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Research Article

Multiple Solutions of a Nonlocal Problem with Nonlinear Boundary Conditions

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In this article, we consider a class of nonlocal $p(x)$-Laplace equations with nonlinear boundary conditions. When the nonlinear boundary involves critical exponents, using the concentration compactness principle, mountain pass lemma, and fountain theorem, we can prove the existence and multiplicity of solutions.

1. Introduction

In this article, we study the following problem:

$$\begin{cases} -A\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\mathrm{d}x\right)\Delta_{p(x)}u = B\left(\int_\Omega F(x,u)dx\right)f(x,u), & x\in\Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = g(x,u), & x\in\partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^N\,(N\geq 3)$ is a bounded domain with smooth boundary, $\partial/\partial v$ is the outer unit normal derivative, $\Delta_{p(x)}u = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator, and $p(x)$ is a continuous function on Ω, $1<\inf_{x\in\Omega}p(x)\leq\sup_{x\in\Omega}p(x)<N$.

There are many relevant conclusions about the study of p-Laplace equations with critical exponentials (see [1–3] and references therein). In [1], the authors studied the following problem:

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(x,u), & x\in\Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial v} = g(x,u), & x\in\partial\Omega, \end{cases} \tag{2}$$

where $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $1<p<N$. Under several conditions on f and g, the authors proved the existence of infinitely solutions of problem (2). In (2), When the function $g(x,u)=\eta|u|^{p-2}u$, $1<p<N$, the relevant results were obtained in [2].

In [4, 5], the general operator (p, q)-Laplacian was considered and also concentration results were produced, while in [6], the existence in bounded sets was proved for a p-Laplacian Dirichlet problem via blowup technique. In [7], the generalized critical Schrödinger equations were considered.

As we know, the Lions concentration compactness principle (see [8]) is a basic tool to prove the existence of solutions when handling nonlinear elliptic equations with critical growth. In [9, 10], the authors extended the Lions concentration compactness principle to the variable exponent. In [11–13], by applying the concentration compactness principle (see [9, 10]), the existence of solutions to the $p(x)$-Laplace equation with Dirichlet boundary conditions were studied.

In [14], the following problem,

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = f(x,u), & x \in \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = |u|^{q(x)-2}u, & x \in \partial\Omega, \end{cases} \tag{3}$$

was discussed, where $q(x)$ relates to the critical exponent. The authors proved that there are infinitely many small solutions to this problem using the concentration compactness principle (see [5]) and the symmetric mountain pass theorem (see [15]).

With the further study of the problem, Kirchhoff-type equations (also known as nonlocal problems) have also attracted extensive attention from scholars (see [16–19]). In [18], according to the variational method and the (S_+) mapping theorem, he obtained some conclusions on the existence and multiplicity of the problem under weaker assumptions.

However, there are few conclusions for Kirchhoff-type equations with critical growth conditions and nonlinear boundary conditions. Therefore, inspired by the above research, this paper discusses the problem in (1). The main results of this article are the following.

Theorem 1. *Suppose $A(t)$: $\mathbb{R}^+ \longrightarrow \mathbb{R}$ and $B(t)$: $\mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions which satisfy the following conditions:*

(a_1) $\exists a_0 > 0$, $a_1 > 0$, *such that* $a_0 \le A(s) \le a_1$, $s \ge 0$;

(a_2) $\exists \sigma \in [0,1]$, $M_1 > 0$, *such that* $\sigma \widehat{A}(s) \ge A(s)s$, $s \ge M_1$, *where* $\widehat{A}(s) = \int_0^s A(t)dt$;

(a_3) $\exists \alpha > 0$, *such that* $\lim\sup_{t\longrightarrow 0^+} \widehat{A}(t)/t^{\alpha} > 0$;

(b_1) $\exists \gamma > 0$, $D_1 > 0$, *such that* $|\widehat{B}(s)| \le D_1 + D_1|s|^{\gamma}$, $s \in \mathbb{R}$, *where* $\widehat{B}(s) = \int_0^s B(t)\mathrm{d}t$;

(b_2) $\exists \lambda > 1$, $M_2 > 0$, *such that* $0 < \lambda \widehat{B}(s) \le B(s)s$, $s \ge M_2$;

(b_3) $\exists \beta > 0$, *such that* $\lim\inf_{t\longrightarrow 0}\widehat{B}(t)/|t|^{\beta} < +\infty$;

(f_1) f: $\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies the Caratheodory condition, and there exists* $D_2 \ge 0$, *such that*

$$|f(x,s)| \le D_2 + D_2|s|^{\theta(x)-1}, \forall (x,s) \in \Omega \times \mathbb{R}, \tag{4}$$

where $\theta(x) \in C_+(\overline{\Omega})$, $\theta(x) < p^*(x)$, $p^*(x) = \begin{cases} \infty, p(x) \ge N, \\ Np(x)/(N-p(x)), p(x) < N, \end{cases}$

(f_2) $\exists \mu > p^+$, $M_3 > 0$, *such that* $0 < \mu F(x,s) \le f(x,s)s$, $|s| \ge M_3$, $\forall x \in \Omega$;

(f_3) $\exists \tau \in C^0(\overline{\Omega})$, *such that* $1 < \tau(x) < p^*(x)$ *for* $x \in \overline{\Omega}$ *and* $\lim\inf_{t\longrightarrow 0}|f(x,t)|/|t|^{\tau(x)-1} < +\infty$ *uniformly in* $x \in \Omega$;

(g_1) $\exists D_3 \ge 0$, *such that* $|g(x,s)| \le D_3(1+|s|^{q(x)-1})$, $\forall (x,s) \in \partial\Omega \times \mathbb{R}$, *where* $\{q(x) = p^*(x)\} \ne \emptyset$; $\sigma p^+ < \eta^-$, $\gamma\theta^+ < p^-$;

(g_2) $\exists q \in C^0(\overline{\Omega})$, *such that* $1 < q(x) \le p^*(x)$ *for* $x \in \partial\Omega$ *and* $\lim\inf_{t\longrightarrow 0}|g(x,t)|/|t|^{q(x)-1} < +\infty$ *uniformly;*

(g_3) $\exists \kappa > \lambda\mu > 0$, $M_4 > 0$, *such that* $0 < \kappa G(x,t) \le g(x,t)t$, $|t| \ge M_4$, $x \in \partial\Omega$.

