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Research Article

Nonequilibrium Geometric No-Arbitrage Principle and Asset Pricing Theorem

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We find a novel and intimate correspondence in the present paper between the martingale and one-parameter transformation group and develop a nonequilibrium geometric no-arbitrage principle to a frictional financial market via this correspondence. Further, we achieve a fundamental pricing theorem via a geometric pricing transform (generator). Finally, we derive that the nonequilibrium geometric no-arbitrage is equivalent to NFLVR in a frictionless financial market. In addition, we apply the nonequilibrium geometric no-arbitrage condition to a frictional financial market. At the end of this paper, a numerical example confirms the effectiveness of the nonequilibrium geometric no-arbitrage condition.

1. Introduction

The study of no-arbitrage theory in a perfect financial market has been an active field over the past seven decades. As we know, in a perfect market, no market participant could affect the price of whatever commodities he buys or sells. In such a market, the forces of supply and demand will produce an equilibrium. In 1958, Modigliani and Miller [1] put forward that the value of a corporate is independent of its financial policy in a perfect financial market, which implies the no-arbitrage theory. No-arbitrage theory, roughly speaking, is also known as the theory of equilibrium no-arbitrages. Based on the principle of equilibrium no-arbitrage, many remarkable financial theories such as asset pricing and risk measurement have come into being, such as Ross [2] gave the arbitrage pricing theory, Black and Scholes [3] and Merton [4] derived the option pricing theory, and so on. Delbaen and Schachermayer [5] proved and gave the fundamental theorem of asset pricing in continuous time by the martingale theory and stochastic integral theory. Subsequently, they [6] studied the unbounded stochastic processes and gave the condition of no-free-lunch-with-vanishing-risk by sigma-martingale. It is well known that in the real world, there exist many kinds of market frictions such as transaction costs, bid-ask spreads, and so on such that the real financial market is not a real perfect market. This fact makes it difficult for real financial markets to achieve equilibrium. In other words, the principle of equilibrium no-arbitrage does not always work in real financial markets with some frictions. However, many scientific research studies, such as asset pricing, still rely without exceptionally on the equilibrium no-arbitrage principle in the financial market with some frictions. It is inappropriate and unreasonable, but we have no choice. It is not known whether there is a corresponding equilibrium no-arbitrage principle in the frictional market.

A natural question is whether there is a nonequilibrium no-arbitrage principle in the real financial markets with some frictions. The importance of this issue is self-evident.

Fortunately, some scholars in recent years have carried out research studies on asset pricing in nonequilibrium financial markets via differential geometry and the theory of fibre bundles.

Ilinski [7, 8] reconstructed the financial market by the techniques of differential geometry and gauge theories and then viewed arbitrage as the curvature of a gauge connection. Also, Ilinski proposed the Geometric Arbitrage Theory and gave the nonequilibrium pricing in the frictionless financial market based on the theory of fibre bundles. Young [9] presented a correspondence between lattice gauge theories and financial models and viewed arbitrage as the curvature defined on closed loops. After this, Vazquez and Farinelli [10] proved that the connection has zero curvature if and only if the financial market is of no-arbitrage. Farinelli [11] obtained that the no-free-lunch-with-vanishing-risk condition is equal to the zero curvature condition plus the Novikov condition. Farinelli [12] also proposed the link between arbitrage theory and spectral theory of the connection Laplacian on the associated vector bundle and showed that the no-free-lunch-with-vanishing-risk condition is equivalent to zero eigenvalues in the spectrum. Sandhu et al. [13, 14] applied Ricci curvature to study the systemic risk and market fragility, and they showed that the curvature is a "crash hallmark." Hughston [15, 16] applied the information geometry into the theory of interest rates. Choi [17] considered the multidimensional Black–Scholes formula without the constant volatility assumption and then derived a general asymptotic solution by using the heat kernel expansion on a Riemannian metric. Carciola et al. [18] and Tang et al. [19] gave another characterization of the no-arbitrage condition by the Harnack inequality, respectively.

The core problem of this paper is to construct a general no-arbitrage criterion in nonequilibrium frictional financial markets. The source of thought in this paper stems from the discovery of an amazing correspondence between the discounted process in financial markets and the pull-back map in differential geometry. The most important fact is that this correspondence does not require the specific state of financial markets. It was with this surprising discovery that we were able to establish an arbitrage-free analysis principle based on the view of Riemannian geometry in nonequilibrium financial markets.

In a little detail, we will in this paper focus on analyzing the no-arbitrage condition in the nonequilibrium financial market via differential geometry methods (one can see [20–22] for details). We will reconstruct the financial market model from a fully new viewpoint and then first give a reasonable description of the nonequilibrium geometric no-arbitrage condition by the Lie derivative.

We believe that this interesting discovery in the present paper will help deeply understand and analyze the intrinsic characteristics of the no-arbitrage principle in frictional financial markets.

The organization of the paper is as follows: in Section 2, we will give the correspondence between the discounted process and the pull-back map in the differential geometry. Section 3 gives the correspondence between the no-arbitrage condition and the Lie derivative and proposes the nonequilibrium geometric no-arbitrage condition and the nonequilibrium strong geometric no-arbitrage condition. Further, the achieved conclusion motives are the geometric pricing map and the fundamental pricing theorem. In Section 4, we prove that the nonequilibrium strong geometric no-arbitrage condition is equivalent to the NFLVR condition in the frictionless financial market and apply the nonequilibrium geometric no-arbitrage condition to a frictional financial market with the bid-ask spreads and the transaction costs. Moreover, a numerical example confirms the efficiency of the nonequilibrium geometric no-arbitrage condition. Section 5 gives a conclusion.

2. Preliminaries

We will give some necessary terminologies and notations in this section (one can see [20–22] for details).

2.1. The Geometric Background. Let M be a smooth manifold with dimension m. Next, we give the definition of the pull-back map

$$\begin{aligned} & f(p) \overset{\varphi^*}{\longleftarrow} f(\varphi(p)), f(p) \in C_p^\infty, f(\varphi(p)) \in C_{\varphi(p)}^\infty \\ & \uparrow\uparrow \\ & p \overset{\varphi}{\longleftarrow} \varphi(p), p \in M, \varphi(p) \in N, \end{aligned} \tag{1}$$

where C_p^∞ denotes the set of all smooth functions on point $p (\in M)$.

Definition 1. M and N are two smooth manifolds with dimensions m and n, φ: $M \longrightarrow N$ is a smooth map between$M$ and N, $\forall p \in M$, $f \in C_p^\infty$,

$$\varphi^*: C_{\varphi(p)}^\infty \longrightarrow C_p^\infty, f \mapsto \varphi^*(f) = f^\circ \varphi \tag{2}$$

is called the pull-back map.

Remark 1. The push-forward map could also be defined similarly as the pull-back map. In this paper, we focus on comparing the present value and the future value which is pulled back from the present point by a pull-back map.

Definition 2. Let M be a C^k manifold with dimension m, φ: $\mathbb{R} \times M \longrightarrow M$, $(t, p) \mapsto \varphi(t, p)$ be a C^r $(r \leqslant k)$ map, and we denote $\varphi(t, p) = \varphi_p(t) = \varphi_t(p)$.

