{n\longrightarrow\infty}\|z_n - a^*\| \le \limsup_{n\longrightarrow\infty}\|x_n - a^*\| = u. \tag{8}$$

Since a^* is in $D_{\mathscr{A}}$, so Proposition 1 (p_2) can be applied to obtain the following:

$$\begin{aligned}&\|\mathscr{A}x_n - a^*\| \le \|x_n - a^*\|, \\ &\Longrightarrow \limsup_{n\longrightarrow\infty}\|\mathscr{A}x_n - a^*\| \le \limsup_{n\longrightarrow\infty}\|x_n - a^*\| = u.\end{aligned} \tag{9}$$

Now by the proof of Lemma 2, we have the following:

$$\begin{aligned}&\|x_{n+1} - a^*\| \le \|z_n - a^*\|, \\ &\Longrightarrow u = \liminf_{n\longrightarrow\infty}\|x_{n+1} - a^*\| \le \liminf_{n\longrightarrow\infty}\|z_n - a^*\|.\end{aligned} \tag{10}$$

From (8) and (10), we have

$$u = \lim_{n\longrightarrow\infty}\|z_n - a^*\|. \tag{11}$$

By (1) and (11), one has

$$\begin{aligned}u &= \lim_{n\longrightarrow\infty}\|z_n - a^*\| \\ &= \lim_{n\longrightarrow\infty}\|(1-\beta_n)x_n + \beta_n\mathscr{A}x_n - a^*\| \\ &= \lim_{n\longrightarrow\infty}\|(1-\beta_n)(x_n - a^*) + \beta_n(\mathscr{A}x_n - a^*)\|.\end{aligned} \tag{12}$$

If and only if

$$u = \lim_{n\longrightarrow\infty}\|(1-\beta_n)(x_n - a^*) + \beta_n(\mathscr{A}x_n - a^*)\|. \tag{13}$$

Lemma 1, can be applied to obtain,

$$\lim_{n\longrightarrow\infty}\|\mathscr{A}x_n - x_n\| = 0. \tag{14}$$

On the other hand we aim to demonstrate $D_{\mathscr{A}} \neq \varnothing$ under the suppositions of a bounded $\{x_n\}$ in the sense that $\lim_{n\longrightarrow\infty}\|\mathscr{A}x_n - x_n\| = 0$. We may select a point $a^* \in A(\mathscr{A}, \{x_n\})$. If Proposition 1 (p_4) is applied, then we must have the following:

$$\begin{aligned}r(\mathscr{A}a^*, \{x_n\}) &= \limsup_{n\longrightarrow\infty}\|x_n - \mathscr{A}a^*\| \\ &\le \left(\frac{3+\alpha}{1-\alpha}\right)\limsup_{n\longrightarrow\infty}\|\mathscr{A}x_n - x_n\| \\ &\quad + \limsup_{n\longrightarrow\infty}\|x_n - a^*\| \\ &= \limsup_{n\longrightarrow\infty}\|x_n - a^*\| \\ &= r(a^*, \{x_n\}).\end{aligned} \tag{15}$$

We observe that $\mathscr{A}a^* \in A(\mathscr{C}, \{x_n\})$. As this set has only element in any UCBS setting, we deduce $\mathscr{A}a^* = a^*$, hence the set $D_{\mathscr{A}}$ is nonempty. □

Among the convergence results, we first obtain our weak convergence result for the K^*-iterative process in the setting of generalized α-nonexpansive mappings as follows:

Theorem 2. *If $\mathscr{A}$ is generalized α-nonexpansive self-map on a closed convex subset $\mathscr{C}$ of a UCBS $\mathscr{H}$ and $\{x_n\}$ is a sequence of iterates obtained from the K^*-iterative process (1). Subsequently, $\{x_n\}$ is weakly convergent to a point of $D_{\mathscr{A}}$.*

Proof. Using Theorem 1, the provided sequence $\{x_n\}$ is bounded. As $\mathscr{H}$ is UCBS, $\mathscr{H}$ a is reflexive Banach space. That is why, we can built a weakly convergent sequence of $\{x_n\}$ so that $\{x_{n_i}\}$ be the subsequence with $a_1 \in \mathscr{C}$ as weak limit. Applying Theorem 1 on the subsequence, we may have

