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*Research Article*

# Complex Dynamics and Chaos Control of a Discrete-Time Predator-Prey Model

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The objective of this study is to investigate the complexity of a discrete predator-prey system. The discretization is achieved using the piecewise constant argument method. The existence and stability of equilibrium points, as well as transcritical and Neimark–Sacker bifurcations, are all explored. Feedback and hybrid control methods are used to control the discrete system's bifurcating and fluctuating behavior. To validate the theoretical conclusions, numerical simulations are performed. The findings of the study suggested that the discretization technique employed in this investigation preserves bifurcation and displays more effective dynamic consistency in comparison to the Euler method.

## 1. Introduction

Predator-prey models are a class of population models utilized to investigate the dynamics of predator-prey relationships within an ecosystem. The aforementioned models depict the temporal evolution of populations of predators and prey, taking into account variables such as predation, reproduction, and natural mortality. Comprehending the intricate dynamics inherent in predator-prey models is of paramount importance in predicting the repercussions of human activities on animal populations and shaping conservation endeavors. The Lotka–Volterra model is a widely recognized predator-prey model that was formulated separately by Lotka [1] and Volterra [2]. Subsequently, many population models have been formulated to account for intricate factors, including multiple predators, predator competition, and ecological factors such as food availability and changes in the weather.

The authors in [3] analyzed the complex behavior of the following system:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1-x)(x-A) - \dfrac{xy}{m}, \\[2ex] \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(\dfrac{\lambda x}{1+\beta x} - \tau\right). \end{cases} \tag{1}$$

The variables $x(t)$ and $y(t)$ represent the densities of the prey and predators, respectively, at a given time $t$. All parameters, namely, $A$, $m$, $\lambda$, $\beta$, and $\tau$, are positive with the additional constraint that $0 < A < 1$. The investigation focused on the examination of equilibrium points, their presence, and their stability. Furthermore, the authors have demonstrated that the system (1) experiences NS bifurcation at the positive fixed point of the system (1).

The area of discrete-time systems analysis has made great strides in comprehending the complicated dynamics shown by many systems. Classic works in this area, such as the discretized quasiperiodic plasma perturbations model [4, 5], the discretized fractional-order predator-prey system [6–8], the discrete economic system [9–11], and discrete systems describing competition games [12–14], have all played important roles in unraveling the complexities of discrete systems. These seminal studies have aided understanding of phenomena such as the emergence of complex patterns and bifurcations in plasma oscillations, the impact of fractional-order dynamics on predator-prey population dynamics, the nonlinear dynamics underlying economic systems, and the intricate dynamics of competitive interactions.

It is noteworthy that a variety of biological models are typically governed by both continuous and discrete models. In recent years, several authors have significantly contributed to the development of discrete models [15–19]. One possible explanation for this phenomenon is that discrete models tend to be more compelling than continuous models in situations involving nonoverlapping generation. It has been established through extensive research that discrete models have the potential to display more complex dynamical behaviors than continuous models. In addition, numerical solutions for discrete models can be generated with greater ease. This has been demonstrated in various studies [20–24].

The authors employed the Euler method in [24] to investigate the following discrete version of system (1):

$$\begin{cases} x_{n+1} = x_n + \gamma x_n\left(\left(1 - x_n\right)\left(x_n - A\right) - \dfrac{y_n}{m}\right), \\ \\ y_{n+1} = y_n + \gamma y_n\left(\dfrac{\lambda x_n}{1 + \beta x_n} - \tau\right). \end{cases} \tag{2}$$

The investigation focused on the examination of the presence and stability of equilibrium points. The findings presented in [24] indicates that the discretization of system (2) using Euler's method with a large step size leads to period-doubling and bifurcation at the positive equilibrium point. This observation is in contrast to the expected accuracy of the numerical method employed.

A further issue is that the discrete system created using the Euler approach is not nearly as realistic as it should be. This is because some parameters and initial values have negative values for the prey and predator population size. Nevertheless, by using the piecewise constant argument approach, it is possible to eliminate the occurrence of negative values. This motivates us to discretize the system (1) by using the piecewise constant argument method [25–29] to obtain the following discrete system:

$$\begin{cases} x_{n+1} = x_n e^{\left(1-x_n\right)\left(x_n - A\right) - y_n/m}, \\ y_{n+1} = y_n e^{\lambda x_n/1+\beta x_n - \tau}. \end{cases} \tag{3}$$

The present investigation's primary findings and deductions are as follows:

(i) The study investigates the presence and topological categorization of equilibrium points

(ii) The results of our study indicate that the mathematical system represented by equation (3) undergoes both NS and transcritical bifurcations

(iii) The present study investigates the conditions for the existence and direction of NS bifurcation at the positive equilibrium point

(iv) In order to rein in the unpredictability of the (3) system, feedback and hybrid control methods are used.

The subsequent text outlines the format of the document: Section 2 provides a comprehensive analysis of the presence of equilibrium points and their regional stability. Section 3 provides a discussion on the NS bifurcation occurring at the positive equilibrium point. The feedback and hybrid control methodologies are employed in Section 4. We have incorporated a few numerical examples in Section 5 to back up our theoretical findings. Finally, the present study is concluded in Section 6.

## 2. Existence and Stability of Equilibrium Points

The system's equilibrium points can be found by solving the following equations:

$$\begin{cases} x = x e^{(1-x)(x-A) - y/m}, \\ y = y e^{\lambda x/1+\beta x - \tau}. \end{cases} \tag{4}$$

The four equilibrium points obtained are as follows:

$$\begin{aligned} P_0 &= (0, 0), \\ P_1 &= (1, 0), \\ P_2 &= (A, 0), \\ P_3 &= \left(-\frac{\tau}{\beta\tau - \lambda}, -\frac{m(\tau + \beta\tau - \lambda)(\tau + A\beta\tau - A\lambda)}{(\beta\tau - \lambda)^2}\right). \end{aligned} \tag{5}$$

The first trio of equilibrium points, denoted as $P_0$, $P_1$, and $P_2$, are classified as boundary equilibrium points. The equilibrium point $P_3$ of the system (3) is the only positive equilibrium point if $A\lambda/1 + A\beta < \tau < \lambda/1 + \beta$.

Next, we explored the stability analysis of the equilibrium points. To investigate the stability of the equilibrium points, we calculated the variational matrix $J$ of system (3) at any point $(u, v)$ as follows:

$$J(u, v) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}, \tag{6}$$

where

$$
\begin{aligned}
j_{11} &= e^{u-u^2-v/m-A+uA}\left(1+u-2u^2+uA\right),\\
j_{12} &= -\frac{e^{-v/m-(-1+u)(u-A)}u}{m},\\
j_{21} &= \frac{e^{-\tau+u\lambda/1+u\beta}v\lambda}{(1+u\beta)^2},\\
j_{22} &= e^{-\tau+u\lambda/1+u\beta}.
\end{aligned}
\tag{7}
$$

The eigenvalues $\varsigma_{1,2}$ of the variational matrix $J$ are helpful in determining the stability of equilibrium points. The equilibrium point $(u, v)$ is known as a sink if $|\varsigma_{1,2}| < 1$, which is locally asymptotically stable (LAS), and as a source if $|\varsigma_1| > 1$ and $|\varsigma_2| > 1$, which is unstable. Moreover, the equilibrium point $(u, v)$ is a saddle point (SP) if $|\varsigma_1| > 1$ and $|\varsigma_2| < 1$ (or $|\varsigma_1| < 1$ and $|\varsigma_2| > 1$). In the case of a non-hyperbolic point (NHP) $(u, v)$, either $|\varsigma_1| = 1$ or $|\varsigma_2| = 1$.

**Lemma 1 [30].** *Let $\Gamma(\varsigma) = \varsigma^2 + T_1\varsigma + T_0$ and $\Gamma(1) > 0$. Moreover, $\varsigma_1, \varsigma_2$ are the solutions of $\Gamma(\varsigma) = 0$, then we have the following:*

*(i)* $|\varsigma_{1,2}| < 1 \Longleftrightarrow T_0 < 1 \wedge \Gamma(-1) > 0$

*(ii)* $|\varsigma_1| < 1 \wedge |\varsigma_2| > 1$ *(or* $|\varsigma_1| > 1 \wedge |\varsigma_2| < 1$*)* $\Longleftrightarrow \Gamma(-1) < 0$

*(iii)* $|\varsigma_1| > 1 \wedge |\varsigma_2| > 1 \Longleftrightarrow T_0 > 1 \wedge \Gamma(-1) > 0$

*(iv)* $\varsigma_1 = -1 \wedge |\varsigma_2| \neq 1 \Longleftrightarrow T_1 \neq 0, 2 \wedge \Gamma(-1) = 0$

*(v)* $\varsigma_{1,2}$ *are complex and* $|\varsigma_{1,2}| = 1 \Longleftrightarrow T_1^2 - 4T_0 < 0 \wedge T_0 = 1$.

**Proposition 2.** *The equilibrium point $P_0$ is LAS.*

*Proof.* The variational matrix $J(P_0)$ is calculated as follows:

$$
J(P_0) = \begin{bmatrix} e^{-A} & 0 \\ 0 & e^{-\tau} \end{bmatrix}. \tag{8}
$$

Clearly, $\varsigma_1 = e^{-A} < 1$ and $\varsigma_2 = e^{-\tau} < 1$.

The local asymptotic stability of the equilibrium point $P_0$ has been derived. Consequently, it is possible for both the predator and prey populations to become extinct simultaneously. □

**Proposition 3.** *The equilibrium point $P_1$ is as follows:*

*(i) LAS if $\tau > \lambda/1 + \beta$*

*(ii) SP if $\tau < \lambda/1 + \beta$*

*(iii) NHP if $\tau = \lambda/1 + \beta$.*

*Proof.* The variational matrix $J(P_1)$ is calculated as follows:

$$
J(P_1) = \begin{bmatrix} A & \dfrac{-1}{m} \\ 0 & e^{-\tau+\lambda/1+\beta} \end{bmatrix}. \tag{9}
$$

Clearly, $\varsigma_1 = A < 1$ and $\varsigma_2 = e^{-\tau+\lambda/1+\beta}$. It is obtained that

$$
\left|e^{-\tau+\lambda/1+\beta}\right| \begin{cases} <1 & \text{if } \tau > \dfrac{\lambda}{1+\beta}, \\ =1 & \text{if } \tau = \dfrac{\lambda}{1+\beta}, \\ >1 & \text{if } \tau < \dfrac{\lambda}{1+\beta}. \end{cases} \tag{10}
$$

The stability of the equilibrium point $P_1 = (1, 0)$ has been determined. Thus, the potential exists for the predator to become extinct. □

**Proposition 4.** *The equilibrium point $P_2$ is as follows:*

*(i) Source if $\tau < A\lambda/1 + A\beta$,*

*(ii) SP if $\tau > A\lambda/1 + A\beta$,*

*(iii) NHP if $\tau = A\lambda/1 + A\beta$.*

*Proof.* The variational matrix $J(P_2)$ is calculated as follows:

$$
J(P_2) = \begin{bmatrix} 1 + A(1-A) & \dfrac{-A}{m} \\ 0 & e^{-\tau+A\lambda/1+A\beta} \end{bmatrix}. \tag{11}
$$

Clearly, $\varsigma_1 = 1 + A(1-A) < 1$ and $\varsigma_2 = e^{-\tau+A\lambda/1+A\beta}$. One can easily check that

$$
\left|e^{-\tau+A\lambda/1+A\beta}\right| \begin{cases} <1 & \text{if } \tau > \dfrac{A\lambda}{1+A\beta}, \\ =1 & \text{if } \tau = \dfrac{A\lambda}{1+A\beta}, \\ >1 & \text{if } \tau < \dfrac{A\lambda}{1+A\beta}. \end{cases} \tag{12}
$$

Topological classification of equilibrium point $P_1$ is depicted in Figure 1(a) and $P_2$ in Figure 1(b).