If φ satisfies the following conditions, then we have

(1) $\forall p \in M$, $\varphi_0(p) = p$

(2) For any real value $s, t \in \mathbb{R}$, $\varphi_s{}^\circ \varphi_t = \varphi_{s+t}$

call φ_t is a C^r one-parameter transformation group on M. Here, φ_t is also a push-forward map.

Definition 3. Let X and Y be two smooth vector fields over a smooth manifold M with dimension m, φ_t is a local one-parameter transformation group induced by X; if the limitation

$$\mathscr{L}_X Y = \lim_{t \longrightarrow 0} \frac{\varphi_{-t}^*\left(Y_{\varphi_t(p)}\right) - Y_p}{t}, \tag{3}$$

exists, we call $\mathscr{L}_X Y$ the Lie derivative of the vector field Y with respect to the vector field Y, where φ_{-t}^*: $T_{\varphi_t(p)} M \longrightarrow T_p M$ is a tangent map induced by φ_t. φ_{-t}^* could be viewed as a pull-back map which pulls the tensor from point $\varphi_t(p)$ back to point p.

Definition 4. Let X be a smooth vector field over a smooth manifold M with dimension m, and we let τ be a (r, s)

-tensor field over M, then the Lie derivative of the tensor field τ with respect to the vector field is defined as follows:

$$\mathscr{L}_X \tau = \lim_{t \longrightarrow 0} \frac{(\Phi_{-t})(\tau) - \tau}{t}, \quad (4)$$

where Φ_{-t}: $T^r_s(\varphi_t(p)) \longrightarrow T^r_s(p)$ is a linear homogeneous tensor field induced by φ_t and Φ_{-t} could be viewed as a pull-back map which pulls the tensor from point $\varphi_t(p)$ back to point p.

We notice that the Lie derivative $\mathscr{L}_X\tau$ of tensor field τ along the vector field X is also a (r, s)-tensor field.

2.2. The Classical Financial Market Model. We consider a filtered probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$, where $\mathbb{P}$ is the statistical (physical) probability measure, $\mathscr{F} = \{\mathscr{F}_t\}_{t\in 0,+\infty}$ is an increasing family of sub-σ-algebras of $\mathscr{F}_\infty$, and $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$ is a probability space. The filtration $\mathscr{F}$ is assumed to satisfy the usual conditions as follows:

(1) Right continuity: $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for all $t \in [0, +\infty)$

(2) $\mathscr{F}_0$ contains all null sets of $\mathscr{F}_\infty$

Assume that in a financial market, the uncertainty of the stochastic processes is modelled by the filtered probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$, and any subject such as risk in the financial market could be modelled by this filtered probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$.

We now consider a financial market in detail with one security (including one portfolio strategy x_t) and denote one (random) financial product value as $V(t, \cdots)$ ($V(t)$ for short) which is determined by the information(such as the time, the asset prices, and the portfolio strategy) in this financial market. We assume that this financial product value $V(t, \cdots)$ is adapted to $\mathscr{F}_t$.

$$\begin{array}{ccccl} & & V^\beta(t_0) & & \\ & \nearrow & \uparrow \beta_{t_0,t} & & \\ V(t_0) & & V(t) & & \text{one financial product value} \\ \uparrow & \longrightarrow & \uparrow & & \text{time } (t \geqslant t_0). \\ t_0 & & t & & \end{array} \quad (5)$$

We assume a positive process $\beta_{t_0,t}$ is adapted to $\mathscr{F}_t$, and by this process, we could compare the values $V(t_0)$ and $V(t)$ (convert to $V^\beta(t_0)$ by $\beta_{t_0,t}$). Note that $\beta_{t_0,t}$ plays a role in pulling back such as a discounted process. This phenomenon reflects the time value of a financial product. Now, we denote this financial market with a financial product as $\mathscr{M} = \{V(t)\}$ which is defined on the probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$.

For example, we consider a frictionless financial market including n assets with prices $S_i(t)$ and a portfolio strategy $x_t = (x^1_t, x^2_t, \ldots, x^n_t)$, and we let $\sum_{i=1}^n x_i(t) S_i(t) \widehat{=} V(t)$ be the portfolio value at time t. For convenience, one can view this portfolio strategy as one security and assume that the portfolio strategy is self-financing. It is well known that if there exists a positive semimartingale β_t such that $\mathbb{E}_t[V(T)\beta_T/\beta_t] - V(t) = 0$, then this market is of no-arbitrage; here, $\beta_T/\beta_t \widehat{=} \beta_{t,T}$ plays a role of pulling the portfolio value at time T back to time t, thus we could compare the portfolio values at time T and time t.

Formally, there exists the definition of pricing kernel in [23] and the no-arbitrage characterization by pricing kernel as follows.

Definition 5. We let $\mathscr{E}$ be an economy with n assets defined on the probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$. We say that the strictly positive, $(\{\mathscr{F}^A_t\}, \mathbb{P})$ continuous semimartingale Z is a pricing kernel for the economy $\mathscr{E}$ if the process ZA is an $(\{\mathscr{F}^A_t\}, \mathbb{P})$ martingale. Here, A denotes the asset prices, and $\mathscr{F}^A_t \widehat{=} \sigma(A^{(i)}_u$: $u \leqslant t, 1 \leqslant i \leqslant n)$ denotes the asset filtration.

Theorem 1. *The economy $\mathscr{E}$ is arbitrage-free if and only if there exists a pricing kernel.*

3. No-Arbitrage Analysis Principle

3.1. Correspondence between the Financial Model and the Geometric Framework. In this section, we will redescribe the financial market in a geometric way and give a no-arbitrage condition from a geometric perspective.

We consider a financial market with n assets. The price processes are modelled on a filtered probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$. We assume, except for special statements, that all processes in this paper are adapted to this filtration $\mathscr{F}_\infty$.

Let M be a smooth n dimensional manifold induced, for instance, by the yield surfaces constructed via some assets; then, the $(r, s)-$ tensor field is a smooth map

$$\begin{aligned} \tau\colon \mathbb{R}^+ \times M \times \Omega &\longrightarrow T^r_s(M) \\ (t, p, \omega) &\mapsto \tau(t, p, \omega). \end{aligned} \quad (6)$$

Note that, in this paper, we will assume any $(r, s)-$ tensor field $\tau(t, p, \omega)$ is measurable to $\mathscr{F}_t$ for every fixed $p \in M$, and the map $p \mapsto \tau(t, p, \omega)$ is smooth for almost each $\omega \in \Omega$ and every fixed $t \in \mathbb{R}^+$.

Any different forms of the value of financial products could be denoted as $(r, s)-$ tensor fields. For example, we assume that the asset prices take values on a smooth manifold M, so we could denote the portfolio value as the real-value function on manifold M which is $(0, 0)-$ tensor field and denote the differential of the portfolio value as the $(0, 1)-$ tensor field.

A smooth vector field is a smooth map

$$X\colon\ (t, p, \omega) \mapsto X(t, p, \omega) \in T_p M. \quad (7)$$

Now, we will give some assumptions of the vector field $X(t, p, \omega)$ in order to need.