$\lim_{i \longrightarrow \infty} \|x_{n_i} - \mathscr{A}x_{n_i}\| = 0$. Hence, by Proposition 1 (p_5), we have $a_1 \in D_{\mathscr{A}}$. It is enough to prove that $\{x_n\}$ is weakly convergent to a_1. Thus, if $\{x_n\}$ does not weakly converge to a_1. Then, a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ and $a_2 \in \mathscr{C}$ with $\left\{x_{n_j}\right\}$ converging weakly to a_2 and $a_2 \neq a_1$ exists. Also by Proposition 1 (p_5), $a_2 \in D_{\mathscr{A}}$, by Lemma 2 with Opial's property, we get

$$\begin{aligned}\lim_{n \longrightarrow \infty}\left\|x_n - a_1\right\| &= \lim_{i \longrightarrow \infty}\left\|x_{n_i} - a_1\right\| \\ &< \lim_{i \longrightarrow \infty}\left\|x_{n_i} - a_2\right\| \\ &= \lim_{n \longrightarrow \infty}\left\|x_n - a_2\right\| \\ &= \lim_{j \longrightarrow \infty}\left\|x_{n_j} - a_2\right\| \\ &< \lim_{j \longrightarrow \infty}\left\|x_{n_j} - a_1\right\| \\ &= \lim_{n \longrightarrow \infty}\left\|x_n - a_1\right\|.\end{aligned} \tag{16}$$

This is a contradiction. Hence, we have $a_1 = a_2$. So, $\{x_n\}$ is weakly convergent to $a_1 \in D_{\mathscr{A}}$. □

Now we discuss the strong convergence of the K^*-iterative process for generalized α-nonexpansive mappings on compact domains.

Theorem 3. *If $\mathscr{A}$ is generalized α-nonexpansive self-map on a compact convex subset $\mathscr{C}$ of a UCBS $\mathscr{H}$ and $\{x_n\}$ is a sequence of iterates obtained from the K^*-iterative process (1). Subsequently, $\{x_n\}$ is strongly convergent to a point of $D_{\mathscr{A}}$.*

Proof. As the domain $\mathscr{C}$ is a compact subset of $\mathscr{H}$ and also $\{x_n\} \subseteq \mathscr{C}$ due to the convexity of $\mathscr{C}$. Thus, a subsequence $\left\{x_{n_i}\right\}$ of $\{x_n\}$ exists with $\lim_{i \longrightarrow \infty}\|x_{n_i} - a^{**}\| = 0$ for some $a^{**} \in \mathscr{C}$. In the view of Theorem 1, $\lim_{i \longrightarrow \infty}\|\mathscr{A}x_{n_i} - x_{n_i}\| = 0$. Applying Proposition 1 (p_4), one has

$$\left\|x_{n_i} - \mathscr{A}a^{**}\right\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\left\|x_{n_i} - \mathscr{A}x_{n_i}\right\| + \left\|x_{n_i} - a^{**}\right\|. \tag{17}$$

Hence, if $i \longrightarrow \infty$, then $\mathscr{A}a^{**} = a^{**}$. In the view of Lemma 2, a^{**} is the strong limit of $\{x_n\}$. This finishes the proof. □

A strong convergence of the K^*-iterative process in the setting of generalized α-nonexpansive mappings on a noncompact domain is established as follows. It should be noted that this result holds in general Banach spaces.

Theorem 4. *If $\mathscr{A}$ is generalized α-nonexpansive self-map on a closed convex subset $\mathscr{C}$ of a Banach space $\mathscr{H}$ and $\{x_n\}$ is a sequence of iterates obtained from the K^*-iterative process (1). Subsequently, $\{x_n\}$ is strongly convergent to a point of $D_{\mathscr{A}}$ if $\liminf_{n\longrightarrow\infty} d(x_n, D_{\mathscr{A}}) = 0$.*

Proof. Using Lemma 2, we have existence of $\lim_{n\longrightarrow\infty}\|x_n - a^*\|$, for each fixed point of $\mathscr{A}$. This provides us that $\lim_{n\longrightarrow\infty} d(x_n, D_{\mathscr{A}})$ exist. Accordingly

$$\lim_{n \longrightarrow \infty} d(x_n, D_{\mathscr{A}}) = 0. \tag{18}$$

Two subsequence $\left\{x_{n_i}\right\}$ and $\{a_i\}$ of $\{x_n\}$ and $D_{\mathscr{A}}$ are, respectively, generated by the above limit. Hence

$$\left\|x_{n_i} - a_i\right\| \leq \frac{1}{2^i} \text{ for each } i \geq 1. \tag{19}$$