Next, we focus on the positive equilibrium point $P_3$. We obtain

$$
J(P_3) = \begin{bmatrix} \dfrac{\left(-2-(1+A)\beta+\beta^2\right)\tau^2 + (1+A-2\beta)\tau\lambda + \lambda^2}{(-\beta\tau+\lambda)^2} & \dfrac{\tau}{m\beta\tau - m\lambda} \\ -\dfrac{m(\tau+\beta\tau-\lambda)(\tau+A\beta\tau-A\lambda)}{\lambda} & 1 \end{bmatrix}. \tag{13}
$$

The matrix $J(P_3)$ yields the characteristic polynomial as

$$\Gamma(\varsigma) = \varsigma^2 + \left(\frac{\left(2+\beta+A\beta-2\beta^2\right)\tau^2-(1+A-4\beta)\tau\lambda-2\lambda^2}{(-\beta\tau+\lambda)^2}\right)\varsigma$$
$$+\frac{1}{\lambda(\lambda-\beta\tau)^2}\left(\beta(1+\beta)(1+A\beta)\tau^4+\lambda^3(1-A\tau)-\lambda\tau^2(2+\tau+(1+A)\beta(1+2\tau)\right.$$
$$\left.+\beta^2(-1+3A\tau))+\lambda^2\tau(1-2\beta+\tau+A(1+\tau+3\beta\tau))\right). \quad (14)$$

Thus, we obtain

$$\Gamma(1) = \frac{\tau(\tau+\beta\tau-\lambda)(\tau+A\beta\tau-A\lambda)}{(\beta\tau-\lambda)\lambda},$$
$$\Gamma(0) = \frac{1}{\lambda(\lambda-\beta\tau)^2}\left(\beta(1+\beta)(1+A\beta)\tau^4+\lambda^3(1-A\tau)-\lambda\tau^2(2+\tau+(1+A)\beta(1+2\tau)\right.$$
$$\left.+\beta^2(-1+3A\tau)\right)+\lambda^2\tau(1-2\beta+\tau+A(1+\tau+3\beta\tau))), \quad (15)$$
$$\Gamma(-1) = \frac{1}{\lambda(\lambda-\beta\tau)^2}\left(\beta(1+\beta)(1+A\beta)\tau^4+\lambda^3(4-A\tau)-\lambda\tau^2(4+\tau+2(1+A)\beta(1+\tau)\right.$$
$$\left.+\beta^2(-4+3A\tau)\right)+\lambda^2\tau(2-8\beta+\tau+A(2+\tau+3\beta\tau))).$$

By using Lemma 1, we obtain the following theorem: □

**Theorem 5.** *The equilibrium point $P_3$ of the system (3) is as follows:*

*(i) LAS if*

$$A < \frac{-\beta(1+\beta)\tau^3-\lambda^2(1+\tau)+\lambda\tau(2+\beta+\tau+2\beta\tau)}{-\lambda^3+\beta^2(1+\beta)\tau^3+\lambda^2(1+\tau+3\beta\tau)-\beta\lambda\tau(1+(2+3\beta)\tau)}, \quad (16)$$

*and one of the conditions listed below is satisfied:*

*(a) $\tau < \lambda \leq \tau(2+\tau)/1+\tau$ and $0 < \beta < -\tau+\lambda/\tau$,*
*(b) $\lambda > \tau(2+\tau)/1+\tau$ and $-\tau^2+\lambda+2\tau\lambda-\sqrt{\tau^4+6\tau^2\lambda+\lambda^2}/\,2\tau^2 < \beta < -\tau+\lambda/\tau$,*

*(ii) not a SP,*

*(iii) a source if one of the conditions listed below is satisfied:*

*(a) $\tau < \lambda \leq \tau(2+\tau)/1+\tau, 0 < \beta < -\tau+\lambda/\tau$ and*

$$\frac{-\beta(1+\beta)\tau^3-\lambda^2(1+\tau)+\lambda\tau(2+\beta+\tau+2\beta\tau)}{-\lambda^3+\beta^2(1+\beta)\tau^3+\lambda^2(1+\tau+3\beta\tau)-\beta\lambda\tau(1+(2+3\beta)\tau)} < A < -\frac{\tau}{\beta\tau-\lambda}, \quad (17)$$

*(b) $\lambda > \tau(2+\tau)/1+\tau, \beta \leq -\tau^2+\lambda+2\tau\lambda-\sqrt{\tau^4+6\tau^2\lambda+\lambda^2}/2\tau^2$, and $A < -\tau/\beta\tau-\lambda$,*

*(c) $\lambda > \tau(2+\tau)/1+\tau, -\tau^2+\lambda+2\tau\lambda-\sqrt{\tau^4+6\tau^2\lambda+\lambda^2}/2\tau^2 < \beta < -\tau+\lambda/\tau$, and*

$$\frac{-\beta(1+\beta)\tau^3-\lambda^2(1+\tau)+\lambda\tau(2+\beta+\tau+2\beta\tau)}{-\lambda^3+\beta^2(1+\beta)\tau^3+\lambda^2(1+\tau+3\beta\tau)-\beta\lambda\tau(1+(2+3\beta)\tau)} < A < -\frac{\tau}{\beta\tau-\lambda}, \quad (18)$$

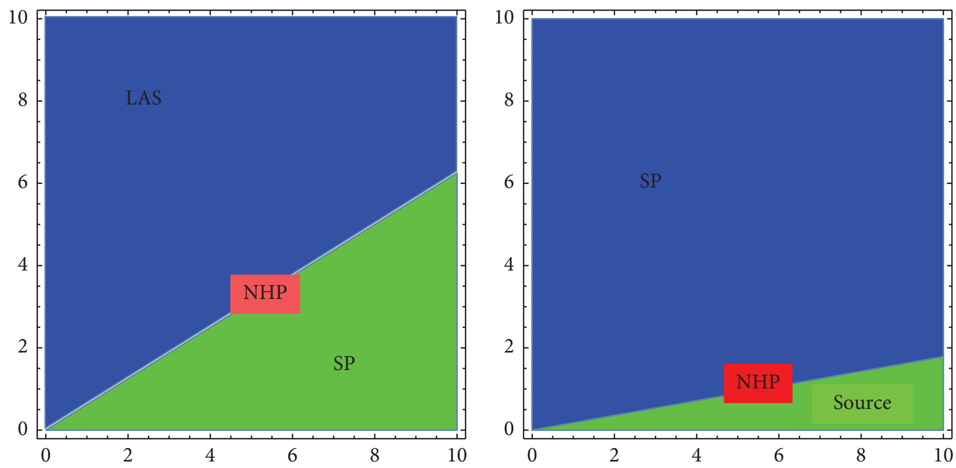



FIGURE 1: Topological classification of (a) $P_1$ and (b) $P_2$ by setting $m = 1.3, \beta = 0.6, A = 0.2, \tau \in [0, 10]$, and $\lambda \in [0, 10]$.

*(iv) NHP if*

$$A = \frac{-\beta(1+\beta)\tau^3 - \lambda^2(1+\tau) + \lambda\tau(2+\beta+\tau+2\beta\tau)}{-\lambda^3 + \beta^2(1+\beta)\tau^3 + \lambda^2(1+\tau+3\beta\tau) - \beta\lambda\tau(1+(2+3\beta)\tau)}, \tag{19}$$

*and one of the conditions listed below is satisfied:*

*(a)* $\tau < \lambda \le \tau(2+\tau)/1+\tau$ *and* $0 < \beta < -\tau + \lambda/\tau$,
*(b)* $\lambda > \tau(2+\tau)/1+\tau$ *and* $-\tau^2 + \lambda + 2\tau\lambda - \sqrt{\tau^4 + 6\tau^2\lambda + \lambda^2}/2\tau^2 < \beta < -\tau + \lambda/\tau$.

Topological classification of equilibrium point $P_3$ is depicted in Figure 2.

It can be inferred that in the event that condition (iv) of Theorem 5 is satisfied, the eigenvalues of $J(P_3)$ are complex numbers with a modulus of one. The system represented by equation (3) undergoes NS bifurcation at point $P_3$ when the parameters are altered in the vicinity of $\Omega_1$ or $\Omega_2$, where

$$\Omega_1 = \left\{ m, \lambda, \beta, \tau \in \mathbb{R}_+ \wedge A \in (0,1) \,\middle|\, \frac{A\lambda}{1+A\beta} < \tau < \frac{\lambda}{1+\beta}, \text{(iv} - \text{a) of Theorem 1 holds} \right\}, \tag{20}$$

$$\Omega_2 = \left\{ m, \lambda, \beta, \tau \in \mathbb{R}_+ \wedge A \in (0,1) \,\middle|\, \frac{A\lambda}{1+A\beta} < \tau < \frac{\lambda}{1+\beta}, \text{(iv} - \text{b) of Theorem 1 holds} \right\}. \tag{21}$$

Moreover, system (3) experiences period-doubling bifurcation at any equilibrium point $P$ if one of the eigenvalues of $J(P)$ is $-1$ and the other eigenvalue do not lie on the unit disk. Since all the eigenvalues of $J(P_0)$, $J(P_1)$, and $J(P_2)$ are non-negative, therefore there is no possibility of period-doubling bifurcation at $P_0, P_1$, and $P_2$. Moreover, the characteristic polynomial of $J(P_3)$ does not satisfy $\Gamma(-1) = 0$. Thus, condition (iv) of Lemma 1 implies that system (3) does not experience period-doubling bifurcation at $P_3$.

## 3. Neimark–Sacker Bifurcation at $P_3$

This section explores the possibility of bifurcation in the system (3). Bifurcations in predator-prey systems manifest as a result of alterations in the system's parameters. A slight modification of a parameter leads to a bifurcation. Bifurcations in predator-prey systems are a crucial component in forecasting the dynamics of wild populations and formulating viable approaches for their conservation. Improper management of bifurcations has the potential to cause

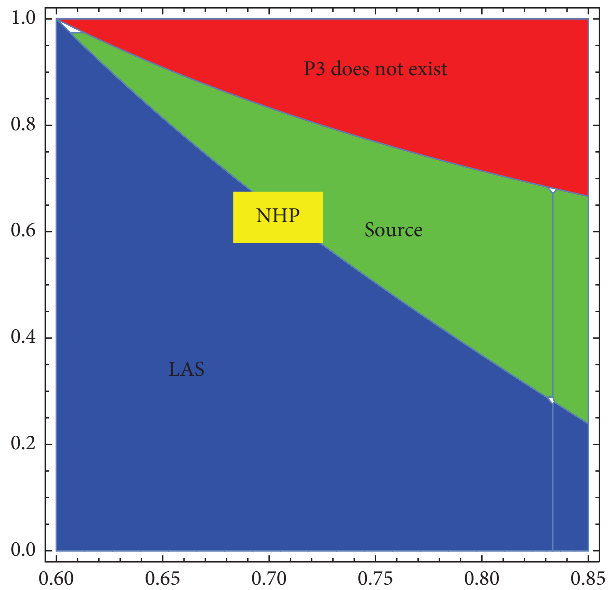



FIGURE 2: Topological classification of $P_3$ by setting $m = 1.3, \tau = 0.5, \beta = 0.2, \lambda \in [0.60, 0.85]$, and $A \in [0, 1]$.

significant disruptions to population dynamics and lead to the destruction of ecosystems. For a detailed bifurcation analysis, we recommend that readers refer to the sources [31–35].

The positive equilibrium point $P_3$ holds noteworthy biological significance as it denotes a crucial point in time where both predator and prey species are able to coexist and flourish. The study placed significant emphasis on comprehending the dynamics and behaviors associated with this particular fixed point. Our study is concentrated exclusively on the bifurcation phenomena that take place at $P_3$. The NS bifurcation located at point $P_3$ is analyzed through the utilization of the bifurcation parameter $A$ for the set $\Omega_1$. An analogous study can be established regarding the set denoted by $\Omega_2$.