(C1) The map $t \mapsto X(t, p, \omega)$ is measurable for every fixed $p \in M$

(C2) The map $p \mapsto X(t, p, \omega)$ is smooth for every fixed $t \in \mathbb{R}^+$ and almost each $\omega \in \Omega$

(C3) For every system of coordinates defined in an open set $U \subset M$ and every compact $K \in U$ and compact interval $I \subset \mathbb{R}^+$, there exist two functions $c(t)$ and $k(t)$ in $L^\infty(I)$ such that for all $(t, x), (t, y) \in I \times K$

$$\|X(t,x)\| \leqslant c(t), \|X(t,x) - X(t,y)\| \leqslant k(t)\|x - y\|, \quad (8)$$

where the norm is given by the Riemannian metric. $L^\infty(I)$ denotes the set of all essentially bounded functions on I.

Note that, in the deterministic situation, if the vector field $X(t, p)$ satisfies the assumptions (C1)–(C3), $X(t, p)$ is called the nonautonomous vector field (refer to [20]).

In this paper, we mainly consider a financial value (such as the portfolio value, net value, and so on) which is a real-value function V on a smooth manifold M in a financial market

$$V: \mathbb{R}^+ \times M \times \Omega \longrightarrow C^\infty(M), \quad (9)$$

where $C^\infty(M)$ denotes the set of all smooth functions on M, i.e., $C^\infty(M) = \{f | f: M \longrightarrow \mathbb{R} \text{ is smooth}\}$. We assume that $V(t, p, \omega)$ (sometimes denote $V(t, p)$ for short) is measurable to $\mathcal{F}_t$ and $\mathbb{E}[|V(t)|^p] < \infty$, where $|\cdot|$ denotes the absolute value.

We consider a frictional financial market with a financial product, and the value of the financial product is a real-valued stochastic process V which is defined on the probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ and a smooth manifold M. We denote this financial market as $\mathcal{M} = \{V; \mathcal{F}_\infty, \mathbb{P}\}$. In addition, we assume that the manifold M is smooth in this paper. For example, we consider a financial market with n assets, the price processes $S(\omega)$ are $M-$ valued stochastic processes, then this financial market could be denoted as $\mathcal{M} = \{x_t \cdot S_t: t \in \mathbb{R}^+, S \in M; \mathcal{F}_\infty, \mathbb{P}\}$, where x_t is a portfolio strategy.

According to the original definition of arbitrage opportunity, that is, the investors could not gain positive cash with nonpositive cash input. Then, we give the following definition.

Definition 6. The financial market $\mathcal{M} = \{V; \mathcal{F}_\infty, \mathbb{P}\}$ is of no-arbitrage (NA) if, for a given time $t \geqslant 0$ and $\forall s > t$, there does not exist a self-financing portfolio such that the financial product value $V(t)$ satisfies

$$\begin{cases} V(t,p) \leqslant 0, \\ V(s,p,\omega) \geqslant 0, a.s. \\ V(t,p) \neq 0 \text{ or } V(s,p,\omega) \neq 0, a.s.. \end{cases} \quad (10)$$

Definition 7 (see[24]). We let M be a C^k manifold with dimension m, $\varphi_{t,s}(p,\omega)$ be a $M-$ valued random field such that for each $t, s, t \leqslant s$ and $p \in M$, $\omega \in \Omega$, and $\varphi_{t,s}(\cdot,\omega): M \longrightarrow M$ is a measurable map. If

(1) $\varphi_{t,s}(p,\cdot)$ is continuous in probability with respect to (t, s, p)

(2) $\varphi_{t,t}$ is an identity map a.s. for each t

(3) $\varphi_{t,s} = \varphi_{t,t_0}\varphi_{t_0,s}$ a.s. for each $t < t_0 < s$.

We call $\varphi_{t,s}$ a stochastic flow of the measurable map.

According to the definition of stochastic flow, we give the stochastic pull-back map.

Definition 8. We define a strictly positive, smooth measure map $\varphi^*_{t,s}$ satisfying $V\varphi_{t,s} = \varphi^*_{t,s}V: \mathbb{R}^+ \times M \times \Omega \longrightarrow \mathbb{R}$, and $V \in C^\infty_{\varphi_s(p)}, \varphi^*_{t,s}V \in C^\infty_{\varphi_t(p)}$, we call this map the pull-back map and $\varphi^*_{t,s}(V(s,p,\omega))$ the pull-back financial value from time s to time t, which is adapted to $\mathcal{F}_s$.

Definition 9. In a financial market $\mathcal{M} = \{V; \mathcal{F}_\infty, \mathbb{P}\}$, for a given pull-back map $\varphi^*_{t_0,t}$, we define

$$R(V, \varphi; t_0, t) = \varphi^*_{t_0,t}(V(t)) - V(t_0). \quad (11)$$

We call $R(V, \varphi; t_0, t)$ a change of the financial product $V(t)$.

Next, we give the definitions of the no-arbitrage condition.

Definition 10. A financial market $\mathcal{M} = \{V; \mathcal{F}_\infty, \mathbb{P}\}$ is of strong no-arbitrage if there exists a pull-back map $\varphi^*_{t_0,t}$ such that $\mathbb{E}_{t_0}[R(V, \varphi; t_0, t)] = 0, a.s.$; that is, $V_{t_0} = \mathbb{E}_{t_0}[\varphi^*_{t_0,t}(V_t)], a.s.$, where $\mathbb{E}_{t_0}[\cdot] \widehat{=} \mathbb{E}^{\mathbb{P}}[\cdot | \mathcal{F}_{t_0}]$.

Now, we will give the definition of the Lie derivative in a financial market. Based on a vector field $X(t, p)$, we could construct a flow $\varphi^*_{t,s}$.

Definition 11. In a financial market $\mathcal{M} = \{V; \mathcal{F}_\infty, \mathbb{P}\}$, we let $X(t, p)$ be a vector field that satisfies the assumption (C1), (C2), and (C3), and the Lie derivative of a financial value $V(t, p)$ with respect to a vector field $X(t, p)$ is defined in terms of the conditional expectation by

$$\mathcal{L}_X V \widehat{=} \lim_{\Delta t \longrightarrow 0} \frac{\mathbb{E}_{t_0}\left[\varphi^*_{t_0,t_0+\Delta t}\left(V_{t_0+\Delta t}\right) - V_{t_0}\right]}{\Delta t} = \frac{d}{dt}\mathbb{E}_{t_0}\left[\varphi^*_{t_0,t}(V_t)\right]|_{t=t_0}, \quad (12)$$

where $\varphi^*_{t_0,t} = \varphi^*_{-t}(\varphi^*_{-t_0})^{-1}$ and $\varphi^*_{-t}: C^\infty_{\varphi_t(p)} \longrightarrow C^\infty_p$ are pull-back maps induced by $X(t, p)$ which satisfies $\varphi^*_0(f) = f$ a.s. and $\varphi^*_{-t_1}\varphi^*_{-t_2}(f) = \varphi^*_{-(t_1+t_2)}(f)$ a.s., $f \in C^\infty(M)$.

Remark 2. In a financial market, we could compare the values of any two times by a discounted process. In the sense of basic functions, one can confirm that the role of the discounted process is exactly equivalent to that of the pull-back map.

Then, we achieve the following geometrical interpretation to invariant-prices: the discounted process plays the role of "transport" of the money amount through time (same currency).