From the proof of Lemma 2, we get nonincreasing $\{x_n\}$, that is why

$$\left\|x_{n_{i+1}} - a_i\right\| \leq \left\|x_{n_i} - a_i\right\| \leq \frac{1}{2^i}. \tag{20}$$

It follows that

$$\begin{aligned}\left\|a_{i+1} - a_i\right\| &\leq \left\|a_{i+1} - x_{n_{i+1}}\right\| + \left\|x_{n_{i+1}} - a_i\right\| \\ &\leq \frac{1}{2^{i+1}} + \frac{1}{2^i} \\ &\leq \frac{1}{2^{i-1}} \longrightarrow 0, \text{as } i \longrightarrow \infty.\end{aligned} \tag{21}$$

Consequently, $\{a_i\}$ is Cauchy sequence in $D_{\mathscr{A}}$ as $\lim_{i \longrightarrow \infty}\|a_{i+1} - a_i\| = 0$; thus, $\{a_i\}$ converges to a^{**}. Using Proposition 1 (p_3), $D_{\mathscr{A}}$ is closed; thus, $a^{**} \in D_{\mathscr{A}}$. By Lemma 2, $\lim_{n\longrightarrow\infty}\|x_n - a^{**}\|$ exists so, a^{**} is the strong limit of $\{x_n\}$. □

Theorem 5. *If $\mathscr{A}$ is generalized α-nonexpansive self-map on a closed convex subset $\mathscr{C}$ of a UCBS $\mathscr{H}$ and $\{x_n\}$ is a sequence of iterates obtained from the K^*-iterative process (1). Subsequently, $\{x_n\}$ is strongly convergent to a point of $D_{\mathscr{A}}$ if $\mathscr{A}$ satisfies condition (I).*

Proof. Using Theorem 1, we can have

$$\liminf_{n\longrightarrow\infty}\left\|\mathscr{A}x_n - x_n\right\| = 0. \tag{22}$$

Due to condition (I) of $\mathscr{A}$, we have

$$\left\|x_n - \mathscr{A}x_n\right\| \geq f\left(\mathrm{d}\left(x_n, D_{\mathscr{A}}\right)\right). \tag{23}$$

Applying (22) on (23), we get

$$\liminf_{n\longrightarrow\infty} f\left(\mathrm{d}\left(x_n, D_{\mathscr{A}}\right)\right) = 0. \tag{24}$$

It follows

$$\liminf_{n\longrightarrow\infty} \mathrm{d}\left(x_n, D_A\right) = 0. \tag{25}$$

Now applying Theorem 4, $\{x_n\}$ converges strongly to $\mathscr{D}_{\mathscr{A}}$. □

3. Application to Fractional Differential Equations

Fractional calculus is important and an active field of research on its own [30–32]. It is well-known that fractional calculus has a crucial role in fluid, electromagnetic theory, and, especially, in electrical networks. In recent years, many papers appeared on the existence and approximation of solutions for certain FDEs (see e.g., Karapinar et al. [33] and others). However, all these authors used the concept of nonexpansive mappings to achieve the main objective that are continuous on their domain of definitions.

Our alternative in this paper is to solve a FDE in the setting of generalized α-nonexpansive mappings that are in general discontinuous. Unlike, other iterative schemes, we suggest the K^* iterative scheme (1) to find the solution for the following FDE.

Now we consider the following FDE and also assume that S is a solution set of it:

$$\left.\begin{aligned} &D^{\xi}h(u) + \Upsilon(u, h(u)) = 0,\\ &h(0) = h(1) = 0, \end{aligned}\right\}, \tag{26}$$

where $(0 \le u \le 1)$, $(1 < \xi < 2)$, and D^{ξ} stands for the Caputo fractional derivative endowed with the order ξ and $\Upsilon\colon [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$.

