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Research Article

Dynamical Behavior and Expressions of Solutions of a Class of Higher-Order Nonlinear Difference Equations

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The aim of this article is to get the forms of the solutions of the following nonlinear higher-order difference equations $\chi_{s+1} = (\chi_{s-3k+1}/\pm 1 \pm \Pi_{l=1}^{k}\chi_{s-3l+1})$, $s = 0, 1, 2, \ldots$, where the initial conditions χ_{-j}, $j = 0, 1, 2, \ldots, 3k-1$ and $k \in \{1, 2, \ldots\}$ are arbitrary real numbers. Also, we examine stability, boundedness, oscillation, and the periodic nature of these solutions.

1. Introduction

Difference equations have played a principal role in the structure and examination of mathematical models of biology, ecology, and physics. The study of nonlinear rational difference equations of higher-order is of prime importance. Lately, there has been a lot of interest in studying the global attractivity, the boundedness, and the solution form of these equations. For more results in this field, see [1–13].

In [14, 15], respectively, the authors obtained the solutions of the following equations:

$$\chi_{s+1} = \frac{\chi_{s-14}}{\pm 1 \pm \chi_{s-2}\chi_{s-5}\chi_{s-8}\chi_{s-11}\chi_{s-14}}, \quad s = 0, 1, 2, \ldots, \tag{1}$$

with initial conditions $\chi_{-j} \in \mathbb{R}$ and $j = 0, 1, 2, \ldots, 14$ and

$$\chi_{s+1} = \frac{\chi_{s-2k+1}}{\pm 1 \pm \Pi_{j=1}^{k}\chi_{s-2j+1}}, \quad s = 0, 1, 2, \ldots, \tag{2}$$

with initial conditions $\chi_{-j} \in \mathbb{R}$, $j = 0, 1, 2, \ldots, 2k-1$ and $k \in \{1, 2, \ldots\}$.

In [16], the authors obtained the solutions of

$$\chi_{s+1} = \frac{\chi_{s-11}}{\pm 1 \pm \chi_{s-2}\chi_{s-5}\chi_{s-8}\chi_{s-11}}, \quad s = 0, 1, 2, \ldots, \tag{3}$$

where $\chi_{-j} \in \mathbb{R}$ and $j = 0, 1, 2, \ldots, 11$.

In this work, we get the solutions of the next difference equations

$$\chi_{s+1} = \frac{\chi_{s-3k+1}}{\pm 1 \pm \Pi_{j=1}^{k}\chi_{s-3j+1}}, \quad s = 0, 1, 2, \ldots, \tag{4}$$

where the initial conditions $\chi_{-j} \in \mathbb{R}$, $j = 0, 1, 2, \ldots, 3k-1$ and $k \in \{1, 2, \ldots\}$. Following that, we investigate the behavior of these solutions. All over this paper, we define mod $(j, 3) = j - 3[j/3]$ where $[\chi]$ is the greatest integer less than or equal to $\chi \in \mathbb{R}$.

2. The Difference Equation $\chi_{s+1} = (\chi_{s-3k+1}/1 + \prod_{l=1}^{k}\chi_{s-3l+1})$

In this portion, we give an express shape of the solutions of the following equation:

$$\chi_{s+1} = \frac{\chi_{s-3k+1}}{1+\prod_{l=1}^{k}\chi_{s-3l+1}}, \quad s = 0, 1, 2, \ldots, \tag{5}$$

where $\chi_{-l} \in \mathbb{R}, l = 0, 1, 2, \ldots, 3k-1; k \in \{1, 2, \ldots\}$. Also, we discuss the stability and boundedness of these solutions.

Theorem 1. *Assume that $\{\chi_s\}_{s=-3k+1}^{\infty}$ is a solution of equation (5), then, for $s = 0, 1, 2, \ldots$,*

$$\chi_{3k-j} = a_j \prod_{l=0}^{s-1}\left(\frac{1+(kl+M_j-1)P_j}{1+(kl+M_j)P_j}\right), \quad j = 0, 1, 2, \ldots, 3k-1, \tag{6}$$

where $P_j = \prod_{r=0}^{k-1} a_{\mathrm{mod}(j,3)+3r}$, $M_j = k - [j/3]$, and $\chi_{-j} = a_j$ with $tP_j \neq -1$ such that $t \in \{1, 2, 3, \ldots\}$.