Remark 3. The Lie derivative is defined as the infinitesimal version of the pull-back of the vector field Y along the flow of the vector field X. In detail, it measures how much Y is modified by the flow of X. Then, we could give a geometrical interpretation; the Lie derivative could measure how much the purchasing power is changed by the flow of the vector field.

In detail, on the probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$

(i) If the Lie derivative is zero almost surely in any $D \in \mathscr{F}_\infty$ with $\mathbb{P}(D) > 0$, then we say that the average pull-back financial value conserves

(ii) If the Lie derivative is nonpositive almost surely in any $D \in \mathscr{F}_\infty$ with $\mathbb{P}(D) > 0$, we say that the average pull-back financial value is nonincreasing

(iii) If the Lie derivative is nonnegative almost surely in any $D \in \mathscr{F}_\infty$ with $\mathbb{P}(D) > 0$, we say that the average pull-back financial value is nondecreasing

Above all, we find some correspondences between a financial market and a manifold intuitively as shown in Table 1.

3.2. Nonequilibrium No-Arbitrage Analysis Principle. The analogy between the discounted process and the pull-back map leads to the following conclusion.

Theorem 2. *The financial market $\mathscr{M} = \{V; \mathscr{F}_\infty, \mathbb{P}\}$ is of no-arbitrage(NA) if there exists a vector field $X(t, p)$ over M which satisfies the assumption (C1), (C2), and (C3), such that the average pull-back financial value is nonincreasing, where the pull-back map is induced by this vector field $X(t, p)$.*

Proof. Based on the vector field $X(t, p)$, we could construct an integral curve (also viewed as a stochastic flow generated by the vector field X) by the following equation:

$$\begin{cases} d\gamma(t) = X(\gamma(t), dt), \\ \gamma(t_0) = p_0. \end{cases} \tag{13}$$

According to the assumptions of the vector field, the previous stochastic differential equation has a unique solution $\gamma(t; t_0, p_0) = p_0 + \int_{t_0}^{t} X(\gamma(u), du)$.

On the other hand, $\gamma(t; t_0, p_0)$ is also the trajectory of the vector field $X(t, p)$. If we take $t_0 = 0$ and $p(0) = p$, $\varphi_t \widehat{=} \gamma(t; 0, p)$ is a local one-parameter transformation group induced by the vector field $X(0, p)$.

Then, for any nonautonomous vector field $X(t, p)$, there exists a unique flow $\varphi_t(p_0) \widehat{=} \gamma(t; t_0, p_0)$ satisfying the following relations:

(1) $\varphi_{t_0}(p_0) = I$

(2) $\varphi_{t_2}(p(t_3))\varphi_{t_1}(p(t_2)) = \varphi_{t_1}(p(t_3))$

We define a pull-back map as $\varphi^*_{t_0,t}: C^\infty_{\varphi_t(p_0)} \longrightarrow C^\infty_{\varphi_{t_0}(p_0)}$, which is a positive map induced by $\varphi_t(p(t_0)) = \varphi_t(p_0)$.

Then, the Lie derivative of the financial value V with respect to the vector field X is as follows:

$$\mathscr{L}_X V = \lim_{\Delta t \longrightarrow 0} \frac{\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V(t+\Delta t)) - V(t)\right]}{\Delta t} = \frac{d}{ds}\mathbb{E}_t\left[\varphi^*_{t,s}(V(s))\right]|_{s=t}. \tag{14}$$

By using the Lagrange mean value theorem, we have

$$\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))\right] - \mathbb{E}_t\left[\varphi^*_{t,t}(V(t))\right] = \frac{d}{ds}\mathbb{E}_{t'}\left[\varphi^*_{t',s}(V_s)\right]|_{s=t'}(\Delta t), t < t' < t + \Delta t. \tag{15}$$

We assume the probability space $(\Omega, \mathscr{F}_\infty, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$, and D is a set in $\mathscr{F}_t$ with $\mathbb{P}(D) > 0$.

If the average pull-back financial value is nonincreasing, that is, in any set $D \in \mathscr{F}_t$ with $\mathbb{P}(D) > 0$, $\mathscr{L}_X V \leqslant 0$ a.s., then $\int_D \mathbb{E}_t[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))] - \mathbb{E}_t[\varphi^*_{t,t}(V(t))]d\mathbb{P} \leqslant 0$, i.e., $\int_D \mathbb{E}_t[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))]d\mathbb{P} \leqslant \int_D \mathbb{E}_t[\varphi^*_{t,t}(V(t))]d\mathbb{P} = \int_D V(t)d\mathbb{P}$.

Then, there holds

$$\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))\right] \leqslant V(t), a.s.. \tag{16}$$

Acting $\varphi^*_{0,t}$ on (16), we have that $\varphi^*_{0,t}(V(t))$ is a supermartingale.

We consider an investment horizon $[0, T]$ and divide $[0, T]$ into n time intervals; that is, $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = T$. Then,

$$\mathbb{E}_{t_{n-1}}\left[\varphi^*_{0,t_n}(V(t_n))\right] \leqslant \varphi^*_{0,t_{n-1}}(V(t_{n-1})), a.s.. \tag{17}$$

Using the law of iterated expectations and iterating (17), we have

$$\mathbb{E}_{t_0}\left[\varphi^*_{0,T}(V(T))\right] \leqslant \varphi^*_{0,t_0}(V(t_0)) = V(0), a.s.. \tag{18}$$

If the financial market exhibits an arbitrage opportunity, the financial value satisfies

$$\begin{cases} V(0) \leqslant 0, \\ V(T, \omega) \geqslant 0, a.s. \\ V(0) \neq 0 \; or \; V(T, \omega) \neq 0, a.s.. \end{cases} \tag{19}$$

With a positive map $\varphi^*_{0,T}$, $\varphi^*_{0,T}(V(T, \omega)) \geqslant 0, a.s.$ holds.

If $V(0) < 0$, then $\mathbb{E}_{t_0}[\varphi^*_{0,T}(V(T, \omega))] \geqslant 0, a.s.$, which is in contradiction with (18).

If $V(0) \leqslant 0$, then $V(T, \omega) > 0$ with a positive probability. Now, we have $\mathbb{E}_{t_0}[\varphi^*_{0,T}(V(T, \omega))] > 0$ with a positive probability, which contradicts (18).

Above all, this financial market is of no-arbitrage. □

Table 1: The correspondence between finance and geometry.

Value of a financial product		Tensor field
A discounted process		A pull-back map
The pull-back value of a financial product	$\leftrightarrow$	A pull-back map acting on a tensor field
Conservation of the average pull-back financial value		Lie derivative is zero almost surely
...		...

Remark 4. The no-arbitrage condition given by the Lie derivative is not only applied to the frictionless financial market but also the frictional financial market.

For example, for a given time t which is viewed as the present time, the wealth process of a portfolio x is denoted as $V(t)\widehat{=}x(t)\cdot S(t)$. At time s, the wealth process is denoted as $V(s)\widehat{=}x(s)\cdot S(s)-G$, where G is the cumulative consumption due to friction during time t and time s. It also could be proved that the financial market is no-arbitrage if there exists a pull-back map such that the average pull-back financial value is nonincreasing.