By introducing a small change, denoted by $\gamma$ (where $|\gamma| \lll 1$), into the bifurcation parameter $A$, system (3) can be written as follows:

$$\begin{cases} x_{n+1} = x_n e^{(1-x_n)(x_n-(A+\gamma))-y_n/m}, \\ y_{n+1} = y_n e^{\lambda x_n/1+\beta x_n-\tau}. \end{cases} \tag{22}$$

We suppose that $x_n = u_n - \tau/\beta\tau - \lambda, y_n = v_n - m(\tau+\beta\tau-\lambda)(\tau+(A+\gamma)\beta\tau-(A+\gamma)\lambda)/(\beta\tau-\lambda)^2$ to translate the equilibrium point $P_3$ to origin. Due to the aforementioned transformation, the system represented by equation (22) transforms to

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 1 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \phi(u_n, v_n) \\ \varphi(u_n, v_n) \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned} a_{11} &= \frac{\lambda^3 + \left(1-\beta^3+\beta^2\gamma+\beta(2+\gamma)\right)\tau^3 + \lambda^2(-1-3\beta\tau+\gamma\tau) + \lambda\tau\left(\beta-2\tau+3\beta^2\tau-2\beta\gamma\tau-\gamma(1+\tau)\right)}{(\lambda-\beta\tau)\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)}, \\ a_{12} &= -\frac{\tau}{m\lambda - m\beta\tau}, \\ a_{21} &= -\frac{m(\lambda-(1+\beta)\tau)\left(\gamma\lambda^3-\beta^2(1+\beta)\gamma\tau^3-\lambda^2(-1+\gamma(1+\tau+3\beta\tau))+\lambda\tau\left(-1+3\beta^2\gamma\tau+\beta(-1+\gamma+2\gamma\tau)\right)\right)}{\lambda\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)}, \\ \phi(u_n, v_n) &= a_1 v_n^2 + a_2 v_n^3 + a_3 u_n v_n^2 + a_4 u_n v_n + a_5 u_n^2 v_n + a_6 u_n^2 + a_7 u_n^3, \\ \varphi(u_n, v_n) &= b_1 u_n v_n + b_2 u_n^2 v_n + b_3 u_n^2 + b_4 u_n^3, \end{aligned} \tag{24}$$

where

$$
\begin{aligned}
a_1 &= \frac{\tau}{2m^2(\lambda-\beta\tau)}, \\
a_2 &= -\frac{\tau}{6m^3(\lambda-\beta\tau)}, \\
a_3 &= \frac{\left(\lambda^3+\left(1-\beta^3+\beta^2\gamma+\beta(2+\gamma)\right)\tau^3+\lambda^2(-1-3\beta\tau+\gamma\tau)+\lambda\tau\left(\beta-2\tau+3\beta^2\tau-2\beta\gamma\tau-\gamma(1+\tau)\right)\right)}{\left(2m^2(\lambda-\beta\tau)\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)\right)}, \\
a_4 &= \frac{\left(-\lambda^3+(1+\beta)\left(-1+\beta^2-\beta(1+\gamma)\right)\tau^3+\lambda^2(1+3\beta\tau-\gamma\tau)+\lambda\tau\left(\gamma+2\tau-3\beta^2\tau+\gamma\tau+\beta(-1+2\gamma\tau)\right)\right)}{\left(m(\lambda-\beta\tau)\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)\right)}, \\
b_1 &= \frac{(\lambda-\beta\tau)^2}{\lambda}, \\
b_2 &= \frac{(\lambda-\beta\tau)^3(\lambda-\beta(2+\tau))}{2\lambda^2}, \\
a_5 &= \frac{\begin{pmatrix}
-2\lambda^5+(1+\beta)^2\left(-1-2\beta+3\beta^2+2\beta^3\right)\tau^5+\lambda^4(2+(7+10\beta)\tau)+2\lambda^2\tau\left(1+\left(3+8\beta+3\beta^2\right)\tau+\left(-1+9\beta+21\beta^2+10\beta^3\right)\tau^2\right) \\
-2(1+\beta)\lambda\tau^3\left(3\beta-2\tau+5\beta^3\tau+\beta^2(1+9\tau)\right)-2\lambda^3\tau\left(4+3\tau+10\beta^2\tau+\beta(3+14\tau)\right) \\
-\gamma^2\tau\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)^2-2\gamma\left(\lambda^5-\beta(1+\beta)^2\left(-1-\beta+\beta^2\right)\tau^5-\lambda^4(2+\tau+5\beta\tau)+\lambda^3\left(1+\tau+6\beta\tau+2\left(-1+2\beta+5\beta^2\right)\tau^2\right)\right. \\
\left.+(1+\beta)\lambda\tau^3\left(-1-\beta^2(-2+\tau)-\tau+5\beta^3\tau-\beta(1+5\tau)\right)-\lambda^2\tau\left(10\beta^3\tau^2+6\beta^2\tau(1+\tau)-\tau(2+3\tau)+\beta\left(1+2\tau-6\tau^2\right)\right)\right)
\end{pmatrix}}{\left(2m(\lambda-\beta\tau)\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)^2\right)}, \\
a_6 &= \frac{\begin{pmatrix}
2\lambda^5-(1+\beta)^2\left(-1-2\beta+3\beta^2+2\beta^3\right)\tau^5-\lambda^4(2+(7+10\beta)\tau) \\
-2\lambda^2\tau\left(1+\left(3+8\beta+3\beta^2\right)\tau+\left(-1+9\beta+21\beta^2+10\beta^3\right)\tau^2\right) \\
+2(1+\beta)\lambda\tau^3\left(3\beta-2\tau+5\beta^3\tau+\beta^2(1+9\tau)\right)+2\lambda^3\tau\left(4+3\tau+10\beta^2\tau+\beta(3+14\tau)\right) \\
+\gamma^2\tau\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)^2+2\gamma\left(\lambda^5-\beta(1+\beta)^2\left(-1-\beta+\beta^2\right)\tau^5\right. \\
-\lambda^4(2+\tau+5\beta\tau)+\lambda^3\left(1+\tau+6\beta\tau+2\left(-1+2\beta+5\beta^2\right)\tau^2\right) \\
+(1+\beta)\lambda\tau^3\left(-1-\beta^2(-2+\tau)-\tau+5\beta^3\tau-\beta(1+5\tau)\right) \\
\left.-\lambda^2\tau\left(10\beta^3\tau^2+6\beta^2\tau(1+\tau)-\tau(2+3\tau)+\beta\left(1+2\tau-6\tau^2\right)\right)\right)
\end{pmatrix}}{\left(2(\lambda-\beta\tau)\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)^2\right)}, \\
b_3 &= -\frac{\left(m(\lambda-\beta\tau)(\lambda-(1+\beta)\tau)(\lambda-\beta(2+\tau))\begin{pmatrix}\gamma\lambda^3-\beta^2(1+\beta)\gamma\tau^3-\lambda^2(-1+\gamma(1+\tau+3\beta\tau)) \\ +\lambda\tau\left(-1+3\beta^2\gamma\tau+\beta(-1+\gamma+2\gamma\tau)\right)\end{pmatrix}\right)}{\left(2\lambda^2\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)\right)}, \\
b_4 &= -\frac{\begin{pmatrix}m(\lambda-\beta\tau)^2(\lambda-(1+\beta)\tau)\left(\lambda^2-2\beta\lambda(3+\tau)+\beta^2\left(6+6\tau+\tau^2\right)\right) \\ \left(\gamma\lambda^3-\beta^2(1+\beta)\gamma\tau^3-\lambda^2\left((-1+\gamma(1+\tau+3\beta\tau))+\lambda\tau\left(-1+3\beta^2\gamma\tau+\beta(-1+\gamma+2\gamma\tau)\right)\right)\right)\end{pmatrix}}{\left(6\lambda^3\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)\right)}, \\
a_7 &= \frac{\begin{pmatrix}
3\left(-1+2\gamma+\gamma^2\right)\lambda^7-(1+\beta)^3\left(-1-3\beta(1+\gamma)-3\beta^2\left(-2+2\gamma+\gamma^2\right)+3\beta^4\left(-1+2\gamma+\gamma^2\right)-\beta^3\left(-11-9\gamma+3\gamma^2+\gamma^3\right)\right)\tau^7 \\
+\lambda^6\left(15+(-2+21\beta)\tau+\gamma^3\tau-3\gamma^2(3+(2+7\beta)\tau)-3\gamma(4+(9+14\beta)\tau)\right) \\
-3\lambda^5\begin{pmatrix}6+(4+25\beta)\tau+\left(-10-4\beta+21\beta^2\right)\tau^2+\gamma^3\tau(1+\tau+2\beta\tau) \\ -\gamma^2\left(3+(4+15\beta)\tau+\left(-1+12\beta+21\beta^2\right)\tau^2\right)-\gamma\left(2+(17+20\beta)\tau+\left(13+54\beta+42\beta^2\right)\tau^2\right)\end{pmatrix} \\
+3(1+\beta)^2\lambda\tau^5\begin{pmatrix}-2\tau+7\beta^4\left(-1+2\gamma+\gamma^2\right)\tau-\gamma(1+\tau)-\beta\left(-5+2\gamma^2(1+\tau)+\gamma(2+8\tau)\right) \\ -\beta^2\left(\gamma(-9+\tau)+\gamma^3(1+\tau)-3(2+7\tau)+\gamma^2(2+8\tau)\right)+\beta^3\left(-5+\gamma^2(3-2\tau)+18\tau-2\gamma^3\tau+\gamma(4+26\tau)\right)\end{pmatrix} \\
-3(1+\beta)\lambda^2\tau^3\begin{pmatrix}2+5\tau-2\tau^2+21\beta^4\left(-1+2\gamma+\gamma^2\right)\tau^2-\gamma^2(1+\tau)^2-\gamma\tau(4+5\tau)+\beta^3\tau\left(-25+31\tau-5\gamma^3\tau+3\gamma^2(5+3\tau)+\gamma(20+93\tau)\right) \\ -\beta\left(-2-27\tau-21\tau^2+\gamma^3(1+\tau)^2+\gamma^2\left(1+10\tau+11\tau^2\right)+\gamma\left(-8-12\tau+13\tau^2\right)\right) \\ +\beta^2\left(-6+9\tau+69\tau^2-\gamma^3\tau(4+5\tau)+\gamma^2\left(3+\tau-19\tau^2\right)+\gamma\left(2+48\tau+37\tau^2\right)\right)\end{pmatrix} \\
+3\lambda^4\begin{pmatrix}2+2(2+9\beta)\tau+2\left(-6+8\beta+25\beta^2\right)\tau^2+5\left(-3-10\beta-2\beta^2+7\beta^3\right)\tau^3+\gamma^3\tau\left(1+(2+4\beta)\tau+\left(1+5\beta+5\beta^2\right)\tau^2\right) \\ -\gamma^2\left(1+(2+9\beta)\tau+\left(-3+16\beta+30\beta^2\right)\tau^2+5\left(-1-\beta+6\beta^2+7\beta^3\right)\tau^3\right)-\gamma\tau\left(10+20\tau+4\tau^2+70\beta^3\tau^2+5\beta^2\tau(8+27\tau)+\beta\left(6+68\tau+65\tau^2\right)\right)\end{pmatrix} \\
+\lambda^3\tau\begin{pmatrix}105\beta^4\left(-1+2\gamma+\gamma^2\right)\tau^3-\gamma^3(1+\tau)^3-6\gamma^2\tau\left(1+3\tau+2\tau^2\right)+\tau\left(12+48\tau+19\tau^2\right)+6\gamma\left(1+4\tau+2\tau^2-3\tau^3\right) \\ -10\beta^3\tau^2\left(15-4\tau+2\gamma^3\tau-3\gamma^2(3+4\tau)-6\gamma(2+9\tau)\right)-3\beta^2\tau\left(18+24\tau-100\tau^2+2\gamma^3\tau(3+5\tau)+\gamma^2\left(-9-24\tau+10\tau^2\right)-2\gamma\left(3+51\tau+65\tau^2\right)\right) \\ -3\beta\left(2+8\tau-36\tau^2-60\tau^3+2\gamma^3\tau\left(1+3\tau+2\tau^2\right)-4\gamma\tau\left(5+15\tau+4\tau^2\right)+\gamma^2\left(-1-4\tau+9\tau^2+20\tau^3\right)\right)\end{pmatrix}
\end{pmatrix}}{\left(6(\lambda-\beta\tau)\left(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)\right)^3\right)},
\end{aligned}
\tag{25}
$$

Let,

$$\varsigma^2 - p(\gamma)\varsigma + q(\gamma) = 0. \quad (26)$$

Be the characteristic equation of the variational matrix of system (23) calculated at origin, where.