Proof. For $s = 0$, the consequence holds. Now, let $s > 0$ and our hypothesis holds for $s-1$. Then,

$$\chi_{3ks-3k-j} = a_j \prod_{l=0}^{s-2}\left(\frac{1+(kl+M_j-1)P_j}{1+(kl+M_j)P_j}\right). \tag{7}$$

From equation (5) and using equation (7), we get

$$\begin{aligned}
\chi_{3ks-(3k-1)} &= \frac{\chi_{3ks-6k+1}}{1+\chi_{3ks-3k-2}\chi_{3ks-3k-5}\cdots\chi_{3ks-6k+1}} \\
&= \frac{a_{3k-1}\prod_{l=0}^{s-2}(1+(kl+M_{3k-1}-1)P_{3k-1}/1+(kl+M_{3k-1})P_{3k-1})}{1+\prod_{j=0}^{k-1}\left(a_{3j+2}\prod_{l=0}^{s-2}\left(1+\left(kl+M_{3j+2}-1\right)P_{3j+2}/1+\left(kl+M_{3j+2}\right)P_{3j+2}\right)\right)} \\
&= \frac{a_{3k-1}\prod_{l=0}^{s-2}(1+(kl+M_{3k-1}-1)P_{3k-1}/1+(kl+M_{3k-1})P_{3k-1})}{1+a_2a_5\ldots a_{3k-1}\prod_{l=0}^{s-2}(1+(kl+M_{3k-1}-1)P_{3k-1}/1+(kl+k)P_{3k-1})}.
\end{aligned} \tag{8}$$

Hence, we have

$$\chi_{3ks-(3k-1)} = a_{3k-1}\prod_{l=0}^{s-1}\left(\frac{1+(kl+M_{3k-1}-1)P_{3k-1}}{1+(kl+M_{3k-1})P_{3k-1}}\right). \tag{9}$$

Similarly, we get

$$\begin{aligned}
\chi_{3ks-(3k-2)} &= \frac{\chi_{3ks-6k+2}}{1+\chi_{3ks-3k-1}\chi_{3ks-3k-4}\cdots\chi_{3ks-6k+2}} \\
&= \frac{a_{3k-2}\prod_{l=0}^{s-2}(1+(kl+M_{3k-2}-1)P_{3k-2}/1+(kl+M_{3k-2})P_{3k-2})}{1+\prod_{j=0}^{k-1}\left(a_{3j+1}\prod_{l=0}^{s-2}\left(1+\left(kl+M_{3j+1}-1\right)P_{3j+1}/1+\left(kl+M_{3j+1}\right)P_{3j+1}\right)\right)} \\
&= \frac{a_{3k-2}\prod_{l=0}^{s-2}(1+(kl+M_{3k-2}-1)P_{3k-2}/1+(kl+M_{3k-2})P_{3k-2})}{1+a_1a_4\ldots a_{3k-2}\prod_{l=0}^{s-2}(1+(kl+M_{3k-2}-1)P_{3k-2}/1+(kl+k)P_{3k-2})}.
\end{aligned} \tag{10}$$

Hence, we have

$$\chi_{3ks-(3k-2)} = a_{3k-2}\prod_{l=0}^{s-1}\left(\frac{1+(kl+M_{3k-2}-1)P_{3k-2}}{1+(kl+M_{3k-2})P_{3k-2}}\right). \quad (11)$$

Once again, from equation (5) and using equation (7), we get

$$\begin{aligned}\chi_{3ks-(3k-3)} &= \frac{\chi_{3ks-6k+3}}{1+\chi_{3ks-3k}\chi_{3ks-3k-3}\cdots\chi_{3ks-6k+3}}\\ &= \frac{a_{3k-3}\prod_{l=0}^{s-2}\left(1+(kl+M_{3k-3}-1)P_{3k-3}/1+(kl+M_{3k-3})P_{3k-3}\right)}{1+\prod_{j=0}^{k-1}\left(a_{3j}\prod_{l=0}^{s-2}\left(1+\left(kl+M_{3j}-1\right)P_{3j}/1+\left(kl+M_{3j}\right)P_{3j}\right)\right)}\\ &= \frac{a_{3k-3}\prod_{l=0}^{s-2}\left(1+(kl+M_{3k-3}-1)P_{3k-3}/1+(kl+M_{3k-3})P_{3k-3}\right)}{1+a_0a_3\ldots a_{3k-3}\prod_{l=0}^{s-2}\left(1+(kl+M_{3k-3}-1)P_{3k-3}/1+(kl+k)P_{3k-3}\right)}.\end{aligned} \quad (12)$$