Definition 12. In a financial market $\mathscr{M}=\{V;\mathscr{F}_\infty,\mathbb{P}\}$, this financial market is of nonequilibrium geometric no-arbitrage(NE-GNA) if there exists a vector field $X(t,p)$ which satisfies the assumption (C1), (C2), and (C3) such that the Lie derivative $\mathscr{L}_X V$ is not positive almost surely in any set $D\in\mathscr{F}_t$ with $\mathbb{P}(D)>0$.

Corollary 1. *In a financial market $\mathscr{M}=\{V;\mathscr{F}_\infty,\mathbb{P}\}$, the nonequilibrium geometric no-arbitrage (NE-GNA) is equivalent to the optimal value of the following optimal problem with the pull-back map $\varphi^*_{t,s}$ in the Lie derivative:*

$$\min\left\{V(\theta,t,p)-\min_{\omega\in\Omega}\varphi^*_{t,s}(V(\theta,s,p,\omega))\right\}, \tag{20}$$

is zero, and the optimal solution θ^* satisfies

$$V(\theta^*,t,p)=\varphi^*_{t,s}(V(\theta^*,s,p,\omega)), a.s.. \tag{21}$$

Proof. ($\Rightarrow$) If the financial market is of nonequilibrium geometric no-arbitrage, then there exists a positive process $\varphi^*_{t,s}$ such that

$$\mathbb{E}_t\left[V(\theta,t,p)-\varphi^*_{t,s}(V(\theta,s,p,\omega))\right]\geqslant 0, a.s., \tag{22}$$

that is,

% $$V(\theta,t,p)\geqslant\mathbb{E}_t\left[\varphi^*_{t,s}(V(\theta,s,p,\omega))\right]\geqslant\min_{\omega\in\Omega}\varphi^*_{t,s}(V(\theta,s,p,\omega)). \tag{23}$$

Then,

$$V(\theta,t,p)-\min_{\omega\in\Omega}\left\{\varphi^*_{t,s}(V(\theta,s,p,\omega))\right\}\geqslant 0, \tag{24}$$

and by taking the minimum with respect to the portfolio strategy, we have

$$\min\left\{V(\theta,t,p)-\min_{\omega\in\Omega}\left\{\varphi^*_{t,s}(V(\theta,s,p,\omega))\right\}\right\}=0. \tag{25}$$

Then, there exists an optimal solution θ^* such that

$$V(\theta^*,t,p)=\min_{\omega\in\Omega}\left\{\varphi^*_{t,s}(V(\theta^*,s,p,\omega))\right\}\leqslant\varphi^*_{t,s}(V(\theta^*,s,p,\omega)). \tag{26}$$

If $V(\theta^*,t,p)<\varphi^*_{t,s}(V(\theta^*,s,p,\omega_k))$ with a positive probability. Then, it follows $V(\theta^*,t,p)<\mathbb{E}_t[\varphi^*_{t,s}(V(\theta^*,s,p,\omega))]$, *a.s.*, which contradicts with (22).

Above all, $V(\theta^*,t,p)=\varphi^*_{t,s}(V(\theta^*,s,p,\omega))$, *a.s.*

($\Leftarrow$) We suppose by contradiction that, with a positive process, the Lie derivative is positive. Then,

$$\mathbb{E}_t\left[V(\theta,t,p)-\varphi^*_{t,s}(V(\theta,s,p,\omega))\right]<0, a.s.. \tag{27}$$

Now, we have $V(\theta,t,p)-\varphi^*_{t,s}(V(\theta,s,p,\omega))<0, \forall\theta$ with a positive probability, which is in contradiction with (21). □

Next, we will give a stronger condition of no-arbitrage in a financial market.

Theorem 3. *The financial market $\mathscr{M}=\{V;\mathscr{F}_\infty,\mathbb{P}\}$ is of strong no-arbitrage(SNA) if the conservation of average pull-back financial value holds, where the pull-back map is generated by the vector field $X(t,p)$.*

Proof. Based on the vector field $X(t,p)$, we could construct an integral curve (also viewed as a stochastic flow generated by the vector field X) by the following equation:

$$\begin{cases} d\gamma(t)=X(\gamma(t),dt), \\ \gamma(t_0)=p_0, \end{cases} \tag{28}$$

according to the assumptions of the vector field, the abovementioned stochastic differential equation has a unique solution $\gamma(t;t_0,p_0)=p_0+\int_{t_0}^{t}X(\gamma(u),du)$.

On the other hand, $\gamma(t;t_0,p_0)$ is also the trajectory of the vector field $X(t,p)$. Then, for any nonautonomous vector field $X(t,p)$, there exists a unique flow $\varphi_t(p_0)\widehat{=}\gamma(t;t_0,p_0)$.

We define a pull-back map as $\varphi^*_{t_0,t}\colon C^\infty_{\varphi_t(p_0)}\longrightarrow C^\infty_{\varphi_{t_0}(p_0)}$, which is a positive linear homogeneous map induced by $\varphi_t(p_0)$.

Then, in a financial market $\mathscr{M}=\{V(t,p)\colon t\in\mathbb{R}^+, p\in M;\mathscr{F}_\infty,\mathbb{P}\}$, the Lie derivative of the financial value is

$$\mathscr{L}_X V=\lim_{\Delta t\longrightarrow 0}\frac{\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))-V(t)\right]}{\Delta t}=\frac{d}{ds}\mathbb{E}_t\left[\varphi^*_{t,s}(V_s)\right]|_{s=t}. \tag{29}$$

By using the Lagrange mean value theorem, we have

$$\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))\right]-\mathbb{E}_t\left[\varphi^*_{t,t}(V(t))\right] = \frac{d}{ds}\mathbb{E}_{t'}\left[\varphi^*_{t',s}(V_s)\right]|_{s=t'}(\Delta t), t<t'<t+\Delta t. \tag{30}$$

We assume the probability space $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, and D is a set in $\mathcal{F}_t$ with $\mathbb{P}(D)>0$. If the conservation of average pull-back financial value holds, that is, in any set $D\in\mathcal{F}_t$, with $\mathbb{P}(D)>0$, $\mathcal{L}_X V=0$ a.s.,

then $\int_D \mathbb{E}_t[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))]d\mathbb{P}=\int_D \mathbb{E}_t[\varphi^*_{t,t}(V(t))]d\mathbb{P} = \int_D V(t)d\mathbb{P}$. Now, we have

$$\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))\right]=V(t), a.s.. \tag{31}$$

We give a map $\varphi^*_{0,t}$ acting on the abovementioned equation as

$$\begin{aligned}\varphi^*_{0,t}\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V(t+\Delta t))\right] &= \mathbb{E}_t\left[\varphi^*_{0,t}\circ\varphi^*_{t,t+\Delta t}(V(t+\Delta t))\right]\\ &= \mathbb{E}_t\left[\varphi^*_{0,t+\Delta t}(V(t+\Delta t))\right], a.s.,\end{aligned} \tag{32}$$

that is,

$$\mathbb{E}_t\left[\varphi^*_{0,t+\Delta t}(V(t+\Delta t))\right]=\varphi^*_{0,t}(V(t)), a.s.. \tag{33}$$

Under a positive map $\varphi^*_{0,t}$, $\varphi^*_{0,t}(V(t))$ is a martingale, which implies this financial market is of strong no-arbitrage.