$$p(\gamma) = -\frac{\left(\left(-2\lambda^3 + (1+\beta)\left(-1 + 2\beta^2 - \beta(1+\gamma)\right)\tau^3 + \lambda^2(2 + \tau + 6\beta\tau - \gamma\tau) + \lambda\tau\left(\gamma + 2\tau - 6\beta^2\tau + \gamma\tau + 2\beta(-1 + (-1+\gamma)\tau)\right)\right)}{\left(\left(\lambda - \beta\tau\right)\left(\lambda^2 + \beta(1+\beta)\tau^2 - \lambda(1 + \tau + 2\beta\tau)\right)\right)},$$

$$q(\gamma) = \frac{-\beta(1+\beta)\gamma\tau^3 + \lambda^2(1 - \gamma\tau) + \lambda\tau(-\beta + \gamma + \gamma\tau + 2\beta\gamma\tau)}{\lambda(\lambda - \beta\tau)}. \quad (27)$$

The solutions of (26) are as follows:

$$\varsigma_{1,2} = \frac{p(\gamma)}{2} \pm \frac{i}{2}\sqrt{4q(\gamma) - p^2(\gamma)}. \quad (28)$$

Because $(A, m, \lambda, \beta, \tau) \in \Omega_1$, thus we have $|\varsigma_{1,2}| = 1$ and

$$\left(\frac{\mathrm{d}|\varsigma_1|}{\mathrm{d}\gamma}\right)_{\gamma=0} = \left(\frac{\mathrm{d}|\varsigma_2|}{\mathrm{d}\gamma}\right)_{\gamma=0} = \frac{\tau\left(\lambda^2 + \beta(1+\beta)\tau^2 - \lambda(1 + \tau + 2\beta\tau)\right)}{2\lambda(\beta\tau - \lambda)} > 0. \quad (29)$$

In addition, it is required that $\varsigma_{1,2}^i \neq 1$ $(i = 1, 2, 3, 4)$ when $\gamma$ equals zero. This condition is comparable to $p(0) \neq \pm 2, -1, 0$. Because $(A, m, \lambda, \beta, \tau) \in \Omega_1$, therefore,

$$p(0) = \frac{2\lambda^3 - \left(-1 - 2\beta + \beta^2 + 2\beta^3\right)\tau^3 - \lambda^2(2 + \tau + 6\beta\tau) + 2\lambda\tau\left(\beta - \tau + \beta\tau + 3\beta^2\tau\right)}{(\lambda - \beta\tau)\left(\lambda^2 + \beta(1+\beta)\tau^2 - \lambda(1 + \tau + 2\beta\tau)\right)} \neq \pm 2, -1, 0. \quad (30)$$

Subsequently, the transformation employed to convert the linear part of equation (23) into canonical form at $\gamma = 0$ is as follows:

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} a_{12} & 0 \\ \xi - a_{11} & -\eta \end{bmatrix}\begin{bmatrix} e_n \\ f_n \end{bmatrix}, \quad (31)$$

where

$$\xi = -\frac{-2\lambda^3 + \left(-1 - 2\beta + \beta^2 + 2\beta^3\right)\tau^3 + \lambda^2(2 + \tau + 6\beta\tau) - 2\lambda\tau\left(\beta - \tau + \beta\tau + 3\beta^2\tau\right)}{2(\lambda - \beta\tau)\left(\lambda^2 + \beta(1+\beta)\tau^2 - \lambda(1 + \tau + 2\beta\tau)\right)},$$

$$\eta = \frac{1}{2(\lambda - \beta\tau)}\sqrt{4(\lambda - \beta\tau)^2 - \frac{\left(-2\lambda^3 + \left(-1 - 2\beta + \beta^2 + 2\beta^3\right)\tau^3 + \lambda^2(2 + \tau + 6\beta\tau) - 2\lambda\tau\left(\beta - \tau + \beta\tau + 3\beta^2\tau\right)\right)^2}{\left(\lambda^2 + \beta(1+\beta)\tau^2 - \lambda(1 + \tau + 2\beta\tau)\right)^2}}. \quad (32)$$

When (23) is transformed by (31), the resulting system is as follows:

$$\begin{bmatrix} e_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} \xi & -\eta \\ \eta & \xi \end{bmatrix}\begin{bmatrix} e_n \\ f_n \end{bmatrix} + \begin{bmatrix} \chi(e_n, f_n) \\ \Upsilon(e_n, f_n) \end{bmatrix}, \quad (33)$$

where

$$\chi(e_n, f_n) = c_1 f_n^2 + c_2 f_n^3 + c_3 e_n f_n + c_4 e_n f_n^2 + c_5 e_n^2 + c_6 e_n^2 f_n + c_7 e_n^3,$$
$$\Upsilon(e_n, f_n) = d_1 f_n^2 + d_2 f_n^3 + d_3 e_n f_n^2 + d_4 e_n f_n + d_5 e_n^2 f_n + d_6 e_n^2 + d_7 e_n^3, \quad (34)$$

where

$$
\begin{aligned}
& c_1 = -\frac{\eta^2}{2m}, \\
& c_2 = -\frac{\eta^3}{6m^2}, \\
& c_3 = \frac{\eta\xi}{m}, \\
& c_4 = \frac{\eta^2\xi}{2m^2}, \\
& c_5 = \frac{-\lambda^2(-1+\xi^2) + 2\beta\lambda(-1+\xi^2)\tau + (2-\beta^2(-1+\xi^2))\tau^2}{2m(\lambda-\beta\tau)^2}, \\
& c_6 = -\frac{\eta(\lambda^2(-1+\xi^2) - 2\beta\lambda(-1+\xi^2)\tau + (-2+\beta^2(-1+\xi^2))\tau^2)}{2m^2(\lambda-\beta\tau)^2}, \\
& c_7 = \frac{\lambda^2(2-3\xi+\xi^3) - 2\beta\lambda(2-3\xi+\xi^3)\tau + (-6\xi+\beta^2(2-3\xi+\xi^3))\tau^2}{6m^2(\lambda-\beta\tau)^2}, \\
& d_1 = \frac{\eta(-\xi+\lambda^3+(1+2\beta-\beta^3)\tau^3 - \lambda^2(1+3\beta\tau) + \lambda\tau(\beta-2\tau+3\beta^2\tau)/(\lambda-\beta\tau)(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)))}{2m}, \\
& d_2 = \frac{\eta^2(-\xi+\lambda^3+(1+2\beta-\beta^3)\tau^3 - \lambda^2(1+3\beta\tau) + \lambda\tau(\beta-2\tau+3\beta^2\tau)/(\lambda-\beta\tau)(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)))}{6m^2}, \\
& d_3 = \frac{(\eta\xi(\lambda^3(-1+\xi) - (1+\beta)(1+\beta+\beta^2(-1+\xi))\tau^3 - \lambda^2(-1-3\beta\tau+\xi(1+\tau+3\beta\tau)) + \lambda\tau 2\tau + 3\beta^2(-1+\xi)\tau + \beta(-1+\xi+2\xi\tau)))}{(2m^2(\lambda-\beta\tau)(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)))}, \\
& d_4 = \frac{(\lambda^4(-\xi+\xi^2-\tau) - \beta^3(1+\beta)\tau^5 + \lambda\tau^3(-\xi-2\beta\xi+\beta^2(1-\xi^2+3\tau)+\beta^3(\xi-\xi^2+4\tau)) + \lambda^3(\xi+3\beta\xi\tau-\xi^2(1+\tau+3\beta\tau)+\tau(1+\tau+4\beta\tau)) + \lambda^2\tau(2\xi\tau+3\beta^2(-\xi+\xi^2-2\tau)\tau+\beta(-\xi+\xi^2(1+2\tau)-\tau(2+3\tau))))}{(m\lambda(\lambda-\beta\tau)(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)))}, \\
& \quad (-\beta^6(1+\beta)\tau 0^8(2+\tau) + \lambda^7 - 1 + \xi + \xi^2 - \xi^3 + \tau^2) + \beta^5\lambda\tau^6(2+(11+12\beta)\tau+(6+7\beta)\tau^2) \\
& \quad +\lambda^6(1-\tau^2-\tau^3-\xi^2(1+5\beta\tau)-\xi(1+\tau+5\beta\tau)+\xi^3(1+\tau+5\beta\tau)+\beta\tau(5-2\tau-7\tau^2)) \\
& \quad -\lambda^2\tau^5(2+4\beta-2\beta^3(-1+\xi)\xi+\beta^2(1+2\xi-\xi^2)+\beta^4(8+\xi-\xi^3+25\tau+15\tau^2)+\beta^5(-1+\xi+\xi^2-\xi^3+30\tau+21\tau^2)) \\
& \quad +\lambda^3\tau^3(4\tau-6\beta^2(-1+\xi)\xi\tau+2\beta(-1+\xi+\tau+2\xi\tau-\xi^2\tau)+5\beta^4\tau(-1+\xi+\xi^2-\xi^3+8\tau+7\tau^2)+\beta^3(-1+\xi+\xi^2+12\tau+4\xi\tau+30\tau^2+20\tau^3-\xi^3(1+4\tau))) \\
& \quad +\lambda^5\tau(-2(-1+\xi)\xi\tau+\beta^2\tau(-10+10\xi+10\xi^2-10\xi^3+12\tau+21\tau^2) \\
& \quad +\beta(-3+3\xi^2+2\tau+7\tau^2+6\tau^3+\xi(3+4\tau)-\xi^3(3+4\tau)) - \lambda^4\tau^2, \\
& d_5 = \frac{(-2+\tau-6\beta(-1+\xi)\xi\tau-\xi^2\tau+2\xi(1+\tau)+5\beta^3\tau(-2+2\xi+2\xi^2-2\xi^3+6\tau+7\tau^2)+\beta^2(-3+3\xi^2+8\tau+20\tau^2+15\tau^3-3\xi^3(1+2\tau)+\xi(3+6\tau)))}{(2m^2\lambda^2(\lambda-\beta\tau)^3(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)))} \\
& \quad -\lambda^6(-1+\xi)(-1+\xi^2-2\tau)+\beta^3(1+\beta)\tau^6(-2+\beta(-2+\tau)+\tau+2\beta^2(-1+\xi)\tau) \\
& \quad +\lambda^5(1+(2+5\beta)\tau+(1+12\beta)\tau^2-\xi^2(1+5\beta\tau)+\xi^3(1+\tau+5\beta\tau)-\xi(1+(3+5\beta)\tau+2(1+6\beta)\tau^2))+\lambda\tau^5 \\
& \quad (-2-4\beta+\beta^5(-1+\xi)(-1+\xi^2-12\tau)+2\beta^3(6-\xi+\xi^2-4\tau)+\beta^2(3-2\xi+\xi^2-3\tau)+\beta^4(\xi^3+5(2+\tau)-\xi(3+10\tau))) \\
& \quad +\lambda^4\tau(-10\beta^2(-1+\xi)(-1+\xi^2-3\tau)\tau+2\tau(\xi-\xi^2+\tau)+\beta(-3+3\xi^2-10\tau-5\tau^2-\xi^3(3+4\tau)+\xi(3+12\tau+10\tau^2))) \\
& \quad +\lambda^2\tau^3(4\tau-5\beta^4(-1+\xi)(-1+\xi^2-6\tau)\tau+6\beta^2\tau(-2+\xi-\xi^2+2\tau)+\beta(-2-2\xi^2\tau+3\tau^2+\xi(2+4\tau))+\beta^3(-1+\xi^2-20\tau-10\tau^2-\xi^3(1+4\tau)+\xi(1+12\tau+20\tau^2)))), \\
& d_6 = \frac{-\lambda^3\tau^2(-2+\tau-\xi^2\tau-10\beta^3(-1+\xi)(-1+\xi^2-4\tau)\tau+\tau^2+2\xi(1+\tau)+2\beta\tau(-2+3\xi-3\xi^2+4\tau)+\beta^2(-3+3\xi^2-20\tau-10\tau^2-3\xi^3(1+2\tau)+\xi(3+18\tau+20\tau^2)))}{(2m\eta\lambda(\lambda-\beta\tau)^3(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)))} \\
& \quad \lambda^7(-1+\xi)(2-3\xi+\xi^3-3\tau^2)+\beta^4(1+\beta)\tau^7(3\beta^2(-1+\xi)\tau(2+\tau)+2(-3+\tau^2)+2\beta(-3+\tau^2)) \\
& \quad -\lambda^6-2+3\tau^2+\tau^3-\xi^3(1+5\beta\tau)-3\xi^2(1+\tau+5\beta\tau)+\xi^4(1+\tau+5\beta\tau)+\beta\tau(-10+6\tau+21\tau^2) \\
& \quad +\xi(5+(2+25\beta)\tau-3(1+2\beta)\tau^2-3(1+7\beta)\tau^3)+\beta^3\lambda\tau^6 \\
& \quad (12-8\tau^2-3\beta^3(-1+\xi)\tau(12+7\tau)-4\beta(-9+5\tau^2)+\beta^2(30+33\tau+6\tau^2-3\xi(2+11\tau+6\tau^2)))+\lambda^2\tau^5 6\xi+12\beta\xi-\beta^5(-1+\xi) \\
& \quad (2-3\xi+\xi^3-90\tau-63\tau^2)-2\beta^3(20-3\xi^2+\xi^3-20\tau^2)+\beta^2(-8+3\xi+6\xi^2-\xi^3+12\tau^2)-\beta^4 \\
& \quad (-3\xi^2+\xi^4+15(4+5\tau+\tau^2)-\xi(22+75\tau+45\tau^2))+\lambda^5\tau(\beta^2(-1+\xi)\tau(20-30\xi+10\xi^3-36\tau-63\tau^2)) \\
& \quad +2\tau(2-3\xi^2+\xi^3-2\tau^2)+\beta(-6-3\xi^3+6\tau+21\tau^2+6\tau^3-3\xi^2(3+4\tau)+\xi^4(3+4\tau)+\xi(15+2\tau-21\tau^2-18\tau^3)) \\
& \quad +\lambda^4\tau^2(3\xi(-2+\tau)-\xi^3\tau+6\xi^2(1+\tau)-5\beta^3(-1+\xi)\tau(4-6\xi+2\xi^3-18\tau-21\tau^2)) \\
& \quad +2\tau(-1+\tau^2)+2\beta\tau(-6+9\xi^2-3\xi^3+10\tau^2)-3\beta^2(-2-\xi^3+10\tau+20\tau^2+5\tau^3-3\xi^2(1+2\tau)+\xi^4(1+2\tau)-\xi(-5+4\tau+20\tau^2+15\tau^3)) \\
& \quad -\lambda^3\tau^3(12\xi\tau-5\beta^4(-1+\xi)\tau(2-3\xi+\xi^3-24\tau-21\tau^2)) \\
& \quad +2\beta^2\tau(-12+9\xi^2-3\xi^3+20\tau^2)+\beta(6\xi(-1+\tau)-4\tau-2\xi^3\tau+8\tau^3+6\xi^2(1+2\tau))+\beta^3, \\
& d_7 = \frac{(2+\xi^3-60\tau-90\tau^2-20\tau^3+3\xi^2(1+4\tau)-\xi^4(1+4\tau)+\xi(-5+28\tau+90\tau^2+60\tau^3))}{(6m^2\eta\lambda^2(\lambda-\beta\tau)^3(\lambda^2+\beta(1+\beta)\tau^2-\lambda(1+\tau+2\beta\tau)))}
\end{aligned}
\tag{35}
$$