Hence, we have

$$\chi_{3ks-(3k-3)} = a_{3k-3}\prod_{l=0}^{s-1}\left(\frac{1+(kl+M_{3k-3}-1)P_{3k-3}}{1+(kl+M_{3k-3})P_{3k-3}}\right). \quad (13)$$

Similarly, one can readily get the other relations for equation (6). The proof is finished. □

Theorem 2. *If* $\chi_{-3k+1},\chi_{-3k+2},\ldots,\chi_0\in[0,\infty)$ *where* $k\in\{1,2,\ldots\}$, *then every solution of equation (5) is bounded.*

Proof. Suppose that $\{\chi_s\}_{s=-3k+1}^{\infty}$ is a solution of equation (5). Then,

$$0\le\chi_{s+1} = \frac{\chi_{s-3k+1}}{1+\Pi_{l=1}^{k}\chi_{s-3l+1}}\le\chi_{s-3k+1}\quad\text{for all } s\ge 0. \quad (14)$$

Hence, the sequence $\{\chi_{3ks-l}\}_{s=0}^{\infty}$, $l=0,1,\ldots,3k-1$ is decreasing and thus is bounded from above by $M=\max\{\chi_{-3k+1},\chi_{-3k+2},\ldots,\chi_0\}$. □

Theorem 3. *Equation (5) has only equilibrium point* $\bar{\chi}=0$.

Proof. By using equation (5), we have

$$\bar{\chi} = \frac{\bar{\chi}}{1+\bar{\chi}^{k}} \Rightarrow \bar{\chi}+\bar{\chi}^{k+1} = \bar{\chi}. \quad (15)$$

Thus,

$$\bar{\chi}^{k+1} = 0. \quad (16)$$

The proof is finished. □

Theorem 4. *Assume that* $\chi_{-3k+1},\chi_{-3k+2},\ldots,\chi_0\in[0,\infty)$ *where* $k\in\{1,2,\ldots\}$, *then* $\bar{\chi}=0$ *is locally stable.*

Proof. Let $\epsilon>0$ and $\{\chi_s\}_{s=-3k+1}^{\infty}$ be a solution of equation (5) such that

$$\sum_{j=0}^{3k-1}\left|\chi_{-j}\right|<\epsilon. \quad (17)$$

It suffices to show that $|\chi_1|<\epsilon$. Now,

$$0<\chi_1 = \frac{\chi_{-3k+1}}{1+\Pi_{l=1}^{k}\chi_{-3l+1}}\le\chi_{-3k+1}<\epsilon. \quad (18)$$

The proof is finished. □

Theorem 5. *Let* $\chi_{-3k+1},\chi_{-3k+2},\ldots,\chi_0\in[0,\infty)$ *where* $k\in\{1,2,\ldots\}$. *Then,* $\bar{\chi}=0$ *is globally asymptotically stable.*

Proof. We teach via Theorem 4 that $\bar{\chi}=0$ is locally stable. Now, let $\{\chi_s\}_{s=-3k+1}^{\infty}$ be a positive solution of equation (5). It is enough to show that

$$\lim_{s\longrightarrow\infty}\chi_s = \bar{\chi} = 0. \quad (19)$$

By Theorem 2, we own $\chi_{s+1}<\chi_{s-3k+1}$ $\forall s\ge 0$. Thus, $\{\chi_{3ks-l}\}_{s=0}^{\infty}$ and $l=0,1,\ldots,3k-1$ are decreasing and bounded which implies that $\{\chi_{3ks-l}\}_{s=0}^{\infty}$ converge to limit (say $F_l>0$). Consequently,

$$\begin{aligned}F_{3k-1} &= \frac{F_{3k-1}}{1+F_2F_5\cdots F_{3k-1}},\\ F_{3k-2} &= \frac{F_{3k-2}}{1+F_1F_4\cdots F_{3k-2}},\ldots,F_0 = \frac{F_0}{1+F_0F_3\cdots F_{3k-3}},\end{aligned} \quad (20)$$

which implies that $F_0=F_1=\ldots=F_{3k-1}=0$, from which the result follows. □

3. The Difference Equation $\chi_{s+1}=(\chi_{s-3k+1}/1-\Pi_{l=1}^{k}\chi_{s-3l+1})$

In this portion, we give an express shape of the solutions for the following equation:

$$\chi_{s+1} = \frac{\chi_{s-3k+1}}{1-\Pi_{l=1}^{k}\chi_{s-3l+1}},\quad s=0,1,2,\ldots, \quad (21)$$

where $\chi_{-l}, l = 0, 1, 2, \ldots, 3k-1$ and $k \in \{1, 2, \ldots\}$ are arbitrary real numbers.