When the conditions $\alpha p^+ < \beta\tau^-$, $\alpha p^+ < q^-$, and $\sigma p^+ < \lambda\mu$ are satisfying, equation (1) has a nontrivial solution.

Theorem 2. *Under the condition that Theorem 1 holds, the following hypotheses are also satisfied:*

(f_4) *when* $x \in \Omega$, $s \in \mathbb{R}$, *we have* $f(x,-s) = -f(x,s)$

(g_4) *when* $x \in \partial\Omega$, $s \in \mathbb{R}$, *we have* $g(x,-s) = -g(x,s)$

Then, we obtain infinitely many solutions $\{\pm w_n\}$ to equation (1), and $J(\pm w_n) \longrightarrow +\infty$ as $n \longrightarrow \infty$, where c_i, d_i, C_i, and D_i $(i = 1, 2, \ldots)$ denote different positive constants.

2. Preliminaries

In this section, we give some properties and definitions of $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ to deal with equation (1).

Let $\Omega \subset \mathbb{R}^N$ be a bounded region, and let

$$\begin{aligned} C_+(\overline{\Omega}) &= \{\theta(x)\colon \theta(x) \in C(\overline{\Omega}), \theta(x) > 1, \forall x \in \overline{\Omega}\}, \\ \theta^+ &= \max\{\theta(x)\colon x \in \overline{\Omega}\}, \\ \theta^- &= \min\{\theta(x)\colon x \in \overline{\Omega}\}, \\ L^{p(x)}(\Omega) &= \left\{w\colon w \text{ is measurable real} - \text{valued function}, \int_{\Omega} |w(x)|^{p(x)}\mathrm{d}x < \infty\right\}. \end{aligned} \tag{5}$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|w|_{L^{p(x)}(\Omega)} := |w|_{p(x)} = \inf\left\{\kappa > 0\colon \int_{\Omega}\left|\frac{w(x)}{\kappa}\right|^{p(x)} dx \le 1\right\}, \tag{6}$$

which is a Banach space.

The definition of space $W^{1,p(x)}(\Omega)$ is as follows:

$$W^{1,p(x)}(\Omega) = \left\{w \in L^{p(x)}(\Omega)\colon |\nabla w| \in L^{p(x)}(\Omega)\right\}, \tag{7}$$

if the following norm is introduced:

$$\|w\| = \inf\left\{\kappa > 0\colon \int_{\Omega}\left|\frac{w(x)}{\kappa}\right|^{p(x)} + \left|\frac{\nabla w(x)}{\kappa}\right|^{p(x)} dx \le 1\right\}. \tag{8}$$

It is well known that $W^{1,p(x)}(\Omega)$ is also a Banach space. Specifically, its dual space is $W^{1,p^*(x)}(\Omega)$, where $1/p^*(x) + 1/p(x) = 1$. For every $w \in W^{1,p(x)}(\Omega)$ and $v \in W^{1,p^*(x)}(\Omega)$, we have

$$\left|\int_{\Omega} wv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^{*-}}\right)|w|_{1,p(x)}|v|_{1,p^*(x)}. \tag{9}$$

By virtue of Hölder inequality holds (see [20, 21]).

Proposition 3 (see [20, 21]). *Let $\chi(w) = \int_{\Omega}|w|^{p(x)}dx$, $\forall w \in L^{p(x)}(\Omega)$; then, we have*

(1) $|w|_{p(x)} < 1\,(= 1;\ > 1) \Leftrightarrow \chi(w) < 1\,(= 1;\ > 1)$

(2) $|w|_{p(x)} > 1 \Rightarrow |w|_{p(x)}^{p^-} \le \chi(w) \le |w|_{p(x)}^{p^+}$; $|w|_{p(x)} < 1 \Rightarrow |w|_{p(x)}^{p^+} \le \chi(w) \le |w|_{p(x)}^{p^-}$

(3) $|w_n - w|_{p(x)} \longrightarrow 0 \Leftrightarrow \chi(w_n - w) \longrightarrow 0$

Proposition 4 (see [20, 21])

(1) $W^{1,p(x)}(\Omega)$ is a reflexive, separable Banach space

(2) If $p \in C_+(\overline{\Omega})$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{p(x)}(\Omega)$ is continuous and compact

Proposition 5 (see [22]). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded region with a Lipschitz boundary.*

Assume that $p \in C^0(\overline{\Omega})$, $1 < p^- \le p^+ < N$, and that $v \in C^0(\partial\Omega)$ satisfies the condition.

$$1 \le v(x) < \frac{(N-1)p(x)}{N-p(x)}, \quad \forall x \in \partial\Omega. \tag{10}$$

Then, the boundary trace embedding from $W^{1,p(x)}(\Omega)$ to $L^{v(x)}(\partial\Omega)$ is compact, with S is the embedding constant.

In this paper, we denote $X\colon = W^{1,p(x)}(\Omega)$, $X^*\colon = (W^{1,p(x)}(\Omega))^*$, and we let "$\rightharpoonup$" and "$\longrightarrow$" represent weak convergence and strong convergence, respectively.

Below, we give the definition of weak solutions for equation (1).

Definition 6. A function $w_0 \in X$ is a weak solution of equation (1), if, for any $v \in X$,

$$A\left(\int_{\Omega}\frac{|\nabla w_0|^{p(x)}}{p(x)}dx\right)\int_{\Omega}|\nabla w_0|^{p(x)-2}\nabla w_0 \cdot \nabla v dx - B\left(\int_{\Omega}F(x,w_0)dx\right)\int_{\Omega}f(x,w_0)v dx - \int_{\partial\Omega}g(x,w_0)v dS = 0, \tag{11}$$

where $F(x,s) = \int_0^s f(x,t)dt$ and dS is the surface measure on $\partial\Omega$.