For the occurrence of NS bifurcation in the system (33), it is necessary that the aforementioned quantity is not equal to zero:

$$L = \left[ -\mathrm{Re}\left( \frac{(1-2\varsigma_1)\varsigma_2^2}{1-\varsigma_1} N_{20} N_{11} \right) - \frac{1}{2}\left|N_{11}\right|^2 - \left|N_{02}\right|^2 + \mathrm{Re}\left(\varsigma_2 N_{21}\right) \right]_{\gamma=0}, \tag{36}$$

where

$$\begin{aligned} N_{20} &= \frac{1}{8}\left[\chi_{e_n e_n} - \chi_{f_n f_n} + 2\Upsilon_{e_n f_n} + i\left(\Upsilon_{e_n e_n} - \Upsilon_{f_n f_n} - 2\chi_{e_n f_n}\right)\right], \\ N_{11} &= \frac{1}{4}\left[\chi_{e_n e_n} + \chi_{f_n f_n} + i\left(\Upsilon_{e_n e_n} + \Upsilon_{f_n f_n}\right)\right], \\ N_{02} &= \frac{1}{8}\left[\chi_{e_n e_n} - \chi_{f_n f_n} - 2\Upsilon_{e_n f_n} + i\left(\Upsilon_{e_n e_n} - \Upsilon_{f_n f_n} + 2\chi_{e_n f_n}\right)\right], \\ N_{21} &= \frac{1}{16}\left[\chi_{e_n e_n e_n} + \chi_{e_n f_n f_n} + \Upsilon_{e_n e_n f_n} + \Upsilon_{f_n f_n f_n} + i\left(\Upsilon_{e_n e_n e_n} + \Upsilon_{e_n f_n f_n} - \chi_{e_n e_n f_n} - \chi_{f_n f_n f_n}\right)\right]. \end{aligned} \tag{37}$$

Based on the analytical approach discussed earlier, we can assert the following theorem as a result.

**Theorem 6.** *We assume that* $(A, m, \lambda, \beta, \tau) \in \Omega_1$ *and* $L \neq 0$, *then (3) experiences NS bifurcation at* $P_3$ *when the bifurcation parameter* $A$ *is close to*

$$A_0 = \frac{-\beta(1+\beta)\tau^3 - \lambda^2(1+\tau) + \lambda\tau(2+\beta+\tau+2\beta\tau)}{-\lambda^3 + \beta^2(1+\beta)\tau^3 + \lambda^2(1+\tau+3\beta\tau) - \beta\lambda\tau(1+(2+3\beta)\tau)}. \tag{38}$$

Furthermore, in the case of $L$ being negative, an attracting invariant curve emerges from $P_3$ when $A$ exceeds $A_0$, while in the case of $L$ being positive, a distancing invariant curve emerges from $P_3$ when $A$ is less than $A_0$.

## 4. Bifurcation and Chaos Control

The utilization of chaos control techniques is prevalent across various domains of applied research and engineering. The objective is to minimize disorder and enhance the efficiency of dynamic systems based on specific performance measures. Historically, unstable fluctuations and bifurcations have been viewed as unfavorable occurrences that hinder the propagation of biological populations and are deemed undesirable in the field of mathematical biology. The construction of a controller capable of modifying the bifurcation characteristics of a specific nonlinear dynamical system with the aim of inducing order from the chaotic state that arises from NS bifurcation is a feasible undertaking. Consequently, the attainment of specific dynamic characteristics becomes feasible.

We used the feedback and hybrid control techniques for controlling bifurcation and chaos in the system. Initially, the state feedback control approach is employed [36, 37]. For this, corresponding to (3), we consider the following controlled model:

$$\begin{cases} x_{n+1} = x_n e^{(1-x_n)(x_n-A) - y_n/m} - U_n, \\ y_{n+1} = y_n e^{\lambda x_n/1+\beta x_n - \tau}, \end{cases} \tag{39}$$

where $U_n = H(x_n + \tau/\beta\tau - \lambda) + P(y_n + m(\tau + \beta\tau - \lambda)\ (\tau + A\beta\tau - A\lambda)/(\beta\tau - \lambda)^2)$ is feedback controlling force and $H$ and $P$ are feedback gains. The Jacobian matrix of (39) at $P_3$ is given by

$$J(P_3) = \begin{bmatrix} \dfrac{-(-1+H)\lambda^2 + (1+A+2(-1+H)\beta)\lambda\tau - \left(2+(1+A)\beta+(-1+H)\beta^2\right)\tau^2}{(\lambda-\beta\tau)^2} & -P - \dfrac{\tau}{m\lambda - m\beta\tau} \\ \dfrac{m(\lambda-(1+\beta)\tau)(-\tau+A(\lambda-\beta\tau))}{\lambda} & 1 \end{bmatrix}. \quad (40)$$

The characteristic equation of $J(E_2)$ is given by

$$\varsigma^2 + T_1\varsigma + T_0 = 0, \quad (41)$$

where

$$T_1 = \frac{(-2+H)\lambda^2 - (1+A+2(-2+H)\beta)\lambda\tau + \left(2+(1+A)\beta+(-2+H)\beta^2\right)\tau^2}{(\lambda-\beta\tau)^2},$$

$$\begin{aligned} T_0 = \frac{1}{\lambda(\lambda-\beta\tau)^2}&\Big(-AmP\lambda^4 - \beta(1+\beta)(1+A\beta)(-1+mP\beta)\tau^4 + \lambda^3(1-H+mP\tau+A(-1+mP \\ &+ 4mP\beta)\tau) + \lambda^2\tau\left(1+A+\tau-mP\tau+A\left(1+(3-3mP)\beta-6mP\beta^2\right)\tau+\beta(-2+2H-3mP\tau)\right) \\ &+ \lambda\tau^2\left(-2-\tau+4AmP\beta^3\tau-\beta(1+A+2A\tau+(2-2mP)\tau)+\beta^2(1-H+3mP\tau+3A(-1+mP)\tau)\right)\Big). \end{aligned} \quad (42)$$

Let $\varsigma_1$ and $\varsigma_2$ be the roots of (41), then we have

$$\varsigma_1 + \varsigma_2 = \frac{(-2+H)\lambda^2 - (1+A+2(-2+H)\beta)\lambda\tau + \left(2+(1+A)\beta+(-2+H)\beta^2\right)\tau^2}{(\lambda-\beta\tau)^2}, \quad (43)$$

$$\varsigma_1\varsigma_2 = \frac{1}{\lambda(\lambda-\beta\tau)^2}\Big(-AmP\lambda^4 - \beta(1+\beta)(1+A\beta)(-1+mP\beta)\tau^4 + \lambda^3(1-H+mP\tau+A(-1+mP \quad (44)$$

$$\begin{aligned} &+ 4mP\beta)\tau) + \lambda^2\tau\left(1+A+\tau-mP\tau+A\left(1+(3-3mP)\beta-6mP\beta^2\right)\tau+\beta(-2+2H-3mP\tau)\right) \\ &+ \lambda\tau^2\left(-2-\tau+4AmP\beta^3\tau-\beta(1+A+2A\tau+(2-2mP)\tau)+\beta^2(1-H+3mP\tau+3A(-1+mP)\tau)\right)\Big). \end{aligned} \quad (45)$$

To derive the lines of marginal stability, it is necessary to solve the equations $\varsigma_1 = \pm 1$ and $\varsigma_1\varsigma_2 = 1$. These constraints ensure that the absolute values of $\varsigma_1$ and $\varsigma_2$ are both less than one. Assuming that $\varsigma_1\varsigma_2 = 1$, it follows from equation (44) that

$$L_1\colon\ -H + \left(-\frac{m(\lambda-(1+\beta)\tau)(-\tau+A(\lambda-\beta\tau))}{\lambda}\right)P + L_{10} = 0, \quad (46)$$

where $L_{10} = \tau(-A\lambda^3 + \beta(1+\beta)(1+A\beta)\tau^3 - \lambda\tau(2+\tau + 3A\beta^2\tau + (1+A)\beta(1+2\tau)) + \lambda^2(1+\tau+A(1+\tau+3\beta\tau)))/\lambda(\lambda-\beta\tau)^2$.

Next, we suppose that $\varsigma_1 = 1$ and using equations (43) and (44), we obtain

$$L_2\colon\ \left(\frac{m(\lambda-(1+\beta)\tau)(-\tau+A(\lambda-\beta\tau))}{\lambda}\right)P + \frac{\tau(-\lambda+\tau+\beta\tau)(-A\lambda+\tau+A\beta\tau)}{\lambda(\lambda-\beta\tau)} = 0. \quad (47)$$

Finally, if $\varsigma_1 = -1$ and using equations (43) and (44), we obtain

$$L_3\colon \ -2H + \left(-\frac{m(\lambda - (1+\beta)\tau)(-\tau + A(\lambda - \beta\tau))}{\lambda}\right)P + L_{30} = 0, \tag{48}$$

where

$$L_{30} = \frac{1}{\lambda(\lambda - \beta\tau)^2}\left(\beta(1+\beta)(1+A\beta)\tau^4 + \lambda^3(4 - A\tau) - \lambda\tau^2(4 + \tau + 2(1+A)\beta(1+\tau) + \beta^2(-4 + 3A\tau)\right) + \lambda^2\tau(2 - 8\beta + \tau + A(2 + \tau + 3\beta\tau))\Big). \tag{49}$$

It is clear that stable eigenvalues are located inside the triangular area defined by the straight lines $L_1, L_2$, and $L_3$.