Theorem 6. *Let $\{\chi_s\}_{s=-3k+1}^{\infty}$ be a solution of equation (21). Then, for $s = 0, 1, 2, \ldots$,*

$$\chi_{3ks-j} = a_j \prod_{l=0}^{s-1} \left(\frac{-1 + \left(kl + M_j - 1\right) P_j}{-1 + \left(kl + M_j\right) P_j} \right), \quad j = 0, 1, 2, \ldots, 3k-1, \tag{22}$$

where $P_j = \prod_{r=0}^{k-1} a_{\mathrm{mod}(j,3)+3r}$, $M_j = k - [j/3]$, and $\chi_{-j} = a_j$, with $tP_j \neq 1$ such that $t \in \{1, 2, 3, \ldots\}$.

Proof. For $s = 0$, the conclusion holds. Now, let $s > 0$ and that our hypothesis is verified for $s - 1$. Then,

$$\chi_{3ks-3k-j} = a_j \prod_{l=0}^{s-2} \left(\frac{-1 + \left(kl + M_j - 1\right) P_j}{-1 + \left(kl + M_j\right) P_j} \right). \tag{23}$$

By equation (21) and using equation (23), we obtain

$$\begin{aligned}
\chi_{3ks-(3k-1)} &= \frac{\chi_{3ks-6k+1}}{1 - \chi_{3ks-3k-2}\chi_{3ks-3k-5}\cdots\chi_{3ks-6k+1}} \\
&= \frac{a_{3k-1} \prod_{l=0}^{s-2} \left(-1 + (kl + M_{3k-1} - 1)P_{3k-1} / -1 + (kl + M_{3k-1})P_{3k-1}\right)}{1 - \prod_{j=0}^{k-1} \left(a_{3j+2} \prod_{l=0}^{s-2} \left(-1 + \left(kl + M_{3j+2} - 1\right) P_{3j+2} / -1 + \left(kl + M_{3j+2}\right) P_{3j+2}\right)\right)} \\
&= \frac{a_{3k-1} \prod_{l=0}^{s-2} \left(-1 + (kl + M_{3k-1} - 1)P_{3k-1} / -1 + (kl + M_{3k-1})P_{3k-1}\right)}{1 - a_2 a_5 \ldots a_{3k-1} \prod_{l=0}^{s-2} \left(-1 + (kl + M_{3k-1} - 1)P_{3k-1} / -1 + (kl + k)P_{3k-1}\right)}.
\end{aligned} \tag{24}$$

Hence, we have

$$\chi_{3ks-(3k-1)} = a_{3k-1} \prod_{l=0}^{s-1} \left(\frac{-1 + (kl + M_{3k-1} - 1)P_{3k-1}}{-1 + (kl + M_{3k-1})P_{3k-1}} \right). \tag{25}$$

Similarly, we get

$$\begin{aligned}
\chi_{3ks-(3k-2)} &= \frac{\chi_{3ks-6k+2}}{1 - \chi_{3ks-3k-1}\chi_{3ks-3k-4}\cdots\chi_{3ks-6k+2}} \\
&= \frac{a_{3k-2} \prod_{l=0}^{s-2} \left(-1 + (kl + M_{3k-2} - 1)P_{3k-2} / -1 + (kl + M_{3k-2})P_{3k-2}\right)}{1 - \prod_{j=0}^{k-1} \left(a_{3j+1} \prod_{l=0}^{s-2} \left(-1 + \left(kl + M_{3j+1} - 1\right) P_{3j+1} / -1 + \left(kl + M_{3j+1}\right) P_{3j+1}\right)\right)} \\
&= \frac{a_{3k-2} \prod_{l=0}^{s-2} \left(-1 + (kl + M_{3k-2} - 1)P_{3k-2} / -1 + (kl + M_{3k-2})P_{3k-2}\right)}{1 - a_1 a_4 \ldots a_{3k-2} \prod_{l=0}^{s-2} \left(-1 + (kl + M_{3k-2} - 1)P_{3k-2} / -1 + (kl + k)P_{3k-2}\right)}.
\end{aligned} \tag{26}$$

Hence, we have

$$\chi_{3ks-(3k-2)} = a_{3k-2}\prod_{l=0}^{s-1}\left(\frac{-1+(kl+M_{3k-2}-1)P_{3k-2}}{-1+(kl+M_{3k-2})P_{3k-2}}\right). \quad (27)$$