Functional J in X associated to the equation in equation (1):

$$J(w) = \widehat{A}\left(\int_{\Omega}\frac{|\nabla w|^{p(x)}}{p(x)}dx\right) - \widehat{B}\left(\int_{\Omega}F(x,w)dx\right) - \int_{\partial\Omega}G(x,w)dS, \tag{12}$$

where $G(x,s) = \int_0^s g(x,t)dt$.

We define an operator $J'\colon X \longrightarrow X^*$ by

$$\langle J'(w), v\rangle = A\left(\int_{\Omega}\frac{|\nabla w|^{p(x)}}{p(x)}dx\right)\int_{\Omega}|\nabla w|^{p(x)-2}\nabla w \cdot \nabla v dx - B\left(\int_{\Omega}F(x,w)dx\right)\int_{\Omega}f(x,w)v dx - \int_{\partial\Omega}g(x,w)v dS, \quad \forall w, v \in W^{1,p(x)}(\Omega). \tag{13}$$

Definition 7 (see [14]). If any sequence $\{w_n\} \subset X$, which satisfies that $\{J(w_n)\}$ is bounded and $\|J'(w_n)\|_X \longrightarrow 0$ as $n \longrightarrow \infty$, has a convergent subsequence, then J is said to satisfy the Palais–Smale condition ((PS) condition for short).

Theorem 8 (see [23]). *Assume that X is a Banach space;* $J \in C^1(X, \mathbb{R})$ *if J is said to satisfy the (PS) condition and* $J(0) = 0$. *Suppose*

$$\begin{aligned} &(L_1)\, \exists H > 0, h > 0\colon \|u\|_X = H \Rightarrow J(w) > h;, \\ &(L_2)\, \exists v_0 \in X\colon \|v_0\| \geq H \text{ and } J(v_0) < h. \end{aligned} \tag{14}$$

Then, J has a critical value.

$$c = \inf_{\omega \in \Gamma} \max_{0 \leq t \leq 1} J(\omega(t)) \geq h, \tag{15}$$

where

$$\Gamma = \{\omega \in C[0, 1]; \omega(0) = 0, \omega(1) = v_0\}. \tag{16}$$

Let X be a separable, reflexive Banach space; then, $\exists \{l_n\}_{n=1}^{\infty} \subset X$, $\{l_n^*\}_{n=1}^{\infty} \subset X^*$, and we have

$$\begin{aligned} l_n^*(l_m) &= \zeta_{n,m} \\ &= \begin{cases} 0, n \neq m, \\ 1, n = m, \end{cases} \text{and} \\ X &= \overline{\text{span}}\{l_n \mid n = 1, 2, \ldots\}, \\ X^* &= \overline{\text{span}}^{W^*}\{l_n^* \mid n = 1, 2, \ldots\}. \end{aligned} \tag{17}$$

For $j = 1, 2, \ldots$, we have

$$\begin{aligned} X_j &= \text{span}\{e_j\}, \\ Y_j &= \oplus_{i=1}^{j} X_i, \\ Z_j &= \overline{\oplus_{i=j}^{\infty} X_i}. \end{aligned} \tag{18}$$

Theorem 9 (see [24]). *Let* $J \in C^1(X, \mathbb{R})$, $J(-w) = J(w)$. *If, for every* $j \in \mathbb{N}$, *there exists* $\rho_j > r_j > 0$, *such that*

$(N_1) a_j := \max_{u \in Y_j, \|u\| = \rho_j} J(w) \leq 0, j \longrightarrow \infty$

$(N_2) b_j := \inf_{u \in Z_j, \|u\| = r_j} J(w) \longrightarrow \infty, j \longrightarrow \infty$

(N_3) *J satisfies the (PS) condition for every* $c > 0$

Then, J has an unbounded sequence of critical values.

3. Local (PS) Condition

Lemma 10. *Suppose that functions A and B are continuous which satisfy the conditions:* $(a_1)(a_2)(b_1)(b_2)$, f, *and* g *satisfy the conditions* $(f_1)(f_2)(g_1)$, $\sigma p^+ < \eta^-$ *and* $\gamma\theta^+ < p^-$ *hold. Then, all (PS) sequences of J are bounded in* $W^{1,p(x)}(\Omega)$.

According to the conditions of Lemma 10, we can know that the nonlinear boundary of (1) involves critical exponents and, thus, the inclusion from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\partial\Omega)$ loses compactness; we can no longer expect the (PS) condition to hold. However, we can solve this difficulty by using the concentration compactness principle.

We use the following lemma to prove that J satisfies the local (PS) condition:

Lemma 11 (see [10]). *Suppose that* $q(x)$ *and* $p(x)$ *are two continuous functions, such that*

$$1 \leq q(x) \leq p^*(x), 1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) < N \text{ in } \Omega. \tag{19}$$

Let $\{w_j\}_{j \in \mathbb{N}} \rightharpoonup w$ in $W^{1,p(x)}(\Omega)$, such that

$$\left|\nabla w_j\right|^{p(x)} \rightharpoonup \mathrm{d}\mu, \left|w_j\big|_{\partial\Omega}\right|^{q(x)} \rightharpoonup \mathrm{d}\nu \text{ weakly}-^{*} \text{in the sense of measures.} \tag{20}$$

Note that $\mathrm{d}\nu$ is a measure supported on $\partial\Omega$. Assume that $\Lambda = \{x \in \partial\Omega \mid q(x) = p^*(x)\} \neq \varnothing$. Then, for some countable index, set I, we have

$$\begin{aligned} \mathrm{d}\nu &= |w|^{q(x)} + \sum_{i \in I} \nu_i \delta_{x_i}, \quad \nu_i > 0, \\ \mathrm{d}\mu &\geq |\nabla w|^{p(x)} + \sum_{i \in I} \mu_i \delta_{x_i}, \quad \mu_i > 0, \\ S\nu_i^{1/p^*(x_i)} &\leq \mu_i^{1/p(x_i)}, \quad i \in I, \end{aligned} \tag{21}$$

where $\{x_i\}_{i=1}^{l} \subset \Lambda$. In the Sobolev trace embedding theorem, S is the best constant.