Now we consider $\mathscr{C} = C[0,1]$, where $C[0,1]$ is the Banach space of continuous maps on $[0,1]$ to $\mathbb{R}$ equiped with the maximum norm. The corresponding Green's function with (26) is defined by

$$G(u,v) = \begin{cases} \dfrac{1}{\Gamma(\xi)}\big(u(1-v)^{(\xi-1)} - (u-v)^{(\xi-1)}, & \text{if } 0 \le v \le u \le 1,\\[2ex] \dfrac{u(1-v)^{(\xi-1)}}{\Gamma(\xi)}, & \text{if } 0 \le u \le v \le 1. \end{cases} \tag{27}$$

The main result is provided in the following way:

Theorem 6. *If* $\mathscr{C} = C[0,1]$, *then set an operator* $\mathscr{A}\colon \mathscr{C} \longrightarrow \mathscr{C}$ *by the formula*

$$\mathscr{A}(h(u)) = \int_0^1 G(u,v)\Upsilon(v,h(v))\mathrm{d}v, \textit{ for each } h(\mathrm{u}) \in \mathscr{C}. \tag{28}$$

If

$$|\Upsilon(v,h(v)) - \Upsilon(v,g(v))| \le \alpha|h(v) - \mathscr{A}(g(v))| + \alpha|g(v) - \mathscr{A}(h(v))| + (1-2\alpha)|g(v) - h(v)|, \tag{29}$$

where α is some real number in [0, 1). Subsequently, theK^* iterates (1) associated with the $\mathscr{A}$ (as defined above) essentially converges to some point of the solution set S of (26) provided that $\liminf_{n\longrightarrow\infty} \mathrm{d}(x_n, S) = 0$.

Proof. Notice that the element h of $\mathscr{C}$ solves (26) if it solves

$$h(u) = \int_0^1 G(u,v)\Upsilon(v,h(v))\mathrm{d}v. \tag{30}$$

Now for every choice of $h, g \in \mathscr{C}$ and $0 \le u \le 1$, it follows that

$$\begin{aligned} \|\mathscr{A}h(u) - \mathscr{A}g(u)\| &\le \left|\int_0^1 G(u,v)\Upsilon(v,h(v)))\mathrm{d}v - \int_0^1 G(u,v)\Upsilon(v,g(v))\mathrm{d}v\right|\\ &= \left|\int_0^1 G(u,v)[\Upsilon(v,h(v)) - \Upsilon(v,g(v))]\mathrm{d}v\right|\\ &\le \int_0^1 G(u,v)|\Upsilon(v,h(v)) - \Upsilon(v,g(v))|\mathrm{d}v\\ &\le \int_0^1 G(u,v)\,(\alpha|h(v) - \mathscr{A}(g(v))| + \alpha|g(v) - \mathscr{A}(h(v))|\\ &\quad + (1-2\alpha)|g(v) - h(v)|)\,\mathrm{d}v\\ &\le (\alpha\|h(v)) - \mathscr{A}(g(v))\| + \alpha\|g(v) - \mathscr{A}(h(v))\| + (1-2\alpha)\\ &\quad \cdot \|g(v) - h(v)\|)\left(\int_0^1 G(u,v)\mathrm{d}v\right)\\ &\le \alpha\|h(v)) - \mathscr{A}(g(v))\| + \alpha\|g(v) - \mathscr{A}(h(v))\|\\ &\quad + (1-2\alpha)\|g(v) - h(v)\|. \end{aligned} \tag{31}$$

Consequently, we get

$$\|\mathscr{A}h - \mathscr{A}g\| \leq \alpha\|h - \mathscr{A}g\| + \alpha\|g - \mathscr{A}h\| + (1 - 2\alpha)\|g - h\|. \tag{32}$$

Hence, $\mathscr{A}$ is generalized α-nonexpansive mapping. In the view of Theorem 4, the sequence of K^* iterates converges to a fixed point of $\mathscr{A}$ and hence to the solution of the given equation. □

4. Numerical Example

First, we construct a novel example of generalized α-nonexpansive mappings on closed convex subset of a UCBS. Using this example, we perform a comparative numerical experiment using our K^* and other iterative processes of the literature.

Example 1. Let $\mathscr{C} = [0, 6]$ and a self-map on $\mathscr{C}$ by the following rule:

$$\mathscr{A}a = \begin{cases} \dfrac{a + 15}{4}, & \text{if } 0 \leq a \leq 5, \\ 4, & \text{if } 5 < a \leq 6. \end{cases} \tag{33}$$

In this case, we prove that $\mathscr{A}$ is generalized 1/3-nonexpansive but does not satisfy the condition (C).