Once again, from equation (21) and using equation (23), we get

$$\begin{aligned}\chi_{3ks-(3k-3)} &= \frac{\chi_{3ks-6k+3}}{1-\chi_{3ks-3k}\chi_{3ks-3k-3}\cdots\chi_{3ks-6k+3}}\\ &= \frac{a_{3k-3}\prod_{l=0}^{s-2}(-1+(kl+M_{3k-3}-1)P_{3k-3}/-1+(kl+M_{3k-3})P_{3k-3})}{1-\prod_{j=0}^{k-1}\left(a_{3j}\prod_{l=0}^{s-2}\left(-1+\left(kl+M_{3j}-1\right)P_{3j}/-1+\left(kl+M_{3j}\right)P_{3j}\right)\right)}\\ &= \frac{a_{3k-3}\prod_{l=0}^{s-2}(-1+(kl+M_{3k-3}-1)P_{3k-3}/-1+(kl+M_{3k-3})P_{3k-3})}{1-a_0a_3\ldots a_{3k-3}\prod_{j=0}^{s-2}(-1+(kj+M_{3k-3}-1)P_{3k-3}/-1+(kj+k)P_{3k-3})}.\end{aligned} \quad (28)$$

Hence, we have

$$\chi_{3ks-(3k-3)} = a_{3k-3}\prod_{l=0}^{s-1}\left(\frac{-1+(kl+M_{3k-3}-1)P_{3k-3}}{-1+(kl+M_{3k-3})P_{3k-3}}\right). \quad (29)$$

Similarly, one can readily get the other relations for equation (22). The proof is finished. □

Theorem 7. *Equation (21) has only $\bar{\chi}=0$, which is a nonhyperbolic fixed point.*

Proof. By equation (21), we have

$$\bar{\chi} = \frac{\bar{\chi}}{1-\bar{\chi}^{k}} \Rightarrow \bar{\chi}-\bar{\chi}^{k+1} = \bar{\chi}. \quad (30)$$

Thus,

$$\bar{\chi}^{k+1} = 0 \Rightarrow \bar{\chi} = 0. \quad (31)$$

Now, define the function $f(\chi_1,\chi_2,\ldots,\chi_k) = (\chi_1/1-\chi_1\chi_2\ldots\chi_k)$ on I^k where I is a subset of $\mathbb{R}$ such that $0\in I$ and $f(I^k)\subseteq I$. Clearly, f is continuously differentiable on I^k, and we have

$$\begin{aligned}&f_{\chi_1}(\chi_1,\chi_2,\ldots,\chi_k) = \frac{1}{(1-\chi_1\chi_2\cdots\chi_k)^2}, f_{\chi_2}(\chi_1,\chi_2,\ldots,\chi_k) = \frac{\chi_1^2\chi_3\cdots\chi_k}{(1-\chi_1\chi_2\cdots\chi_k)^2},\ldots,\\ &f_{\chi_k}(\chi_1,\chi_2,\ldots,\chi_k) = \frac{\chi_1^2\cdots\chi_{k-1}}{(1-\chi_1\chi_2\cdots\chi_k)^2},\end{aligned} \quad (32)$$

which implies that

$$f_{\chi_1}(\bar{\chi},\bar{\chi},\ldots,\bar{\chi}) = 1, f_{\chi_2}(\bar{\chi},\bar{\chi},\ldots,\bar{\chi}) = \ldots = f_{\chi_k}(\bar{\chi},\bar{\chi},\ldots,\bar{\chi}) = 0. \quad (33)$$

Thus, the linearized equation of equation (21) about $\bar{\chi}=0$ is

$$z_{s+1} = z_{s-3k+1}, \quad (34)$$

and the characteristic equation is

$$\lambda^{3k}-1 = 0 \Rightarrow |\lambda_l| = 1; \quad l = 1,2,\ldots,3k, \quad (35)$$

so $\bar{\chi}$ is a nonhyperbolic equilibrium point. □

Open Question 8. Discuss the global behavior of solutions of equation (21) about $\bar{\chi}=0$.

4. The Difference Equation $\chi_{s+1} = (\chi_{s-3k+1}/-1+\prod_{l=1}^{k}\chi_{s-3l+1})$

In this portion, we give an express shape of the solutions for the following equation:

$$\chi_{s+1} = \frac{\chi_{s-3k+1}}{-1+\prod_{l=1}^{k}\chi_{s-3l+1}}, \quad s = 0,1,2,\ldots, \quad (36)$$

where $\chi_{-l}, l = 0,1,2,\ldots,3k-1$ and $k\in\{1,2,\ldots\}$ are arbitrary real numbers. In addition to this, we examine the oscillation and periodicity of these solutions.