Next, we will prove the pull-back map which is induced by the vector field $X(t,p)$ is unique.

As we know from the previously mentioned proof, for any nonautonomous vector field $X(t,p)$, there exists a unique flow $\varphi_t(p_0)$.

If there exist two different positive maps $(\varphi^*_{t_0,t})^1$ and $(\varphi^*_{t_0,t})^2$ induced by $\varphi_t(p_0)$.

Taking the conditional expectation of the maps which acts on a financial value, we have

$$\mathbb{E}_t\left[\left(\varphi^*_{t,t+\Delta t}\right)^1(V(s))-\left(\varphi^*_{t,t+\Delta t}\right)^2(V(t+\Delta t))\right]\neq 0, a.s.. \tag{34}$$

However, if the conservation of pull-back financial value holds, that is, $\mathcal{L}_X V=0$ a.s. in the set of $\mathcal{F}_t$, there holds $\mathbb{E}_t[(\varphi^*_{t,t+\Delta t})^1(V(t+\Delta t))]=V(t)=\mathbb{E}_t[(\varphi^*_{t,t+\Delta t})^2(V(t+\Delta t))]$, *a.s.*, then we have $\mathbb{E}_t[(\varphi^*_{t,t+\Delta t})^1(V(t+\Delta t))-(\varphi^*_{t,t+\Delta t})^2(V(t+\Delta t))]=0, a.s.$. This contradicts with (34), then the positive map $\varphi^*_{t,t+\Delta t}$ induced by the flow $\varphi_t(p_0)$ which is generated by the vector field $X(t,p)$ in the Lie derivative is unique. □

Remark 5. The Lie derivative of a financial product value V with respect to the vector field X is zero which almost surely implies that the flow of the vector field X keeps the financial value F as the same. This phenomenon shows that the financial value V does not change under the flow of the vector field X, which is consistent with the constant purchasing power in a financial market.

Definition 13. The financial market $\mathcal{M}=\{V;\mathcal{F}_\infty,\mathbb{P}\}$ is of nonequilibrium strong geometric no-arbitrage (NE-SGNA) if there exists a vector field X such that the Lie derivative $\mathcal{L}_X V$ is zero almost surely in any set $D\in\mathcal{F}_t$ with $\mathbb{P}(D)>0$.

Corollary 1 and Theorem 3 motivate the following interesting geometric fundamental pricing theorem.

Definition 14. In a financial market $\mathcal{M}=\{V;\mathcal{F}_\infty,\mathbb{P}\}$, we say that a positive pull-back map $\varphi^*_{t,s}$ plays the role of pricing if a process $\varphi^*_{t,s}(V(s))$ is a supermartingale (martingale), and we call this map $\varphi^*_{t,s}$ a geometric pricing map (generator).

Theorem 4. *(Fundamental Pricing Theorem) A financial market $\mathcal{M}=\{V;\mathcal{F}_\infty,\mathbb{P}\}$ is of nonequilibrium geometric no-arbitrage if and only if there exists a geometric pricing map (generator); that is, for a contingent claim V, there exists a positive pull-back map $\varphi^*_{t,s}$ such that $V(t)\geqslant\mathbb{E}_t[\varphi^*_{t,s}(V(s))]$ a.s. in any set $D\in\mathcal{F}_t$ with $\mathbb{P}(D)>0$.*

4. Geometric No-Arbitrage Analysis for Two Specific Financial Markets

In this section, we will apply the NE-GNA(NE-SGNA) condition to two specific financial markets.

4.1. The Frictionless Financial Market. We consider a financial market including one risky asset $S(t)$ and a stochastic vector field $X(t,p)$. We assume the prices of the asset satisfy the following equation:

$$dS(t)=S(t)\mu(t)dt+S(t)\sigma(t)dW(t), \tag{35}$$

and we assume that the stochastic vector field is characterized by $A(t,p)$ and $B(t,p)$; in detail, $X(t,p)$ induces a stochastic process $\varphi(t)$ as follows:

$$d\varphi(t)=X(\varphi(t),dt)\widehat{=}A(t,\varphi(t))dt+B(t,\varphi(t))dW(t), \tag{36}$$

where $(W_t)_{t\in[0,+\infty}$ is a standard $\mathbb{P}$-Brownian motion in $\mathbb{R}^K$, for $K\in\mathbb{N}$, and $\mu(t)$, $\sigma(t)$ are $\mathbb{R}$, $\mathbb{R}^K$-valued locally bounded predictable stochastic processes on a filtered probability space $(\Omega,\ \mathcal{F}_\infty,\ (\mathcal{F})_{t\geqslant 0},\ \mathbb{P})$. $A(t,p)$: $[0,+\infty]\times M\longrightarrow TM$ and $B(t,p)$: $[0,+\infty]\times M\longrightarrow TM\times\mathbb{R}^K$ satisfy the linear growth condition and Lipchitz continuity.

Now, we denote the financial market as $\mathcal{M}=\{S;\mathcal{F}_\infty,\mathbb{P}\}$, and the vector field $X(t,p)$ is characterized by the pair (A,B).

Theorem 5. *In a financial market $\mathcal{M}=\{S;\mathcal{F}_\infty,\mathbb{P}\}$, the NFLVR (no-free-lunch-with-vanishing-risk) condition is equivalent to that there exists a vector field $X(t,p)=\{(A,B)\}$ such that the conservation of average pull-back financial value holds.*

Proof. ($\Leftarrow$) We take $F(t,p)$: $=S(t)$, and $X(t,p)$ induces a process as follows:

$$\begin{cases} d\varphi(t)=X(\varphi(t),dt)=A(t,\varphi(t))dt+B(t,\varphi(t))dW(t),\\ \varphi(t_0)=\varphi_0.\end{cases} \tag{37}$$

According to the existence and uniqueness theorem for stochastic differential equations [25], (37) has a unique solution $\varphi(t)$, and it is obvious that the process $\varphi(t)$ is a semimartingale.

The infinitesimal change of price $S(t)$ modified by the flow of X could be defined as follows:

$$\mathscr{L}_X S = \lim_{\Delta t \longrightarrow 0} \frac{\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t} S_{\varphi_{t+\Delta t}(p)} - S_{\varphi_t(p)}\right]}{\Delta t} = \frac{d}{ds}\mathbb{E}_t\left[\varphi^*_{t,s}(S_s)\right]|_{s=t}, \tag{38}$$

where $\varphi(t; t_0, \varphi_0)$ is a stochastic flow induced by X and $\varphi^*_{t,s}(t \leqslant s)$ is a pull-back map induced by $\varphi(t; t_0, \varphi_0)$. In detail, we take $\varphi^*_{t,s} = \varphi_t \circ \varphi_s^{-1}$.

If the conservation of average pull-back financial value holds, that is, in any $D \in \mathscr{F}_t$, with $\mathbb{P}(D) > 0$, $\mathscr{L}_X S = 0$ a.s., then there holds $\int_D \mathbb{E}_t[\varphi^*_{t_0,s} S(s)]d\mathbb{P} = \int_D \mathbb{E}_t[\varphi^*_{t_0,t} S(t)]d\mathbb{P} = \int_D \varphi^*_{t_0,t} S(t) d\mathbb{P}$.