Next, the hybrid control approach [38] is employed to regulate chaos through bifurcation effects. This is carried out with the aim of inducing order in the system described by system (3), which exhibits a state of disorder. We assume the following controlled system corresponding to system (3):

$$\begin{cases} x_{n+1} = (1-\rho)x_n + \rho x_n e^{(1-x_n)(x_n - A) - y_n/m}, \\ y_{n+1} = (1-\rho)y_n + \rho y_n e^{\lambda x_n/1+\beta x_n - \tau}, \end{cases} \tag{50}$$

where $\rho \in (0,1)$. Controlled system (50) and uncontrolled system (3) have the same equilibrium points. The variational matrix of the controlled system evaluated at its positive equilibrium point $P_3$ is provided by

$$J(P_3) = \begin{bmatrix} \dfrac{\lambda^2 + \lambda(-2\beta + \rho + A\rho)\tau + \left(\beta^2 - 2\rho - (1+A)\beta\rho\right)\tau^2}{(\lambda - \beta\tau)^2} & -\dfrac{\rho\tau}{m\lambda - m\beta\tau} \\ -\dfrac{m\rho(\lambda - (1+\beta)\tau)(-\tau + A(\lambda - \beta\tau))}{\lambda} & 1 \end{bmatrix}. \tag{51}$$

The characteristic polynomial of matrix $J(P_3)$ is as follows:

$$\Gamma(\varsigma) = \varsigma^2 + T_1\varsigma + T_0, \tag{52}$$

where

$$\begin{aligned} T_1 &= \frac{-2\lambda^2 + \lambda(4\beta - (1+A)\rho)\tau + \left(-2\beta^2 + 2\rho + (1+A)\beta\rho\right)\tau^2}{(\lambda - \beta\tau)^2}, \\ T_0 &= \frac{1}{\lambda(\lambda - \beta\tau)^2}\Big(\beta(1+\beta)(1+A\beta)\rho^2\tau^4 + \lambda^3\left(1 - A\rho^2\tau\right) + \lambda^2\tau\left((1+A)\rho(1+\rho\tau) + \beta\left(-2 + 3A\rho^2\tau\right)\right) \\ &\quad - \lambda\tau^2\left(\rho(2+\rho\tau) + (1+A)\beta\rho(1+2\rho\tau) + \beta^2\left(-1 + 3A\rho^2\tau\right)\right)\Big). \end{aligned} \tag{53}$$

By simple computations, we obtain

$$
\begin{aligned}
\Gamma(1) &= -\frac{\rho^2\tau(-\lambda+\tau+\beta\tau)(-A\lambda+\tau+A\beta\tau)}{\lambda(\lambda-\beta\tau)},\\
\Gamma(-1) &= \frac{1}{\lambda(\lambda-\beta\tau)^2}\Big(\beta(1+\beta)(1+A\beta)\rho^2\tau^4+\lambda^3\big(4-A\rho^2\tau\big)+\lambda^2\tau\big((1+A)\rho(2+\rho\tau)+\beta\big(-8+3A\rho^2\tau\big)\big)\\
&\quad -\lambda\tau^2\big(2(1+A)\beta\rho(1+\rho\tau)+\rho(4+\rho\tau)+\beta^2\big(-4+3A\rho^2\tau\big)\big)\Big),\\
\Gamma(0) &= \frac{1}{\lambda(\lambda-\beta\tau)^2}\Big(\beta(1+\beta)(1+A\beta)\rho^2\tau^4+\lambda^3\big(1-A\rho^2\tau\big)+\lambda^2\tau\big((1+A)\rho(1+\rho\tau)+\beta\big(-2+3A\rho^2\tau\big)\big)\\
&\quad -\lambda\tau^2\big(\rho(2+\rho\tau)+(1+A)\beta\rho(1+2\rho\tau)+\beta^2\big(-1+3A\rho^2\tau\big)\big)\Big).
\end{aligned}
\tag{54}
$$

It is evident that the value of $\Gamma(1)$ is greater than zero. According to Lemma 1, the equilibrium point $P_3$ of system (50) exhibits local asymptotic stability under the condition that $\Gamma(1)>0, \Gamma(-1)>0$, and $\Gamma(0)<1$.

## 5. Numerical Examples

In this section, some numerical simulations are performed to verify the aforementioned theoretical discussion. These simulations are further indicating the interesting complex behavior of system (3). MATLAB was used for the computations and visual illustrations.

*5.1. Bifurcation Analysis by Varying $\tau$.* In this example, we explored the NS bifurcation at $P_3$ by utilizing $\tau$ as a bifurcation parameter while keeping $m=1.3, \lambda=1, \beta=0.6, A=0.2, x_0=0.5$, and $y_0=0.3$ fixed. Specifically, we vary $\tau$ within the range of $0.44<\tau<0.65$. Upon simultaneous solution of the inequalities $\Gamma(1)>0$, $\Gamma(-1)>0$, and $\Gamma(0)<1$, the resulting interval for the variable $\tau$ is $0.463093<\tau<0.625$. Thus, the equilibrium point denoted as $P_3$ exhibits local asymptotic stability provided that the value of $\tau$ falls within the interval of 0.463093 to 0.625. The NS bifurcation value has been determined to be $\tau=0.463093$ through calculations. From calculations, it is obtained that $P_3=(0.641275, 0.205785)$. The eigenvalues of the matrix $J(P_3)$ have been determined to be $\varsigma_{1,2}=0.973531 \pm 0.228554i$, satisfying the condition $|\varsigma_{1,2}|=1$. The aforementioned analysis verifies that the stability of the equilibrium point $P_3$ is affected at $\tau=0.463093$ as a result of the emergence of the NS bifurcation.

Figures 3(a) and 3(b) display the bifurcation diagrams. The bifurcation diagrams illustrate that system (3) undergoes a transcritical bifurcation at $\tau=0.625$. By concurrently solving $\Gamma(1)>0, \Gamma(-1)>0$, and $\Gamma(0)<1$, it can be determined that the equilibrium point $P_1$ is unstable and the equilibrium point $P_3$ is stable for $\tau=0.62$. Furthermore, when $\tau$ equals 0.63, the equilibrium point denoted as $P_1$ exhibits stability, while the equilibrium point labeled as $P_3$ displays instability. At a value of $\tau=0.625$, it can be observed that the equilibrium points $P_1$ and $P_3$ undergo a collision and a subsequent exchange in their respective stability. The evidence supports the presence of a transcritical bifurcation. MLE graph is presented in Figure 3(c).

Figures 4(a)–4(e) depict the phase portraits for various values of $\tau$. The depictions of phase portraits illustrate the bifurcation of a smooth invariant circle from the stable equilibrium point denoted as $P_3$. An invariant curve that encloses the equilibrium point $P_3$ emerges when $\tau$ is less than 0.463093, and its radius expands as $\tau$ decays.

The controlled system (50) is evaluated using the identical parameter and initial values, where $\rho$ is set to 0.5. The stability of the equilibrium point $P_3$ is based upon the range of values for $\tau$, specifically when $0.452515<\tau<0.625$. The bifurcation diagrams of the controlled system exhibit a leftward shift of the NS bifurcation. The figures labeled as 5(a) and 5(b) are presented for reference. The occurrence of NS bifurcation in the controlled system takes place when the value of $\tau$ surpasses the threshold of $\tau=0.452515$. The postponement of the NS bifurcation across a wider range of $\tau$ can be achieved by decreasing the value of the control parameter $\rho$.

*5.2. Bifurcation Analysis by Varying A.* In this example, we explored the NS bifurcation at $P_3$ while keeping $m=1.3, \lambda=1, \beta=0.6, \tau=0.5, x_0=0.7$, and $y_0=0.1$ constant. Specifically, we investigated the impact of varying $A$ within the range of $0.32<A<0.42$ as a bifurcation parameter. Through simultaneous solution of the inequalities $\Gamma(1)>0$, $\Gamma(-1)>0$, and $\Gamma(0)<1$, the value of $A$ is determined to be within the range of 0 to 0.38206. Consequently, the equilibrium point denoted as $P_3$ exhibits local asymptotic stability provided that $0<A<0.38206$. The bifurcation value has been determined to be $A=0.38206$ through mathematical computations. From calculations, it is obtained that $P_3=(0.714286, 0.123398)$. The eigenvalues pertaining to the variational matrix of $P_3$ are given by $\varsigma_{1,2}=0.983389 \pm 0.181512i$, where the magnitude of each eigenvalue is equal to one. The aforementioned statement verifies that the system represented by equation (3) undergoes a NS bifurcation at the point $P_3$ upon the crossing of $A$ over the value of 0.38206. Figures 6(a) and 6(b) display the bifurcation diagrams of the system (3). The diagrams depicting bifurcation demonstrate that the equilibrium point denoted as $P_3$ exhibits stability within the range of

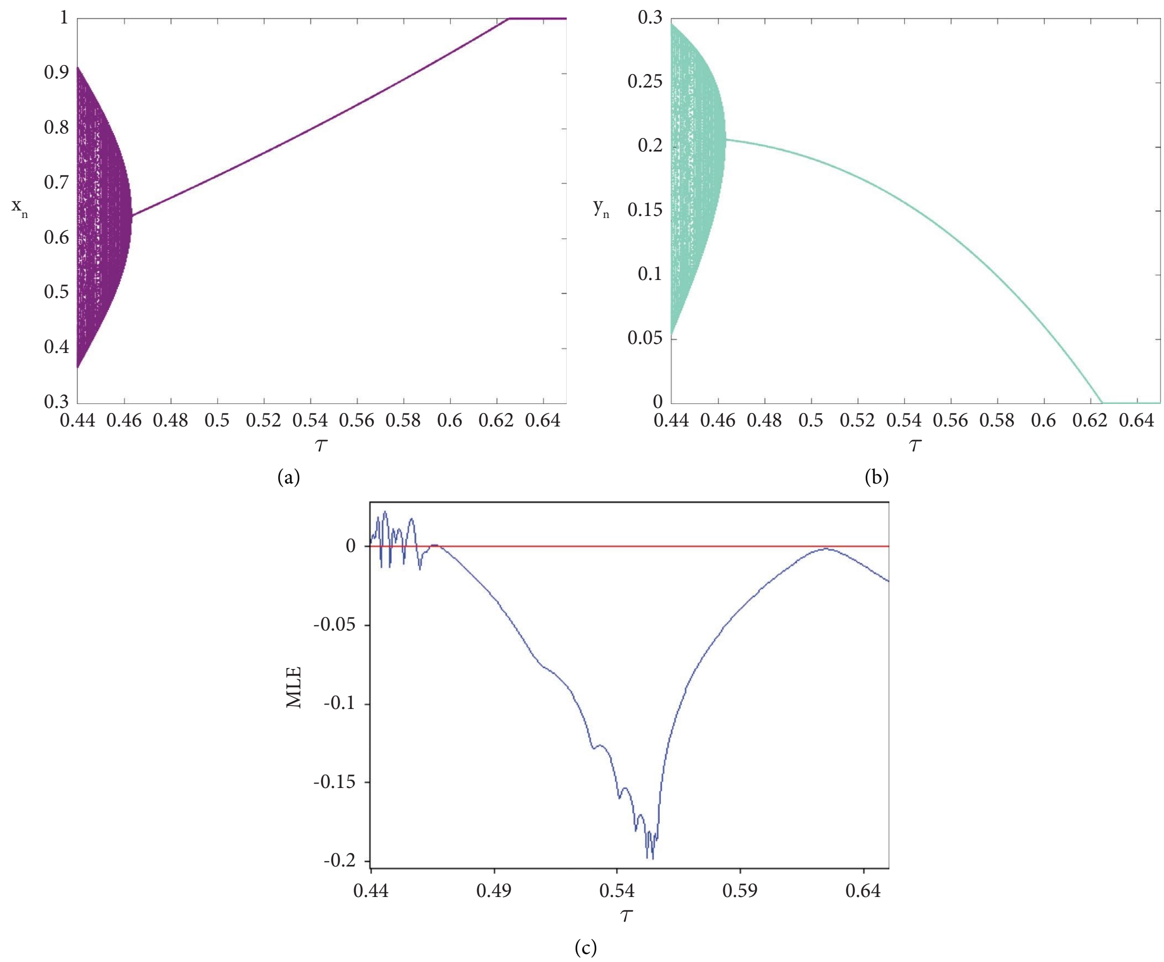



Figure 3: Bifurcation diagrams for system (3) with $m = 1.3, \lambda = 1, \beta = 0.6, A = 0.2, \tau \in [0.44, 0.65]$ and with initial conditions $x_0 = 0.5, y_0 = 0.3$; (a) bifurcation diagram of $x_n$, (b) bifurcation diagram of $y_n$, and (c) MLE graph of system (3).