Theorem 9. *If k is odd and $P_j \neq 2$, then equation (36) has a periodic solution with period $6k$.*

Proof. Using equation (36), we get

$$\chi_{s+6k} = \frac{\chi_{s+3k}}{-1 + \prod_{l=0}^{k-1} \chi_{s+3k+3l}}. \quad (37)$$

Since

$$-1 + \prod_{l=0}^{k-1} \chi_{s+3k+3l} = -1 + \prod_{l=0}^{k-2} \chi_{s+3k+3l} \frac{\chi_{s+3k-3}}{-1 + \prod_{l=0}^{k-1} \chi_{s+3k-3+3l}}, \quad (38)$$

then

$$-1 + \prod_{l=0}^{k-1} \chi_{s+3k+3l} = \frac{1}{-1 + \prod_{l=0}^{k-1} \chi_{s+3k-3+3l}}. \quad (39)$$

Similarly, since

$$-1 + \prod_{l=0}^{k-1} \chi_{s+3k-3+3l} = \frac{1}{-1 + \prod_{l=0}^{k-1} \chi_{s+3k-6+3l}}, \quad (40)$$

then

$$-1 + \prod_{l=0}^{k-1} \chi_{s+3k+3l} = -1 + \prod_{l=0}^{k-1} \chi_{s+3k-6+3l}. \quad (41)$$

Also, since

$$\begin{aligned} -1 + \prod_{l=0}^{k-1} \chi_{s+3k-6+3l} &= -1 + \prod_{l=0}^{k-1} \chi_{s+3k-12+3l}, \\ -1 + \prod_{l=0}^{k-1} \chi_{s+3k-12+3l} &= \frac{1}{-1 + \prod_{l=0}^{k-1} \chi_{s+3k-15+3l}}, \end{aligned} \quad (42)$$

then

$$\chi_{s+6k} = \frac{\left(\chi_s / -1 + \prod_{l=0}^{k-1} \chi_{s+3k-15+3l}\right)}{\left(1/-1 + \prod_{l=0}^{k-1} \chi_{s+3k-15+3l}\right)} = \chi, \quad s = 0, 1, 2, \ldots. \quad (43)$$

□

Theorem 10. *Assume that k is odd, then the periodic $6k$ solution of equation (36) has the form*

$$\chi_{6ks-j} = \frac{a_{j-q_j(3k)^{q_j}}}{\left(-1 + P_{j-3k}\right)^{\alpha_j q_j}}; \quad j = 0, 1, \ldots, 6k-1, P_j \neq 2 \text{ and } s = 1, 2, \ldots, \quad (44)$$

where $\chi_{-t} = a_t, P_t = \prod_{r=0}^{k-1} a_{\bmod(t,3)+3r}$, *with* $P_t \neq 1$, $t = 0, 1, 2, \ldots, 3k-1$; $k \in \{1, 2, \ldots\}$, $P_{-l} = 0$, $l = 1, 2, \ldots, 3k$, $q_j = 1/2\left((-1)^{[j/3k]+1} + 1\right)$, *and* $\alpha_j = (-1)^{[j/3]+1}$.

Proof. From the definition of q_j, we can see that

$$\begin{aligned} q_0 &= q_1 = \ldots = q_{3k-1} = 0, \\ q_{3k} &= q_{3k+1} = \ldots = q_{6k-1} = 1. \end{aligned} \quad (45)$$

Also,

$$\alpha_{l+6r} = -1, \alpha_{l+3+6r} = 1; \quad l = 0, 1, 2 \text{ and } r = 0, 1, 2, \ldots, k-1. \quad (46)$$

So,

$$\begin{aligned} \chi_1 &= \frac{\chi_{-3k+1}}{-1 + \chi_{-2}\chi_{-5} \cdots \chi_{-3k+1}} = \frac{a_{3k-1}}{-1 + a_2 a_5 \ldots a_{3k-1}}, \\ \chi_2 &= \frac{\chi_{-3k+2}}{-1 + \chi_{-1}\chi_{-4} \cdots \chi_{-3k+2}} = \frac{a_{3k-2}}{-1 + a_1 a_4 \ldots a_{3k-2}}, \\ \chi_3 &= \frac{\chi_{-3k+3}}{-1 + \chi_0\chi_{-3} \cdots \chi_{-3k+3}} = \frac{a_{3k-3}}{-1 + a_0 a_3 \ldots a_{3k-3}}, \\ &\vdots \end{aligned}$$