Proof of Lemma 10. Let $g(x, s) = y(x, s) + z(x, s)$ and denote

$$\begin{aligned} Y(x, s) &= \int_0^s y(x, t)\mathrm{d}t, \\ Z(x, s) &= \int_0^s z(x, t)\mathrm{d}t. \end{aligned} \tag{22}$$

Let us make the following assumptions.

$(O_1) \exists M > 0$ and a function $\eta(x) \in C^1(\overline{\Omega})$ satisfying

$$\eta(x) \leq q(x), \quad \forall x \in \overline{\Omega}, \tag{23}$$

such that z satisfies

$$0 < Z(x, s) \leq \frac{s}{\eta(x)} z(x, s), \quad \forall x \in \overline{\Omega}, \quad |s| \geq M. \tag{24}$$

(O_2) For $\eta(x)$ in (O_1), there exists $\delta > 0$ small enough, such that y satisfies

$$|y(x, s)| \leq |s|^{(\eta(x)/1+\delta)-1}, \quad \forall x \in \overline{\Omega}, \quad |s| \geq M. \tag{25}$$

For convenience, we define

$$E(w_n) = \widehat{A}\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\mathrm{d}x\right). \tag{26}$$

For a large enough $\|w_n\|$, according to (a_2), we have

$$\begin{aligned}\sigma p^+ E(w_n) &= \sigma p^+ \widehat{A}\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\mathrm{d}x\right)\\ &\geq A\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\mathrm{d}x\right)\int_\Omega |\nabla w_n|^{p(x)}\mathrm{d}x = E'(w_n)w_n.\end{aligned} \tag{27}$$

Under assumptions (f_1), we obtain $|F(x,s)| \leq D_2 (1+|s|^{\theta(x)})$.

The conditions (O_1), (O_2) imply that $Z(x,s) \geq |s|^{\eta(x)}$, $\forall x \in \overline{\Omega}$, when $|s|$ is large enough,

$$|Y(x,s) - \beta y(x,s)s| \leq D_4 + D_4|s|^{(\eta(x)/1+\delta)}, \quad \forall (x,s) \in \overline{\Omega}\times\mathbb{R}. \tag{28}$$

Let $\{w_n\}$ be a (PS) sequence and assume $\|w_n\| \longrightarrow \infty$. Since $\eta(x) \in C^1(\overline{\Omega})$, we have

$$\begin{aligned}
&c + \|w_n\|_{p(x)}\\
&\geq J(w_n) - \left\langle J'(w_n), \frac{1+\delta}{\eta(x)}w_n\right\rangle\\
&= E(w_n) - \frac{\sigma p^+}{\eta(x)}E(w_n) - A\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\mathrm{d}x\right)\int_\Omega |\nabla w_n|^{p(x)-2}\nabla w_n\cdot\nabla\left(\frac{1+\delta}{\eta(x)}w_n\right)\mathrm{d}x + \frac{\sigma p^+}{\eta(x)}E(w_n)\\
&\quad -\widehat{B}\left(\int_\Omega F(x,w_n)\mathrm{d}x\right) - \int_{\partial\Omega} G(x,w_n)\mathrm{d}S + \frac{1+\delta}{\eta(x)}B\left(\int_\Omega F(x,w_n)\mathrm{d}x\right)\int_\Omega w_n f(x,w_n)\mathrm{d}x + \frac{1+\delta}{\eta(x)}\int_{\partial\Omega} w_n g(x,w_n)\mathrm{d}S\\
&\geq \left(1 - \frac{\sigma p^+}{\eta(x)}\right)E(w_n) + \frac{1}{\eta(x)}\left(\sigma p^+ E(w_n) - (1+\delta)E'(w_n)w_n\right) + \left(\frac{\lambda\mu(1+\delta)}{\eta^+} - 1\right)\widehat{B}\left(\int_\Omega F(x,w_n)dx\right)\\
&\quad +A\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}dx\right)\int_\Omega \frac{1+\delta}{\eta^2(x)}w_n|\nabla w_n|^{p(x)-1}\nabla\eta(x)\mathrm{d}x - \int_{\partial\Omega} G(x,w_n)\mathrm{d}S + \frac{1+\delta}{\eta(x)}\int_{\partial\Omega} w_n g(x,w_n)\mathrm{d}S\\
&\geq a_0\left(\frac{1}{\sigma p^+} - \frac{1}{\eta^-}\right)\int_\Omega |\nabla w_n|^{p(x)}\mathrm{d}x - D_2\left(\frac{\lambda\mu(1+\delta)}{\eta^+} - 1\right)\|w_n\|^{\gamma\theta(x)} - a_1\int_\Omega \frac{(1+\delta)|\nabla\eta(x)|}{\eta^2(x)}|w_n||\nabla w_n|^{p(x)-1}\mathrm{d}x\\
&\quad +\int_{\partial\Omega}\frac{\delta}{\eta(x)}Z(x,w_n)\mathrm{d}S - \int_{\partial\Omega} D_4|w_n|^{(\eta(x)/1+\delta)}\mathrm{d}S - D_4|\Omega|.
\end{aligned} \tag{29}$$

According to the Young inequality, we obtain

$$\frac{(1+\delta)|\nabla\eta(x)|}{\eta^2(x)}|w_n||\nabla w_n|^{p(x)-1} \le d_1|w_n||\nabla w_n|^{p(x)-1} \le d_1\left(\varepsilon_1|\nabla w_n|^{p(x)} + \varepsilon_1^{1-p^+}|w_n|^{p(x)}\right), \tag{30}$$

and

$$|w_n|^{p(x)} \le \varepsilon_2|w_n|^{q(x)} + \varepsilon_2^{-(p^+/(q-p)^-)}. \tag{31}$$

According to the embedding theorem (see [19, 20]), it follows that

$$|w_n|_{q(x)} \le d_2\|w_n\|_{p(x)}. \tag{32}$$

It is not hard to see that

$$\int_{\partial\Omega} D_4|w_n|^{(\eta(x)/1+\delta)}\mathrm{d}S \le d_3 + \int_{\partial\Omega}\frac{\varepsilon_3}{1+\delta}|w_n|^{\eta(x)}\mathrm{d}S. \tag{33}$$