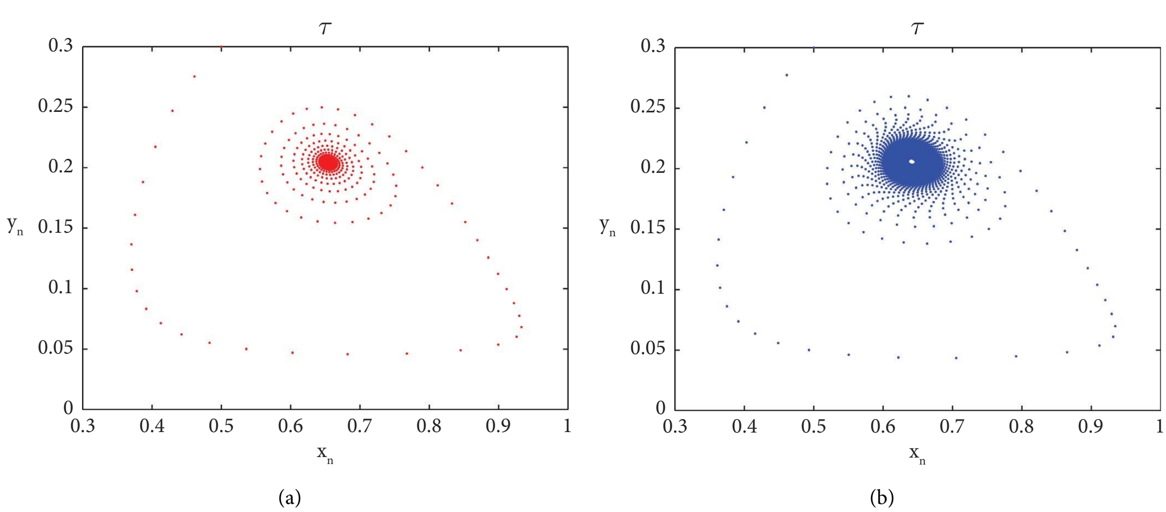



Figure 4: Continued.

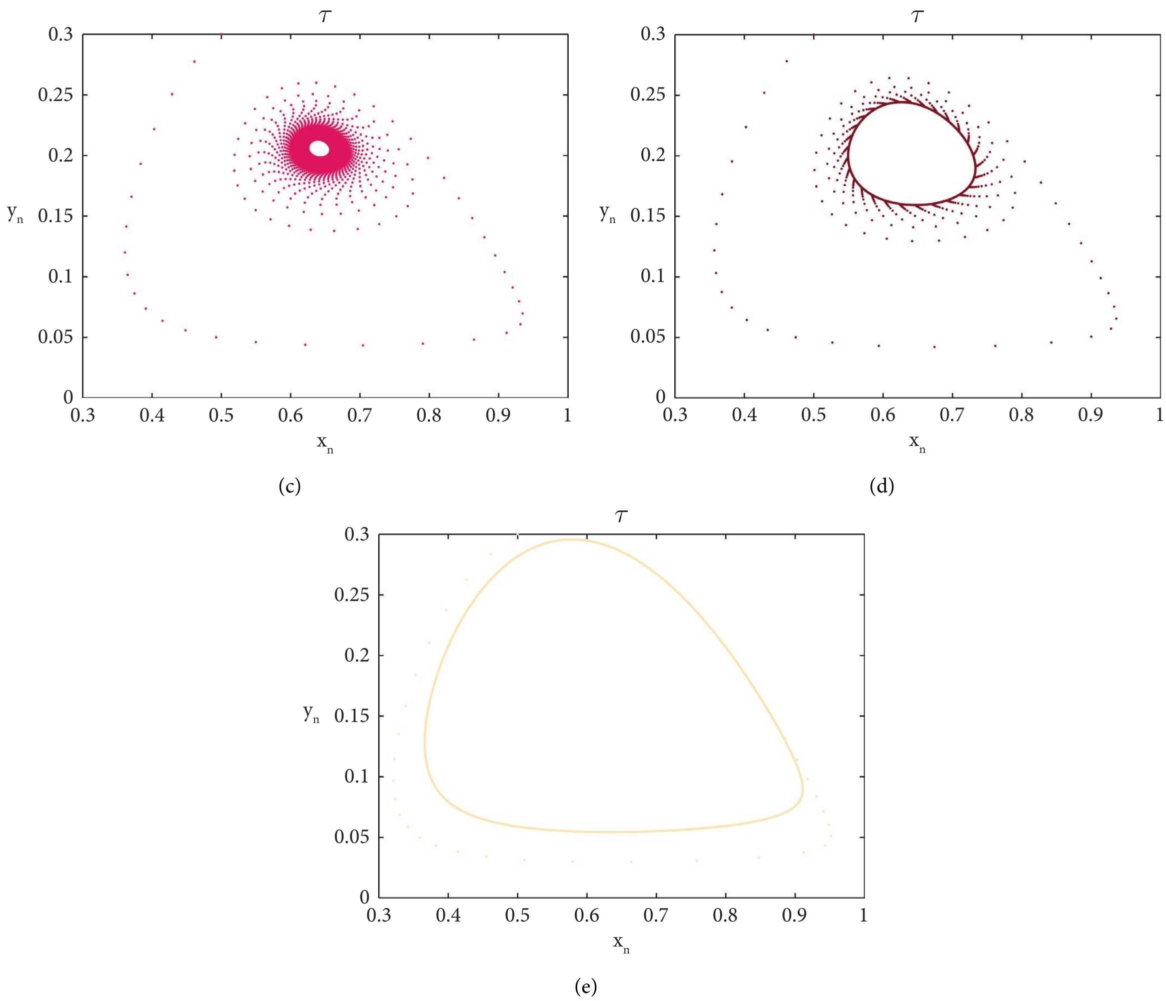



FIGURE 4: Phase portraits of system (3). (a) $\tau = 0.47$, (b) $\tau = 0.4631$, (c) $\tau = 0.4630$, (d) $\tau = 0.46$, and (e) $\tau = 0.44$.

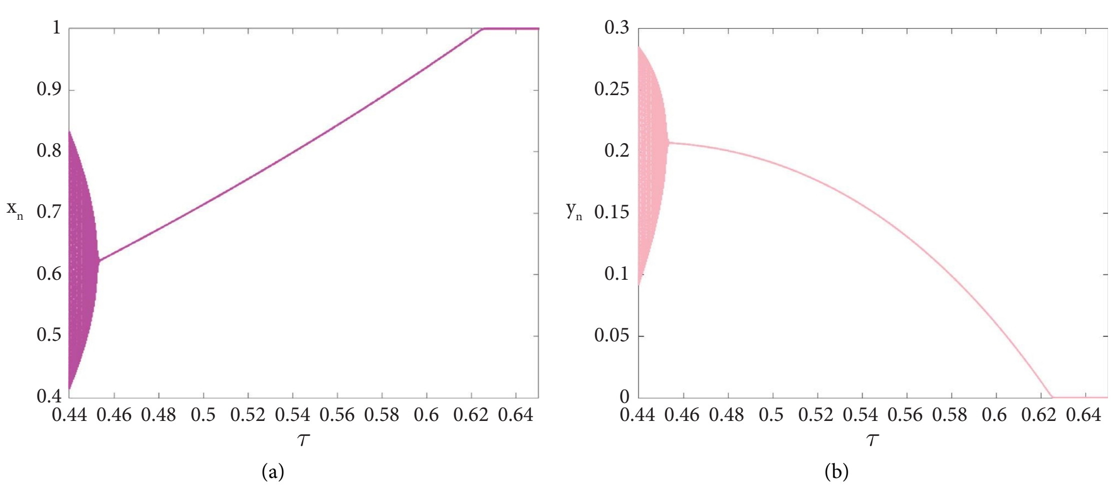
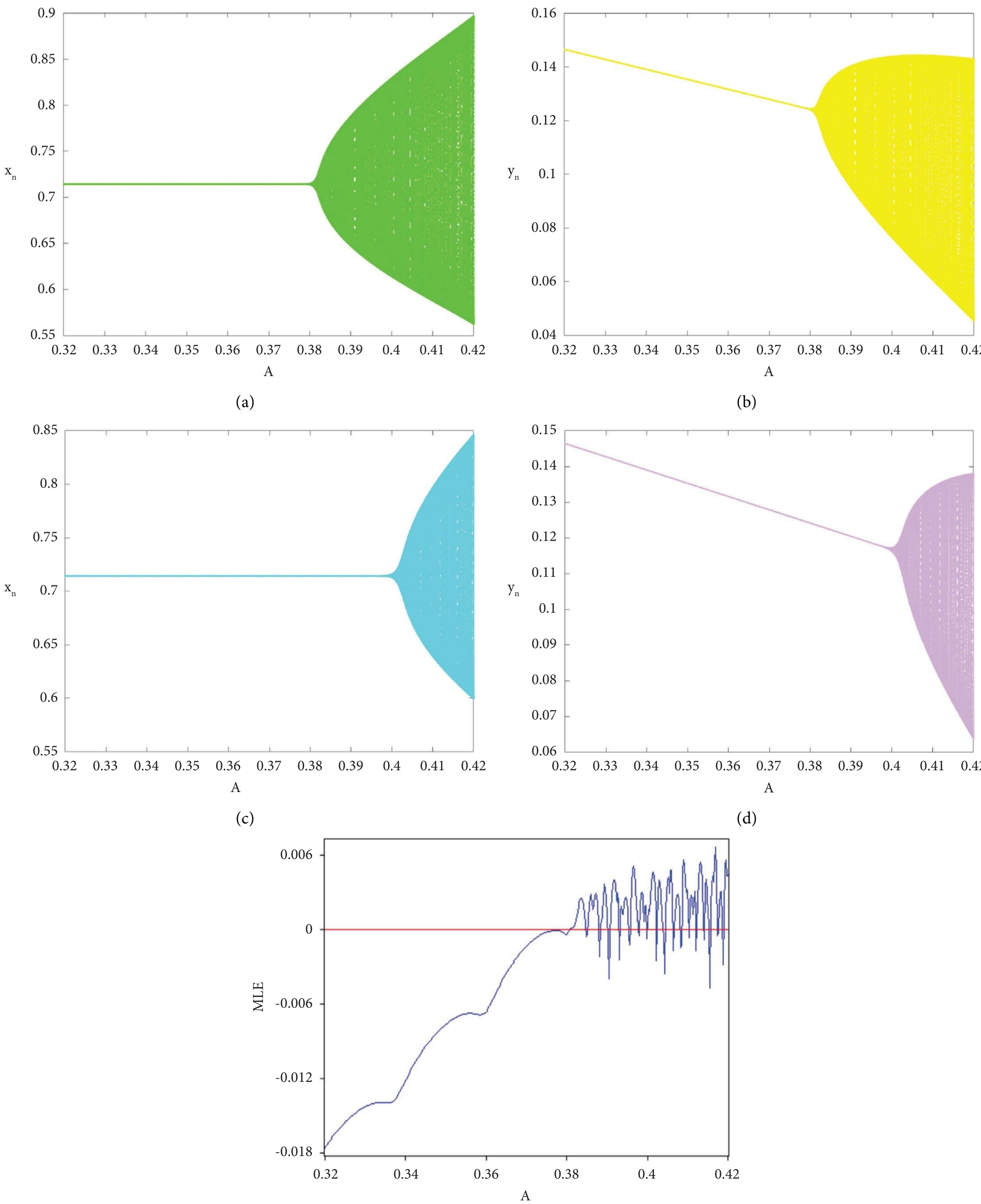



FIGURE 5: Bifurcation diagrams for controlled system (50) with $\rho = 0.5, m = 1.3, \lambda = 1, \beta = 0.6, A = 0.2, \tau \in [0.44, 0.65]$ and with initial conditions $x_0 = 0.5$, $y_0 = 0.3$; (a) bifurcation diagram of $x_n$ and (b) bifurcation diagram of $y_n$.