$$\begin{aligned}\chi_{3k} &= \frac{\chi_0}{-1+\chi_0\chi_3\cdots\chi_{3k-3}} = \frac{a_0}{-1+a_0a_3\ldots a_{3k-3}},\\ \chi_{3k+1} &= \frac{\chi_1}{-1+\chi_1\chi_4\cdots\chi_{3k-2}} = a_{3k-1},\\ \chi_{3k+2} &= \frac{\chi_2}{-1+\chi_2\chi_5\cdots\chi_{3k-1}} = a_{3k-2},\\ \chi_{3k+3} &= \frac{\chi_3}{-1+\chi_3\chi_6\cdots\chi_{3k}} = a_{3k-3},\\ &\cdot\\ &\cdot\\ &\cdot\\ \chi_{6k} &= \frac{\chi_{3k}}{-1+\chi_{6k-3}\chi_{6k-6}\cdots\chi_{3k}} = a_0,\end{aligned} \tag{47}$$

and the result follows by induction. □

Theorem 11. *Assume that k is odd, then equation (36) has $\bar{\chi}_1 = 0$ and $\bar{\chi}_2 = \sqrt[k]{2}$, which are nonhyperbolic equilibrium points.*

Proof. The evidence is identical to the proof of Theorem 7 and shall be neglected. □

Theorem 12. *Let k be even and $\{\chi_s\}_{s=-3k+1}^{\infty}$ be a solution of equation (36). Then, for $s = 0, 1, 2, \ldots$,*

$$\chi_{3ks-j} = \frac{a_j}{\left(-1+P_j\right)^{sq_j}}; \quad j = 0, 1, 2, \ldots, 3k-1, \tag{48}$$

where $P_j = \prod_{r=0}^{k-1} a_{\text{mod}(j,3)+3r}$, $q_j = (-1)^{[j/3]+1}$, and $\chi_{-j} = a_j$, with $P_j \neq 1$.

Proof. For $s = 0$, the conclusion holds. Now, let $s > 0$ and our hypothesis is verified for $s-1$. Then,

$$\chi_{3ks-3k-j} = \frac{a_j}{\left(-1+P_j\right)^{(s-1)q_j}}. \tag{49}$$

From equation (36) and using equation (49), we get

$$\begin{aligned}\chi_{3ks-(3k-1)} &= \frac{\chi_{3ks-6k+1}}{-1+\chi_{3ks-3k-2}\chi_{3ks-3k-5}\cdots\chi_{3ks-6k+1}}\\ &= \frac{\left(a_{3k-1}/(-1+P_{3k-1})^{s-1}\right)}{-1+\left(a_2(-1+P_2)^{s-1}\right)\left(a_5/(-1+P_5)^{s-1}\right)\ldots\left(a_{3k-1}/(-1+P_{3k-1})^{s-1}\right)}\\ &= \frac{a_{3k-1}}{(-1+a_2a_5\ldots a_{3k-1})^{s-1}(-1+a_2a_5\ldots a_{3k-1})}.\end{aligned} \tag{50}$$

Hence, we have

$$\chi_{3ks-(3k-1)} = \frac{a_{3k-1}}{\left(-1 + P_{3k-1}\right)^{s}}. \tag{51}$$

Similarly, we get

$$\begin{aligned}
\chi_{3ks-(3k-2)} &= \frac{\chi_{3ks-6k+2}}{-1 + \chi_{3ks-3k-1}\chi_{3ks-3k-4}\cdots\chi_{3ks-6k+2}} \\
&= \frac{\left(a_{3k-2}/\left(-1 + P_{3k-2}\right)^{s-1}\right)}{-1 + a_1\left(-1 + P_1\right)^{s-1}\left(a_4/\left(-1 + P_4\right)^{s-1}\right)\ldots\left(a_{3k-2}/\left(-1 + P_{3k-2}\right)^{s-1}\right)} \\
&= \frac{a_{3k-2}}{\left(-1 + a_1a_4\ldots a_{3k-2}\right)^{s-1}\left(-1 + a_1a_4\ldots a_{3k-2}\right)}.
\end{aligned} \tag{52}$$

Hence, we have

$$\chi_{3ks-(3k-2)} = \frac{a_{3k-2}}{\left(-1 + P_{3k-2}\right)^{s}}. \tag{53}$$