Substitute equations (30)–(33) into the above equation; then,

$$\begin{aligned}
c + \|w_n\|_{p(x)} &\ge a_0\left(\frac{1}{\sigma p^+} - \frac{1}{\eta^-}\right)\int_\Omega |\nabla w_n|^{p(x)}\mathrm{d}x - D_2\left(\frac{\lambda\mu(1+\delta)}{\eta^+} - 1\right)\|w_n\|^{\gamma\theta(x)} \\
&\quad - a_1\int_\Omega \frac{(1+\delta)|\nabla\eta(x)|}{\eta^2(x)}|w_n||\nabla w_n|^{p(x)-1}\mathrm{d}x + \int_{\partial\Omega}\frac{\delta}{\eta(x)}Z(x,w_n)\mathrm{d}S - \int_{\partial\Omega} D_4|w_n|^{(\eta(x)/1+\delta)}\mathrm{d}S - D_4|\Omega| \\
&\ge a_0\left(\frac{1}{\sigma p^+} - \frac{1}{\eta^-}\right)\int_\Omega |\nabla w_n|^{p(x)}\mathrm{d}x - D_2\left(\frac{\lambda\mu(1+\delta)}{\eta^+} - 1\right)\|w_n\|^{\gamma\theta(x)} - a_1 d_1\varepsilon_1\int_\Omega|\nabla w_n|^{p(x)}dx \\
&\quad - a_1 d_1\varepsilon_1^{1-p^+}\int_\Omega |w_n|^{p(x)}\mathrm{d}x + \int_{\partial\Omega}\frac{\delta}{\eta(x)}|w_n|^{\eta(x)}\mathrm{d}S - \int_{\partial\Omega}\frac{\varepsilon_3}{1+\delta}|w_n|^{\eta(x)}\mathrm{d}S - (d_3 + D_4)|\Omega| \\
&\ge a_0\left(\frac{1}{\sigma p^+} - \frac{1}{\eta^-}\right)\int_\Omega |\nabla w_n|^{p(x)}\mathrm{d}x - D_2\left(\frac{\lambda\mu(1+\delta)}{\eta^+} - 1\right)\|w_n\|^{\gamma\theta(x)} - a_1 d_1\varepsilon_1\int_\Omega|\nabla w_n|^{p(x)}\mathrm{d}x \\
&\quad - a_1 d_1\varepsilon_1^{1-p^+}\varepsilon_2\int_\Omega |w_n|^{q(x)}dx - a_1 d_1\varepsilon_1^{1-p^+}\varepsilon_2^{-(p^+/(q-p)^-)}|\Omega| - (d_3 + D_4)|\Omega| + \left(\frac{\delta}{\eta(x)} - \frac{\varepsilon_3}{1+\delta}\right)\int_{\partial\Omega}|w_n|^{\eta(x)}dS \\
&\ge \left(a_0\left(\frac{1}{\sigma p^+} - \frac{1}{\eta^-}\right) - a_1 d_1\varepsilon_1 - a_1 d_1 {\varepsilon_1}^{1-p^+}\varepsilon_2 d_2\right)\|w_n\|^{p(x)} - D_2\left(\frac{\lambda\mu(1+\delta)}{\eta^+} - 1\right)\|w_n\|^{\gamma\theta(x)} \\
&\quad + \left(\frac{\delta}{\eta(x)} - \frac{\varepsilon_3}{1+\delta}\right)\|w_n\|^{\eta(x)} - \left(a_1 d_1\varepsilon_1^{1-p^+}\varepsilon_2^{-(p^+/(q-p)^-)} + d_3 + D_4\right)|\Omega|.
\end{aligned} \tag{34}$$

When the positive constant $\varepsilon_i\,(1 = 1, 2, 3)$ is small enough, we have

$$\begin{gathered}
a_0\left(\frac{1}{\sigma p^+} - \frac{1}{\eta^-}\right) - a_1 d_1\varepsilon_1 - a_1 d_1 {\varepsilon_1}^{1-p^+}\varepsilon_2 d_2 = d_4 > 0, \\
a_1 d_1 {\varepsilon_1}^{1-p^+}{\varepsilon_2}^{-(p^+/(q-p)^-)} + d_3 + D_4 = d_5 > 0.
\end{gathered} \tag{35}$$

Therefore,

$$\begin{aligned}
c + \|w_n\|_{p(x)} &\ge J(w_n) - \left\langle J'(w_n), \frac{1+\delta}{\eta(x)}w_n\right\rangle \\
&\ge d_4\|w_n\|^{p^-} - D_2\left(\frac{\lambda\mu(1+\delta)}{\eta^+} - 1\right)\|w_n\|^{\gamma\theta^+} - d_5|\Omega|.
\end{aligned} \tag{36}$$

Because $\gamma\theta^+ < p^-$, $\{w_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. □

Theorem 12. *Let $\{w_n\} \subset W^{1,p(x)}(\Omega)$ be a (PS) sequence, with energy level c. If $c \leq d_4 \cdot S^N (a_0/D_3)^{(N/p^*(x_i))} - d_5|\Omega|$, then there exists a subsequence $\{w_n\} \longrightarrow w$ in $W^{1,p(x)}(\Omega)$.*