Proof. Let $a, a' \in \mathscr{A}$, then following are the all possible cases:

Case I. If $5 < a, a' \leq 6$, we have

$$\begin{aligned} &\frac{1}{3}\left|a - \mathscr{A}a'\right| + \frac{1}{3}\left|a' - \mathscr{A}a\right| + \left(1 - 2\left(\frac{1}{3}\right)\right)\left|a - a'\right| \\ &\geq 0 = \left|\mathscr{A}a - \mathscr{A}a'\right|. \end{aligned} \tag{34}$$

Case II. If $0 < a, a' \leq 5$, we have

$$\begin{aligned} &\frac{1}{3}\left|a - \mathscr{A}a'\right| + \frac{1}{3}\left|a' - \mathscr{A}a\right| + \left(1 - 2\left(\frac{1}{3}\right)\right)\left|a - a'\right| \\ &= \frac{1}{3}\left|a - \left(\frac{a' + 15}{4}\right)\right| + \frac{1}{3}\left|a' - \frac{a + 15}{4}\right| + \frac{1}{3}\left|a - a'\right| \geq \frac{1}{4}\left|a - a'\right| \\ &= \left|\mathscr{A}a - \mathscr{A}a'\right|. \end{aligned} \tag{35}$$

Case III. If $0 < a \leq 5$ and $5 < a' \leq 6$, we have

$$\begin{aligned} &\frac{1}{3}\left|a - \mathscr{A}a'\right| + \frac{1}{3}\left|a' - \mathscr{A}a\right| + \left(1 - 2\left(\frac{1}{3}\right)\right)\left|a - a'\right| = \frac{1}{3}|a - 4| + \frac{1}{3}\left|a' - \left(\frac{a + 15}{4}\right)\right| \\ &+ \frac{1}{3}\left|a - a'\right| \geq \frac{1}{4}|a - 1| = \left|\mathscr{A}a - \mathscr{A}a'\right|. \end{aligned} \tag{36}$$

Hence, $|\mathscr{A}a - \mathscr{A}a'| \leq 1/3|a - \mathscr{A}a'| + 1/3|a' - \mathscr{A}a| + (1 - 2(1/3))\,|a - a'|$ for every two points $a, a' \in \mathscr{C}$. Now let $a = 5$ and $a' = 11/2$, then $1/2|a - \mathscr{A}a| \leq |a - a'| \Rightarrow |\mathscr{A}a - \mathscr{A}a'| > |a - a'|$. Thus, $\mathscr{A}$ does not satisfy condition (C). □

To show the high accuracy of the proposed K^* iteration, we compare it with the one-step Mann iteration [7], two-step Ishikawa [9], leading two-stepS-iteration of Agarwal [11] and a leading iterative scheme studied by Thakur [18]. We may take $\alpha_n = 0.85$ and $\beta_n = 0.65$. Table 1 shows some values for the initial value of $x_1 = 4.9$. Additionally, Figure 1 offers detail on the behavior of the different schemes. Moreover, if $\|x_n - 0\| < 10^{-4}$, then further comparison is given in Table 2. For generalized α-nonexpansive mapping it is evident that the K^* iterative method performs better than the other methods.

We finish this section with an example. This example uses a subset of a two-dimensional Euclidian space.

Example 2. Let $\mathscr{C} = [0, 1] \times [0, 1]$ and set a self-map on $\mathscr{A}$ by the following rule:

$$\mathscr{A}\left(a, a'\right) = \begin{cases} \left(\dfrac{a}{2}, \dfrac{a'}{3}\right), & \text{if } \left(a, a'\right) \in \left[0, \dfrac{1}{2}\right) \times \left[0, \dfrac{1}{2}\right), \\ \left(\dfrac{a}{4}, \dfrac{a'}{5}\right), & \text{if } \left(a, a'\right) \in \left[\dfrac{1}{2}, 1\right] \times \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{37}$$

Here $\mathscr{A}$ is a generalized α-nonexpansive mapping with fixed point (0, 0). The numerical results are shown in Table 3.