(e)

Figure 6: Bifurcation diagrams for system (3) and controlled system (50) with $\rho = 0.6, m = 1.3, \lambda = 1, \beta = 0.6, \tau = 0.5, A \in [0.32, 0.42]$ and with initial conditions $x_0 = 0.7$, $y_0 = 0.1$; (a) bifurcation diagram of $x_n$ of (3), (b) bifurcation diagram of $y_n$ of (3), (c) bifurcation diagram of $x_n$ of (50), (d) bifurcation diagram of $y_n$ of (50), and (e) MLE graph for system (3).

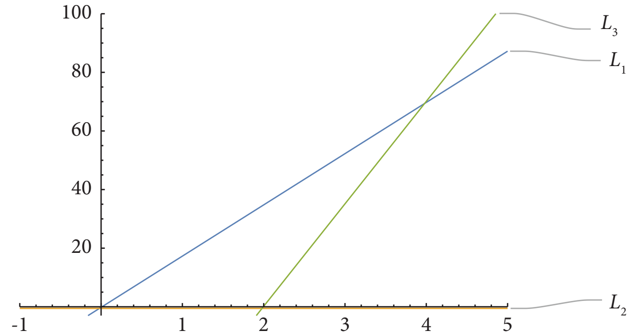



FIGURE 7: Stability region for controlled model (39) with $m = 1.3, \tau = 0.5, \beta = 0.6, \lambda = 1, A = 0.4, x_0 = 0.7, y_0 = 0.1$.

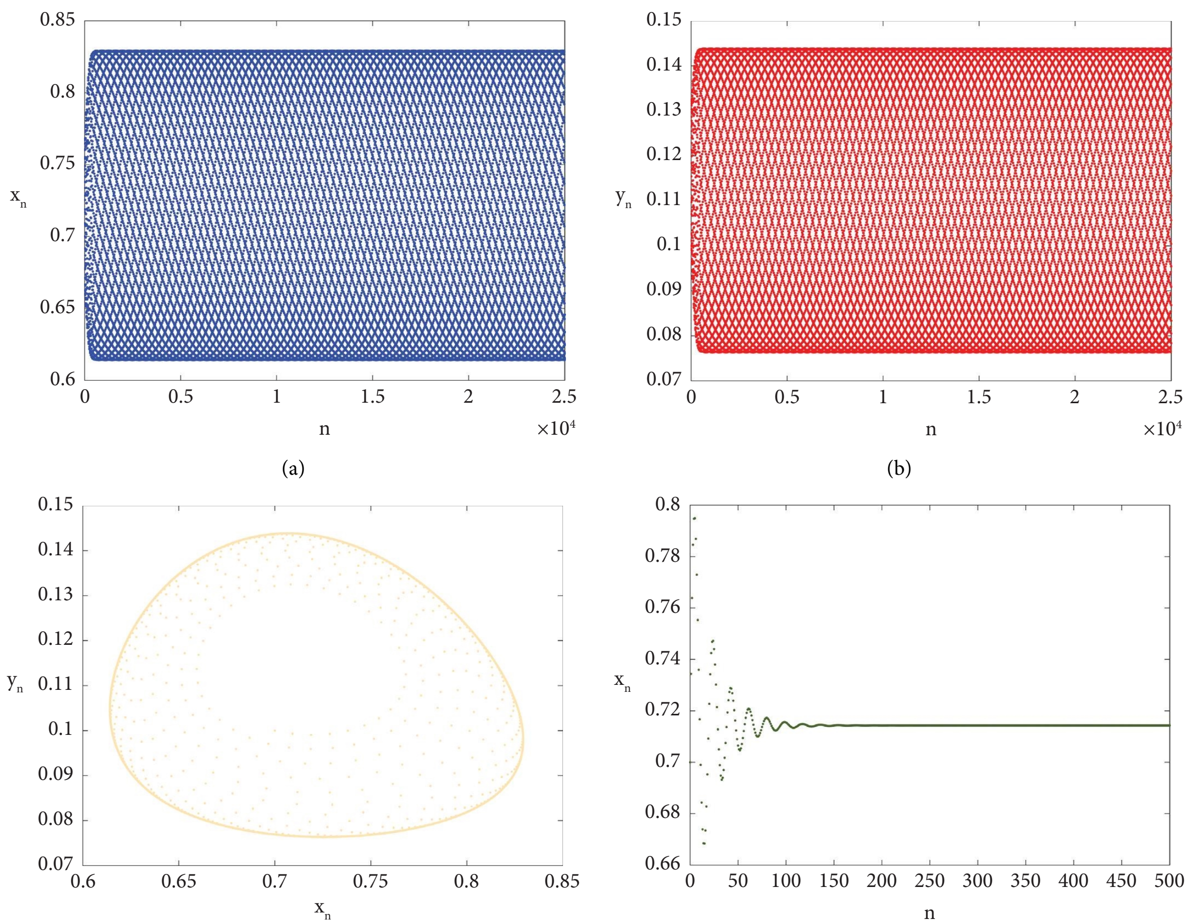
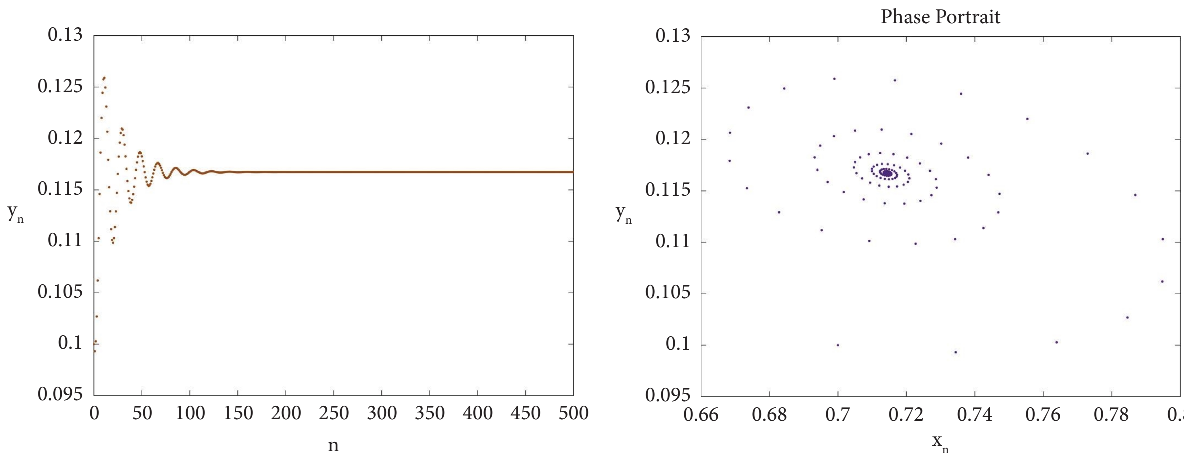



FIGURE 8: Continued.




(e) (f)

FIGURE 8: Plots for model (3) and (39) with $m = 1.3, \tau = 0.5, \beta = 0.6, \lambda = 1, A = 0.4, H = 0.17, P = 1.36, x_0 = 0.7, y_0 = 0.1$. (a) Plot of $x_n$, (b) plot of $y_n$, (c) phase portrait, (d) plot of $x_n$, (e) plot of $y_n$, and (f) phase portrait.

$0 < A < 0.38206$ and loses stability at $A = 0.38206$ as a result of the emergence of NS bifurcation. MLE graph is presented in Figure 6(e).

Now, we want to check the effectiveness of the feedback control method. Considering $m = 1.3, \tau = 0.5, \beta = 0.6, \lambda = 1, A = 0.4, x_0 = 0.7, y_0 = 0.1$ for controlled system (39), we obtain the following lines for its marginal stability:

$$\begin{aligned} L_1&: P = 17.4825(-0.0110204 + H), \\ L_2&: P = -0.549451, \end{aligned} \tag{55}$$

$$L_1: P = 17.4825(-3.99061 + 2H). \tag{56}$$

The stability region bounded by the marginal lines $L_1, L_2$, and $L_3$ for controlled model (39) is presented in Figure 7.

For these parametric values, the positive equilibrium point is $P_3 = (0.714286, 0.123398)$ of model (3) is unstable. Figure 8 depicts the plot of $x_n$ in Figure 8(a), the plot of $y_n$ in Figure 8(b), and the phase portrait in Figure 8(c) for model (3). We considered the corresponding controlled system (39) in which the feedback controlling force is represented by $U_n = H(x_n - 0.714286) + P(y_n - 0.123398)$ with feedback gains $H = 0.17$ and $P = 1.36$. Figure 8 depicts the plot of $x_n$ in Figure 8(d), the plot of $y_n$ in Figure 8(e), and the phase portrait in Figure 8(f) for model (39). Therefore, we conclude that the feedback control method is useful to control bifurcation and chaos.

Next, we want to see the effectiveness of the hybrid control method. Controlled system (50) is evaluated using identical parameters and initial values, where $\rho = 0.6$. For values of $A$ such that $0 < A < 0.402371$, the equilibrium point $P_3$ is stable. The bifurcation diagrams of the controlled system exhibit a delayed NS bifurcation. The figures denoted as 6(c) and 6(d) are presented for reference. The NS bifurcation is observed in the controlled system when the value of $A$ exceeds 0.402371. The postponement of the NS bifurcation across a wider range of $A$ can be achieved by using small control parameter values $\rho$.

## 6. Conclusion

This study examines the dynamic complexity of a discrete-time version of a predator-prey system in continuous time, as represented by equation (1). The discrete system denoted by equation (3) is derived through the piecewise constant argument technique. The complex dynamics of the continuous-time system represented by equation (1) were examined in a previous study conducted by [3]. The study revealed that the system undergoes transcritical and NS bifurcation. Furthermore, the discrete version of the aforementioned system (1) was analyzed in [24] by implementing the Euler technique. The discrete system denoted by equation (2) was demonstrated to undergo NS and period-doubling bifurcation. The authors assert that their discrete system, represented as (3), exhibits dynamic consistency with the continuous-time system (1). This claim is supported by their findings, which indicate that discrete system (3) undergoes transcritical and NS bifurcation. Furthermore, it has been demonstrated that system (3) does not undergo period-doubling bifurcation. Consequently, based on our research, we concluded that the piecewise constant argument approach exhibits greater dynamic consistency than the Euler technique. In addition, the Euler technique produces a discretized system that may display limited realism due to the potential occurrence of negative population sizes for both the predator and prey under specific parameter and initial value conditions. Conversely, the method of piecewise constant argument offers a solution to this issue by guaranteeing the prevention of negative values.

When compared to the Euler approach, the applied discretization methodology, the piecewise constant argument method, exhibits its usefulness in retaining bifurcation features and demonstrating enhanced dynamic consistency. This has been used in ecological modeling and simulation.

Researchers may improve the prediction power and reliability of ecological models by using more precise and reliable discretization approaches. As a result, this can assist in making informed decisions and implementing effective management strategies for conservation efforts and sustainable resource management. The aforementioned results indicated that the utilization of the piecewise constant argument method presented a viable strategy for discretizing predator-prey systems. Nevertheless, there exist potential areas for further investigation in this domain. An area that warrants further investigation is the examination of the influence of supplementary ecological factors, such as fluctuations in the environment or spatial factors, on the discrete system's dynamics.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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