Proof. According to Lemma 10, if $\{w_n\}$ is a PS sequence, it can be concluded that $\{w_n\}$ is bound in $W^{1,p(x)}(\Omega)$. According to Lemma 11, we know that there exists a subsequence $\{w_n\}$ (still denoted as $\{w_j\}$), such that

$$\begin{aligned} &w_j \rightharpoonup \text{w in } W^{1,p(x)}(\Omega),\\ &w_j \longrightarrow \text{w in } L^{r(x)}(\Omega), 1 \leq r(x) < < p^{*(x)},\\ &\left|\nabla w_n\right|^{p(x)} \rightharpoonup d\mu \geq |\nabla w|^{p(x)} + \sum_{j\in I} \mu_i \delta_{x_j}, \quad \mu_j > 0, \end{aligned} \tag{37}$$

$$\left|w_n|_{\partial\Omega}\right|^{q(x)} \rightharpoonup \mathrm{d}\nu = |w|^{q(x)} + \sum_{j\in I} \nu_j \delta_{x_j}, \quad \nu_j > 0, \tag{38}$$

$$S\nu_j^{1/p^*(x_j)} \leq \mu_j^{1/p(x_j)}. \tag{39}$$

Let $\xi(x) \in C_0^\infty(\overline{\Omega})$, and define $\xi(x) = \xi((x - x_i)/\varepsilon)$, such that

$$\begin{aligned} &\xi(x) = 1 \text{ in } B(x_i, \varepsilon),\\ &\xi(x) = 0 \text{ in } B(x_i, 2\varepsilon)^c, |\nabla\xi| \leq \frac{2}{\varepsilon} \text{ in } \Omega. \end{aligned} \tag{40}$$

Consider $\{w_n\xi\}$. As $J'(w_n) \longrightarrow 0$ in $(W^{1,p(x)}(\Omega))^*$, we obtain

$$\lim_{n\longrightarrow\infty} \langle J'(w_n), \xi w_n\rangle = 0, \tag{41}$$

i.e.,

$$\begin{aligned} &A\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\mathrm{d}x\right)\int_\Omega |\nabla w_n|^{p(x)-2}\nabla w_n \cdot \nabla(\xi w_n)\mathrm{d}x - B\left(\int_\Omega F(x, w_n)\mathrm{d}x\right)\int_\Omega f(x, w_n)\xi w_n \mathrm{d}x\\ &-\int_{\partial\Omega} g(x, w_n)\xi w_n \mathrm{d}S \longrightarrow 0, n \longrightarrow \infty. \end{aligned} \tag{42}$$

According to the Hölder inequality, we have

$$\begin{aligned} 0 &\leq \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\left|A\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)}\mathrm{d}x\right)\int_\Omega |\nabla w_n|^{p(x)-2}\nabla w_n \nabla\xi \cdot w_n \mathrm{d}x\right|\\ &\leq a_1 \lim_{\varepsilon\longrightarrow 0}\lim_{n\longrightarrow\infty}\left(\int_\Omega |\nabla w_n|^{p(x)}\right)^{(p(x)-1/p(x))}\left(\int_\Omega |\nabla\xi|^{p(x)}|w_n|^{p(x)}\mathrm{d}x\right)^{1/p(x)}\\ &\leq a_1 C \lim_{\varepsilon\longrightarrow 0}\left(\int_{B(x_i,2\varepsilon)} |\nabla\xi|^{p(x)}|w_n|^{p(x)}\mathrm{d}x\right)^{1/p(x)}\\ &\leq a_1 C \lim_{\varepsilon\longrightarrow 0}\left(\int_{B(x_i,2\varepsilon)} |\nabla\xi|^N \mathrm{d}x\right)^{(1/N)}\left(\int_{B(x_i,2\varepsilon)} |w_n|^{p^*(x)}\mathrm{d}x\right)^{1/p^*(x)}\\ &\leq a_1 C \lim_{\varepsilon\longrightarrow 0}\left(\int_{B(x_i,2\varepsilon)} |w_n|^{p^*(x)}\mathrm{d}x\right)^{1/p^*(x)}\\ &= 0. \end{aligned} \tag{43}$$

It is easy to verify that

$$\lim_{\varepsilon \longrightarrow 0} \lim_{n \longrightarrow \infty} \int_\Omega f(x, w_n) \xi w_n \mathrm{d}x = 0. \tag{44}$$

Hence, from equations (37)–(43), we have

$$\lim_{\varepsilon \longrightarrow 0} \left[A\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)} \mathrm{d}x \right) \int_\Omega |\nabla w_n|^{p(x)} \xi \mathrm{d}x - \int_{\partial\Omega} g(x, w_n) \xi w_n \mathrm{d}S \right] = 0. \tag{45}$$

Then,

$$\begin{aligned} A\left(\int_\Omega \frac{|\nabla w_n|^{p(x)}}{p(x)} \mathrm{d}x \right) \int_\Omega |\nabla w_n|^{p(x)} \xi \mathrm{d}x &= \int_{\partial\Omega} g(x, w_n) \xi w_n \mathrm{d}S, \\ a_0 \int_\Omega |\nabla w_n|^{p(x)} \xi \mathrm{d}x &\le D_3 \int_{\partial\Omega} |w_n|^{q(x)} \xi \mathrm{d}S, \\ a_0 \int_\Omega \xi \mathrm{d}\mu &\le D_3 \int_{\partial\Omega} \xi \mathrm{d}\nu. \end{aligned} \tag{46}$$

When $\varepsilon \longrightarrow 0$, we conclude that $\nu_i \ge a_0/D_3 \mu_i$. Then, through equation (39), we obtain $\mu_i^{1/p(x_i)} \ge S ((a_0/D_3)\mu_i)^{(1/p^*(x_i))}$, which suggests that

$$\mu_i \ge S^N \left(\frac{a_0}{D_3} \right)^{(N/p^*(x_i))} \text{ or } \mu_i = 0. \tag{47}$$

Suppose that the first case $\mu_i \ge S^N (a_0/D_3)^{N/p^*(x_i)}$ is true; for some $i \in I$,

$$\begin{aligned} c &= \lim_{n \longrightarrow \infty} \left(J(w_n) - \left\langle J'(w_n), \frac{1+\delta}{\eta(x)} w_n \right\rangle \right) \\ &\ge d_4 \cdot \lim_{n \longrightarrow \infty} \left(\int_\Omega |\nabla w_n|^{p(x)} \mathrm{d}x \right) - d_5 |\Omega| \\ &= d_4 \cdot \int_\Omega \mathrm{d}\mu - d_5 |\Omega| \\ &\ge d_4 \cdot \int_\Omega |\nabla w|^{p(x)} \mathrm{d}x + d_4 \cdot S^N \left(\frac{a_0}{D_3} \right)^{N/p^*(x_i)} - d_5 |\Omega| \\ &\ge d_4 \cdot S^N \left(\frac{a_0}{D_3} \right)^{N/p^*(x_i)} - d_5 |\Omega|. \end{aligned} \tag{48}$$