Table 1: Comparison of various iterative processes.

n	K^*	Thakur	S	Ishikawa	Mann
1	4.9	4.9	4.9	4.9	4.9
2	4.99883887	4.99633984	4.98535938	4.97410938	4.96375000
3	4.99998652	4.99986603	4.99785652	4.99329676	4.98685938
4	4.99999984	4.99999510	4.99968618	4.99826449	4.99523652
5	5	4.99999982	4.99995405	4.99955067	4.99827324
6	5	4.99999999	4.99999327	4.99988366	4.99937405
7	5	5	4.99999902	4.99996988	4.99977309
8	5	5	4.99999986	4.9999922	4.99991775
9	5	5	4.99999998	4.99999798	4.99997018
10	5	5	5	4.99999948	4.99998919
11	5	5	5	4.99999986	4.99999608
12	5	5	5	4.99999996	4.99999858
13	5	5	5	4.99999999	4.99999940
14	5	5	5	5	4.99999981
15	5	5	5	5	4.99999993
16	5	5	5	5	4.99999998
17	5	5	5	5	4.99999999
18	5	5	5	5	5

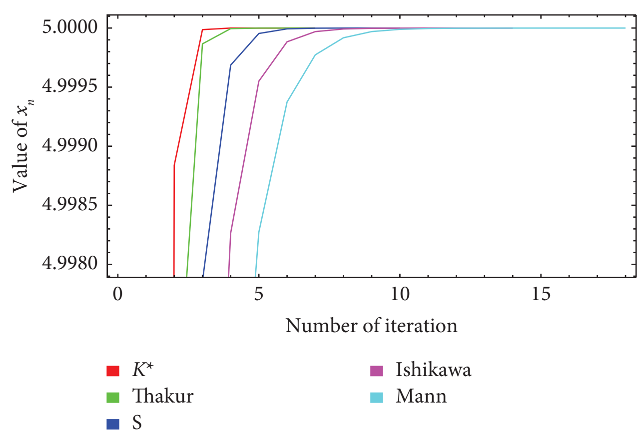

Figure 1: Graphical comparison of various iterative processes.

Table 2: Comparison of the iterates for different choice of parameters and initial points.

Iterations	Initial points					
	0.2	1.2	2.2	3.2	4.2	4.9
For $\alpha_n = n/(n+1)^{10/9}, \beta_n = 1/(n+3)^{2/3}$						
Mann	14	13	13	12	9	9
Ishikawa	13	12	12	12	11	8
S	6	6	6	6	5	9
Thakur	4	4	4	5	5	5
K^*	**3**	**3**	**4**	**4**	**4**	**4**

Table 3: Sequences generated by K^*-, Thakur-, and S-iteration processes.

n	K^*	Thakur	S
1	(0.1, 0.2)	(0.1, 0.2)	(0.1, 0.2)
2	(0.00970, 0.00272)	(0.01962, 0.01585)	(0.03925, 0.04755)
3	(0.00094, 0.00007)	(0.00385, 0.00125)	(0.01540, 0.01130)
4	(0.00009, 0)	(0.00075, 0.00009)	(0.00604, 0.00268)
5	(0, 0)	(0.00014, 0)	(0.00237, 0.00063)
6	(0, 0)	(0.00002, 0)	(0.00093, 0.00015)
7	(0, 0)	(0, 0)	(0.00036, 0.00003)
8	(0, 0)	(0, 0)	(0.00014, 0)
9	(0, 0)	(0, 0)	(0.00005, 0)
10	(0, 0)	(0, 0)	(0, 0)

In this case, it is also clear that our K^* iterative scheme is moving fast to the fixed point (0, 0).

5. Conclusions

In this research, we obtained the following new finding:

(i) We studied the K^* iterative scheme of Ullah and Arshad for approximating fixed points of generalized α-nonexpansive mappings.

(ii) We successfully carried out some weak and strong convergence results under various mild conditions.

(iii) We carried out an application of our main outcome for solution of a FBVP in a Banach space setting.

(iv) A new example of generalized α-nonexpansive mappings is constructed and proved that it exceeds properly the class of mappings with condition (C).

(v) Using our new example, we showed that the K^* iterative scheme is more effective and suggests very high accurate numerical results in the setting of generalized α-nonexpansive mappings in the setting of generalized α-nonexpansive mappings.

(vi) Accordingly, our main outcome improved some recent results of Ullah and Arshad [19] form the case of mappings with condition (C) to the general case of mappings called generalized α-nonexpansive mappings. In a similar way, our results are the improvement and refinements of the results due to Agarwal [11], Abbas [16], Thakur [18], and many others from the setting of nonexpansive and Suzuki mappings to the general setting of generalized α-nonexpansive mappings.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Each author contributed equally and significantly to every part of this article. All authors read and approved the final version of the paper.

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