Then, we have

$$\mathbb{E}_t\left[\varphi^*_{t_0,s} S(s)\right] = \varphi^*_{t_0,t} S(t), a.s. \forall t_0 \leqslant t. \tag{39}$$

Above all, in the financial market $\mathscr{M} = \{S; \mathscr{F}_\infty, \mathbb{P}\}$, for a given vector field $X(t, p)$ that is characterized by the pair (A, B), there exists a map $\varphi^*_{t_0,t}$ such that $\varphi^*_{t_0,t} S(t)$ is a martingale. Note that $\varphi^*_{t_0,t}$ generated by $X(t, p)$ is a semimartingale which could be viewed as a pricing kernel. Then, we get that this market is of NFLVR.

($\Rightarrow$) The market $\mathscr{M} = \{S; \mathscr{F}_\infty, \mathbb{P}\}$ satisfies NFLVR condition, then there exists a positive semimartingale β_t such that $\beta_t S_t$ is a martingale. We take $X(\varphi, dt) = \varphi d \ln \beta_t$; then by a computation, we have $\varphi^*_{t,s}(f) = \beta_s / \beta_t \cdot f$ and $\mathscr{L}_X S = 0, a.s.$ which implied the conservation of the average pull-back financial value. □

It is directly to have the following corollary.

Corollary 2. *In a frictionless financial market, (NFLVR) condition ⇒ (NE-SGNA) condition ⇒ (NE-GNA) condition ⇒ (NA) condition.*

4.2. The Frictional Financial Market. We consider a nonequilibrium financial market with the transaction cost and bid-ask spread including n assets.

At time t, the bid price of asset i is $S_i^a(t)$ and the ask price of asset i is $S_i^b(t)$, which satisfy $0 \leqslant S_i^b(t) \leqslant S_i^a(t)$.

The transaction cost of buying a unit asset i is $\lambda_i^a(t) S_i^a(t)$, and the transaction cost of selling a unit asset i is $\lambda_i^b(t) S_i^b(t)$, satisfying $0 \leqslant \lambda_i^a(t), \lambda_i^b(t) \leqslant 1$.

At time t, the earning of asset i is $r_i(t)(i = 1, 2, \ldots, n)$. We denote the return vector by $R(t) = (r_1(t), \ldots, r_n(t))$.

At time t, the form of costs is defined as follows:

$$C(z,t) = \begin{cases} \left(1 + \lambda_i^a(t)\right) s_i^a(t) z, & z > 0; \\ \left(1 - \lambda_i^b(t)\right) s_i^b(t) z, & z < 0, \end{cases} \tag{40}$$

then

$$C(x,t) = \sum_{i=1}^{n} C(x_i, t), \forall x \in \mathbb{R}^n, \tag{41}$$

where $C(x,t)$ is called the total cost of the strategy $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ at time t, the state payoff vector at time s is $R^T(s)x$, where $s > t$, $x_i > 0$ denotes that the amount of buying asset i is x_i, and $x_i < 0$ denotes that the amount of buying asset i is $-x_i$.

For a given time t which is viewed as the present time and time s which is viewed as the future time, let $V(t)$ be a total cost of portfolio x and $V(s)$ be a return vector of portfolio x. Then, we denote this frictional financial market $\mathscr{M} = \{V; \mathscr{F}, \mathbb{P}\} = \{x^T R(s), C(x,t); \mathscr{F}_\infty, \mathbb{P}\}$, where $s > t$.

Definition 15. In a frictional financial market $\mathscr{M} = \{x^T R(s), C(x,t); \mathscr{F}_\infty, \mathbb{P}\}$, for a given time t, the rate of change of the pull-back portfolio value is defined as follows:

$$\begin{aligned} \mathscr{L}_X V &\hat{=} \lim_{\Delta t \longrightarrow 0} \frac{\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}(V_{t+\Delta t}) - V_t\right]}{\Delta t} \\ &= \lim_{\Delta t \longrightarrow 0} \frac{\mathbb{E}_t\left[\varphi^*_{t,t+\Delta t}\left(x^T R_{t+\Delta t}\right)\right] - C(x,t)}{\Delta t}, \end{aligned} \tag{42}$$

where V: $[0, +(\infty) \times M \times \Omega \longrightarrow \mathbb{R}$ and for almost each $\omega \in \Omega$, $\varphi^*_{t,s}(\omega)$: $C^\infty_{\varphi_s(p)} \longrightarrow C^\infty_{\varphi_t(p)}$ is a stochastic pull-back map induced by the vector field X.

According to the fundamental property of the term structure ($P_{t,t} = 1$ and $P_{t,s} \cdot P_{s,h} = P_{t,h}$), the pull-back map $\varphi^*_{t,s}$ could be viewed as a term structure. We take $\varphi^*_{t,s}(V(s)) \hat{=} P_{t,s} \cdot V(s)$. Then, we gain the following conclusion by the (NE-GNA) condition.

Theorem 6. *If a financial market $\mathscr{M} = \{x^T R(s), C(x, t); \mathscr{F}_\infty, \mathbb{P}\}$ satisfies the NE-GNA condition, then there exists a number r such that the optimal value of the following optimal problem,*

$$\min \left\{(1 + r)C(x,t) - \min_{\omega \in \Omega} x^T R(s)\right\}, \tag{43}$$

is zero, and the optimal solution x^* satisfies

$$(1 + r)C(x^*, t) = \{x^*\}^T R(s), a.s.. \tag{44}$$

Proof. If the financial market is of nonequilibrium geometric no-arbitrage, then there exists a positive process $\varphi^*_{t,s}$ such that

$$\mathbb{E}_t\left[C(x,t) - \varphi^*_{t,s}\left(x^T R\right)\right] = \mathbb{E}_t\left[C(x,t) - P_{t,s} \cdot x^T R\right] \geqslant 0, a.s, \tag{45}$$

that is,

$$C(x,t) \geqslant \mathbb{E}_t\left[P_{t,s} x^T R(s)\right] \geqslant \min_{\omega \in \Omega} \mathbb{E}_t\left[P_{t,s}\right] x^T R(s). \tag{46}$$

We take $r = \mathbb{E}_t[P_{t,s}]^{-1} - 1$, we have

$$(1 + r)C(x,t) \geqslant \min_{\omega \in \Omega} x^T R(s). \tag{47}$$

Then,

$$(1 + r)C(x,t) - \min_{\omega \in \Omega} x^T R(s) \geqslant 0, \tag{48}$$

Table 2: The prices of a net worth product.