This is not true. Consequently, $\mu_i = 0$ for every $i \in I$. Furthermore, when $n \longrightarrow \infty$, we have

$$\int_{\partial\Omega} |w_n|^{q(x)} \mathrm{d}S \longrightarrow \int_{\partial\Omega} |w|^{q(x)} \mathrm{d}S. \tag{49}$$

We have that $\{w_n\} \subset W^{1,p(x)}(\Omega)$ is bounded. Then, for a subsequence $\{w_n\}$ and $w \in W^{1,p(x)}(\Omega)$, we have $\{w_n\} \rightharpoonup w$ in $W^{1,p(x)}(\Omega)$. Observe that

$$\begin{aligned} \|w_n - w\|_{p(x)} = {} & \langle J'(w_n) - J'(w), w_n - w \rangle + B\left(\int_\Omega F(x, w_n - w) \mathrm{d}x \right) \int_\Omega (f(x, w_n) - f(x, w))(w_n - w) \mathrm{d}x \\ & + \int_{\partial\Omega} (g(x, w_n) - g(x, w))(w_n - w) \mathrm{d}S. \end{aligned} \tag{50}$$

In fact, it is clear that

$$\langle J'(w_n) - J'(w), w_n - w \rangle \longrightarrow 0, n \longrightarrow \infty. \tag{51}$$

Using the Hölder inequality and the fact $|w_n - w|_{p(x)} \longrightarrow 0$, $n \longrightarrow \infty$, we obtain

$$\left| \int_\Omega (f(x, w_n) - f(x, w))(w_n - w) \mathrm{d}x \right| \le |f(x, w_n) - f(x, w)|_{q(x)} |w_n - w|_{p(x)} \longrightarrow 0, n \longrightarrow \infty. \tag{52}$$

Because $\{w_n\} \longrightarrow w$ in $L^{q(x)}(\partial\Omega)$, according to Proposition 5, we obtain that $W^{1,p(x)}(\Omega)$ is compactly embedded $L^{q(x)}(\partial\Omega)$. Thus, we obtain

$$\int_{\partial\Omega}(g(x,w_n)-g(x,w))(w_n-w)\mathrm{d}S \longrightarrow 0, n \longrightarrow \infty. \tag{53}$$

Through equations (51)–(53), we can deduce that

$$\begin{aligned}\|w_n-w\|_{p(x)} = A\left(\int_\Omega \frac{|\nabla w_n-\nabla w|^{p(x)}}{p(x)}\mathrm{d}x\right)\int_\Omega\left(|\nabla w_n|^{p(x)-2}\nabla w_n - |\nabla w|^{p(x)-2}\nabla w\right)\\ (\nabla w_n-\nabla w)\mathrm{d}x \longrightarrow 0, n \longrightarrow \infty.\end{aligned} \tag{54}$$

It is known that

$$\left(|u|^{p-2}u-|v|^{p-2}v, u-v\right) \geq \begin{cases} c_p\dfrac{|u-v|^2}{(|u|+|v|)^{2-p}}, & \forall p \leq 2, \\ & \qquad u, v \in \mathbb{R}^N. \\ c_p|u-v|^p, & \forall p \geq 2, \end{cases} \tag{55}$$

Combining equations (54) and (55), we can deduce that

$$\int_\Omega\left(|\nabla w_n-\nabla w|^{p(x)}\right)\mathrm{d}x \longrightarrow 0, n \longrightarrow \infty. \tag{56}$$

Thus, according to Proposition 3 (3), we can prove that $\|w_n-w\|_{p(x)} \longrightarrow 0$, $n \longrightarrow \infty$. □

4. The Proof of Main Results

Proof of Theorem 1. We use the mountain pass theorem to find critical values below level c; thus, we need to verify that the functional J satisfies Theorem 8.

According to Lemma 10, the function J satisfies the local (PS) condition. Apparently, $J(0)=0$.

First, we verify (L_1). If $\|w\|=H$ is small enough, then

$$\begin{aligned} J(w) &= \widehat{A}\left(\int_\Omega \frac{|\nabla w|^{p(x)}}{p(x)}\mathrm{d}x\right) - \widehat{B}\left(\int_\Omega F(x,w)\mathrm{d}x\right) - \int_{\partial\Omega} G(x,w)\mathrm{d}S \\ &\geq C_1\|w\|^{\alpha p^+} - C_2\|w\|^{\beta\tau^-} - \int_{\partial\Omega} C_3|w|^{q(x)}\mathrm{d}S \\ &\geq C_1\|w\|^{\alpha p^+} - C_2\|w\|^{\beta\tau^-} - C_3\|w\|^{q^-}. \end{aligned} \tag{57}$$

We define $T(t)=C_1t^{\alpha p^+}-C_2t^{\beta\tau^-}-C_3t^{q^-}$. Because $\alpha p^+ < \beta\tau^-$ and $\alpha p^+ < q^-$, we can easily obtain that $T(H) > h > 0$ for some H sufficiently small.

Next, we verify (L_2). For sufficiently large $s>0$, from (a_2) it follows that $\widehat{A}(s) \leq C_4s^\sigma$; through (f_1), (f_2), we get that $F(x,s) \geq |s|^\mu$; (b_2) implies that $\widehat{B}(s) \geq C_5|s|^\lambda$; (g_1) and (g_3) imply that $G(x,s) \geq |s|^\kappa \geq |s|^{\lambda\mu}$.

Next, we fix $\varpi \in W^{1,p(x)}(\Omega)\backslash\{0\}$, and then we obtain

$$J(t\varpi) \leq C_6t^{\sigma p^+} - C_7t^{\lambda\mu} - C_8t^{\lambda\mu}. \tag{58}$$

For t large enough, let $v_0 = t\varpi$; because $\sigma p^+ < \lambda\mu$, $J(v_0) = J(t\varpi) \longrightarrow -\infty$ as $t \longrightarrow +\infty$.

We can draw the subsequent results from the Fountain theorem, which is similar to the proof of Theorem 4.8 in [25]. □

Proof of Theorem 2. We prove the result using Theorem 9. From (f_4) and (g_4), it can be known that the functional J is an even energy functional and satisfies the local (PS) condition.

We assume that the $\|w\| > 1$; thus,

$$\begin{aligned} J(w) &= \widehat{A}\left(\int_{\Omega}\frac{|\nabla w|^{p(x)}}{p(x)}\mathrm{d}x\right) - \widehat{B}\left(\int_{\Omega}F(x,w)\mathrm{d}x\right) - \int_{\partial\Omega}G(x,w)\mathrm{d}S \\ &\geq \left(\int_{\Omega}\frac{|\nabla w|^{p(x)}}{p(x)}\mathrm{d}x\right)^{\alpha} - c_1\left(\int_{\Omega}|w|^{\tau(x)}\mathrm{d}x\right)^{\beta} - \int_{\partial\Omega}c_2\left(1+|w|^{q(x)}\right)\mathrm{d}S \\ &\geq \frac{1}{p^+}\|w\|^{\alpha p^-} - c_3\max\left\{|w|^{\beta\tau^+}_{L^{\tau(x)}(\Omega)}, |w|^{\beta\tau^-}_{L^{\tau(x)}(\Omega)}, |w|^{q^+}_{L^{q(x)}(\partial\Omega)}, |w|^{q^-}_{L^{q(x)}(\partial\Omega)}\right\} - c_4. \end{aligned} \tag{59}$$

We obtain the function below if $\max\left\{|w|^{\beta\tau^+}_{L^{\tau(x)}(\Omega)}, |w|^{\beta\tau^-}_{L^{\tau(x)}(\Omega)}, |w|^{q^+}_{L^{q(x)}(\partial\Omega)}, |w|^{q^-}_{L^{q(x)}(\partial\Omega)}\right\} = |w|^{\beta\tau^+}_{L^{\tau(x)}(\Omega)}$.

$$\begin{aligned} J(w) &\geq \frac{1}{p^+}\|w\|^{\alpha p^-} - c_3\max\left\{|w|^{\beta\tau^+}_{L^{\tau(x)}(\Omega)}, |w|^{\beta\tau^-}_{L^{\tau(x)}(\Omega)}, |w|^{q^+}_{L^{q(x)}(\partial\Omega)}, |w|^{q^-}_{L^{q(x)}(\partial\Omega)}\right\} - c_4 \\ &\geq \frac{1}{p^+}\|w\|^{\alpha p^-} - c_3\alpha_k^{\beta\tau^+}\|w\|^{\beta\tau^+} - c_4 \\ &= \|w\|^{\beta\tau^+}\left(\frac{1}{p^+}\|w\|^{\alpha p^- - \beta\tau^+} - c_3\alpha_k^{\beta\tau^+}\right) - c_4. \end{aligned} \tag{60}$$

Now, we take $r_k = \|w\| = (\tau^+ c_3\alpha_k^{\beta\tau^+})^{(1/\alpha p^- - \beta\tau^+)}$; accordingly, we have

$$\begin{aligned} J(w) &\geq \frac{1}{p^+}\|w\|^{\alpha p^-} - c_3\alpha_k^{\beta\tau^+}\|w\|^{\beta\tau^+} - c_4 \\ &= \frac{1}{p^+}\|w\|^{\alpha p^-} - \frac{1}{\tau^+}\|w\|^{\alpha p^- - \beta\tau^+ + \beta\tau^+} - c_4 \\ &= \left(\frac{1}{p^+} - \frac{1}{\tau^+}\right)\|w\|^{\alpha p^-} - c_4. \end{aligned} \tag{61}$$

Because $\alpha_j \longrightarrow 0$, $r_j \longrightarrow \infty$, and $\tau^- > p^+$, $J(w) \longrightarrow \infty$.

For the other cases, using a similar method, we obtain $J(w) \longrightarrow \infty$, since $\alpha_j \longrightarrow 0$, $\beta_j \longrightarrow 0$, $j \longrightarrow \infty$. Thus, (N_2) is true.

According to (f_2) and (g_3), we obtain

$$\begin{aligned} F(x,s) &\geq c_5|s|^{\mu} - c_5, \quad \forall (x,s) \in \Omega\times\mathbb{R}, \\ G(x,s) &\geq c_6|s|^{\kappa} - c_6, \quad \forall (x,s) \in \partial\Omega\times\mathbb{R}. \end{aligned} \tag{62}$$

Let $w \in Y_j$; then, we have

$$\begin{aligned} J(w) &\leq \frac{1}{p^-}\|w\|^{\sigma p^+} - \left(\int_{\Omega}(c_5|w|^{\mu} - c_5)\mathrm{d}x\right)^{\lambda} - \int_{\partial\Omega}(c_6|w|^{\kappa} - c_6)\mathrm{d}S \\ &\leq \frac{1}{p^-}\|w\|^{\sigma p^+} - \left(c_5\int_{\Omega}|w|^{\mu}\mathrm{d}x\right)^{\lambda} - c_6\int_{\partial\Omega}|w|^{\kappa}\mathrm{d}S + c_7. \end{aligned} \tag{63}$$

Because $\dim Y_j < \infty$, all norms on Y_j are equivalent. Therefore,

$$J(w) \leq \frac{1}{p^-}\|w\|^{\sigma p^+} - c_5\|w\|^{\lambda\mu} - c_6\|w\|^{\kappa} + c_7. \tag{64}$$

When $\|w\| \longrightarrow \infty$, we have $a_j \longrightarrow -\infty$ and $\kappa > \lambda\mu > \sigma p^+$. Thus, (N_1) is true.

On the basis of the proofs of (N_1) and (N_2), we let $\rho_j > r_j > 0$. Then, the conclusion is valid. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Each part of this paper is the result of the joint efforts of LJ and MQ. They contributed equally to the final version of the paper. All the authors have read and approved the final manuscript.

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