2020.7.1	2020.7.2	2020.7.3	2020.7.4	2020.7.5	2020.7.6	2020.7.7
1.065	1.0651	1.0654	1.0658	1.0657	1.0657	1.0656
2020.7.8	2020.7.9	2020.7.10	2020.7.11	2020.7.12	2020.7.13	2020.7.14
1.0659	1.0659	1.0661	1.0664	1.0667	1.067	1.0685
2020.7.15	2020.7.16	2020.7.17	2020.7.18	2020.7.19	2020.7.20	2020.7.14
1.0687	1.0691	1.0694	1.0695	1.0696	1.07	1.0702
2020.7.22	2020.7.23	2020.7.24	2020.7.25	2020.7.26	2020.7.27	2020.7.28
1.0704	1.0705	1.0704	1.0705	1.0706	1.0705	1.0703

Table 3: The value of the Lie derivative.

t_0 (day)	$e^{-0.03/365*(t_1-t_0)}S(t_1)-S(t_0)$
2020.7.1	0.0000
2020.7.2	0.0002
2020.7.3	0.0003
2020.7.4	−0.0002
2020.7.5	−0.0001
...	...
2020.7.24	0.0000
2020.7.25	0.0000
2020.7.26	−0.0002
2020.7.27	−0.0003

and by taking the minimum with respect to the portfolio strategy, we have

$$\min\left\{(1+r)C(x,t)-\min_{\omega\in\Omega} x^T R(s)\right\}=0. \quad (49)$$

Then, there exists an optimal solution x^* such that

$$(1+r)C(x^*,t)=\min_{\omega\in\Omega}\left\{\{x^*\}^T R(s)\right\}\leqslant\{x^*\}^T R(s). \quad (50)$$

If there exists at least one state ω_k such that $(1+r)C(x^*,t)<\{x^*\}^T R(s)$. Then, it follows $C(x^*,t)=(1+r)C(x^*,t)\mathbb{E}_t[P_{t,s}]<\mathbb{E}_t[P_{t,s}\{x^*\}^T R(s)]$, which contradicts with (45).

Above all, $(1+r)C(x^*,t)=\{x^*\}^T R(s), a.s.$. □

Remark 6. Theorem 5 states that if we could find a positive process such that, at the end of period, the net investment value is almost surely nonpositive by adjusting the portfolio strategy x, then there does not exist any arbitrage opportunity in this financial market. Then, we could view this positive process in the Lie derivative as playing a role in pricing. In addition, according to the fundamental property of the term structure, the pull-back map in the Lie derivative could be taken by the term structure.

This example further confirms that the pull-back map could play a role in pricing.

4.3. Numerical Example

4.3.1. A Frictionless Financial Market. According to Theorem 3, we could investigate and judge if there is an arbitrage opportunity in a financial market by the value of the Lie derivative.

For convenience, we now consider a kind of net worth product with 28 trading dates for a dummy fund company in Table 2 and take the pull-back map to be regarded as the discounted process with the annual return 3%.

The price of this net worth product is shown in Table 2.

By discretizing the Lie derivative, we have Table 3, and the corresponding figure is shown as Figure 1.

Expect for a few points, $\mathscr{L}_X S|_{t_0}$ is almost zero. Then, there almost does not exist any arbitrage opportunity by buying or selling the kind of net worth product. This numerical example shows that the Lie derivative could be used for the description of the no-arbitrage opportunity.

4.3.2. A Frictional Financial Market. Next, we consider a financial market with one asset. The price at time t is $S(t)=100$. The transaction cost of buying (selling) one unit asset $\lambda=\mu=0.3\%$. At time $t+1$, we assume $\Omega=\{\omega_1,\omega_2\}$; then, the price is as follows:

$$S(t+1,\omega)\widehat{=}R(t+1,\omega)=\begin{cases}105, & \omega=\omega_1;\\ 60, & \omega=\omega_2.\end{cases} \quad (51)$$

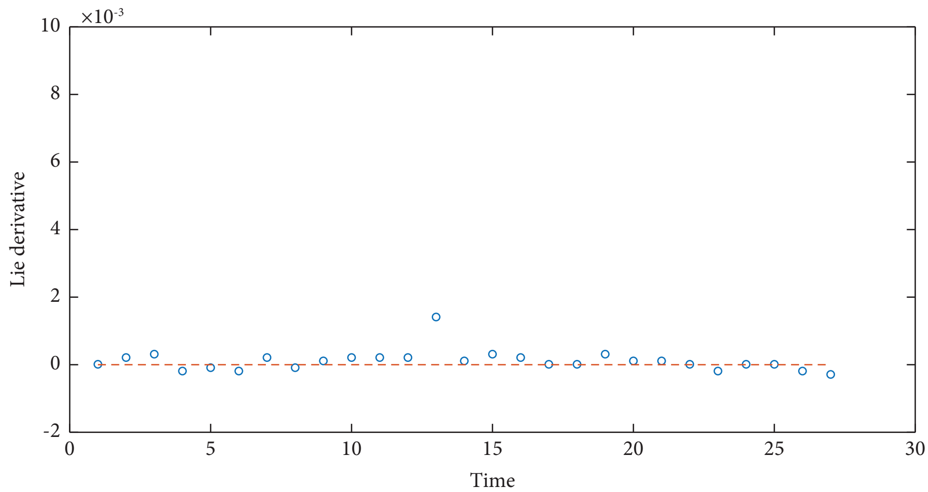

Figure 1: Lie derivative.

With $(S(t+1,\omega_1)=105)=9/10$, $P(S(t+1,\omega_2)=60)=1/10$.

Then,

$$C(x;\ t) = 99{:}\ 7x1fx60g + 100{:}\ 3x1fx>0g;$$
$$E\varphi t,t+1*xTR = \varphi t,t+1*xRt+1,\omega 1;\omega 1P\omega 1+\varphi t,t+1*xRt+1;\omega 2,\omega 2P\omega 2$$
$$= \varphi t,t+1*94.5x;\omega 1+\varphi t,t+1*6x;\omega 2. \quad (52)$$

When $x\geqslant 0$, $100.3x-\varphi^*_{t,t+1}(94.5x;\omega_1)-\varphi^*_{t,t+1}(6x;\omega_2)\geqslant 0$
.

When $x\leqslant 0$, $99.7x-\varphi^*_{t,t+1}(94.5x;\omega_1)-\varphi^*_{t,t+1}(6x;\omega_2)\geqslant 0$. Above all, we have

$$99.7\leqslant\frac{1}{x}\left(\varphi^*_{t,t+1}(94.5x;\omega_1)+\varphi^*_{t,t+1}(6x;\omega_2)\right)\leqslant 100.3. \quad (53)$$

Now, we take $\varphi^*_{t,t+1}=(\varphi^*_{t,t+1}(\cdot;\omega_1),\varphi^*_{t,t+1}(\cdot;\omega_2))$ satisfying (53); this financial market is of no-arbitrage.

This simple example states that the pull-back map could be viewed as a state price vector. It is reasonable to view the pull-back map as a pricing map.

5. Conclusion

Roughly speaking, this paper first gives a new geometric description of the nonequilibrium no-arbitrage condition via the Lie derivative argument. The criteria of no-arbitrage conditions contain some other remarkable works of no-arbitrage conditions such as the NFLVR condition in the frictionless financial market with the asset prices following the geometric Brownian motions. What is more, the nonequilibrium no-arbitrage condition in this paper could also be applied to no-arbitrage characterization in the frictional financial market with the general transaction costs, and the pull-back map could be viewed as a pricing map. At last, a numerical example confirms the efficiency of the description of the no-arbitrage condition by using the Lie derivative.

In future studies, we will continue with the theoretical and empirical study of asset portfolios on the manifold in a frictional financial market. On the other hand, we will study the asset pricing formula based on the invariant under the transformation group.

Data Availability

The data are available at http://www.sse.com.cn/market/.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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