Once again, from equation (36) and using equation (49), we get

$$\begin{aligned}
\chi_{3ks-(3k-3)} &= \frac{\chi_{3ks-6k+3}}{-1 + \chi_{3ks-3k}\chi_{3ks-3k-3}\cdots\chi_{3ks-6k+3}} \\
&= \frac{\left(a_{3k-3}/\left(-1 + P_{3k-3}\right)^{s-1}\right)}{-1 + a_0\left(-1 + P_0\right)^{s-1}\left(a_3/\left(-1 + P_3\right)^{s-1}\right)\ldots\left(a_{3k-3}/\left(-1 + P_{3k-3}\right)^{s-1}\right)} \\
&= \frac{a_{3k-3}}{\left(-1 + a_0a_3\ldots a_{3k-3}\right)^{s-1}\left(-1 + a_0a_3\ldots a_{3k-3}\right)}.
\end{aligned} \tag{54}$$

Hence, we have

$$\chi_{3ks-(3k-3)} = \frac{a_{3k-3}}{\left(-1 + P_{3k-3}\right)^{s}}. \tag{55}$$

Similarly, one can readily get the other relations for equation (48). The proof is finished. □

Theorem 13. *Assume that k is even, then equation (36) has three equilibrium points $\bar{\chi}_1 = 0$, $\bar{\chi}_2 = \sqrt[k]{2}$, and $\bar{\chi}_3 = -\sqrt[k]{2}$, which are nonhyperbolic fixed points.*

Proof. The evidence is identical to the proof of Theorem 7 and shall be neglected. □

Theorem 14. *Equation (36) is periodic of period $3k$ iff $P_j = 2$ and $j = 0, 1, \ldots, 3k-1$ and will be take the form*

$$\chi_{3ks-j} = a_j; \quad j = 0, 1, \ldots, 3k-1 \text{ and } s = 0, 1, 2, \ldots. \tag{56}$$

Proof. The proof follows immediately from Theorems 10 or 12. □

Theorem 15. *Assume that $a_0, a_1, \ldots, a_{3k-1} \in (0, 1)$, then the solution $\{\chi_s\}_{s=-3k+1}^{\infty}$ oscillates about the equilibrium point $\bar{\chi}_1 = 0$, with positive semicycles of length $3k$ and negative semicycles of length $3k$.*

Proof. By Theorems 10 or 12, we have $\chi_1, \chi_2, \ldots, \chi_{3k} < 0$ and $\chi_{3k+1}, \chi_{3k+2}, \ldots, \chi_{6k} > 0$, and the result follows by induction. □

5. The Difference Equation $\chi_{s+1} = (\chi_{s-3k+1}/ -1 - \Pi_{l=1}^{k}\chi_{s-3l+1})$

In this portion, we give an express shape of the solutions for the following equation:

$$\chi_{s+1} = \frac{\chi_{s-3k+1}}{-1 - \Pi_{l=1}^{k}\chi_{s-3l+1}}, \quad s = 0, 1, 2, \ldots, \tag{57}$$

where $\chi_{-j} \in \mathbb{R}$, $j = 0, 1, 2, \ldots, 3k-1$; $k \in \{1, 2, \ldots\}$. In addition to this, we examine the oscillation and periodicity of these solutions.

Theorem 16. *If k is odd and $P_j \neq -2$, then equation (57) has a periodic solution with period $6k$.*

Proof. The evidence is identical to the proof of Theorem 9 and shall be neglected. □

Theorem 17. *Assume that k is odd, the periodic 6k solution of equation (57) has the form*

$$\chi_{6ks-j} = \frac{a_{j-q_j(3k)^{q_j}}}{\left(-1-p_{j-3k}\right)^{\alpha_j q_j}}; \quad j = 0, 1, \ldots, 6k-1, P_j \neq -2 \text{ and } s = 1, 2, \ldots, \tag{58}$$

where $\chi_{-t} = a_t, P_t = \prod_{r=0}^{k-1} a_{\bmod(t,3)+3r}$ *with* $P_t \neq -1$, $t = 0, 1, 2, \ldots, 3k-1$; $k \in \{1, 2, \ldots\}$, $P_{-l} = 0, l = 1, 2, \ldots, 3k$, $q_j = 1/2\left((-1)^{[j/3k]+1}+1\right)$, *and* $\alpha_j = (-1)^{[j/3]+1}$.

Proof. From the definition of q_j, we can see that

$$\begin{aligned} q_0 &= q_1 = \ldots = q_{3k-1} = 0, \\ q_{3k} &= q_{3k+1} = \ldots = q_{6k-1} = 1. \end{aligned} \tag{59}$$

Also,

$$\alpha_{l+6r} = -1, \alpha_{l+3+6r} = 1; \quad l = 0, 1, 2 \text{ and } r = 0, 1, 2, \ldots, k-1. \tag{60}$$

So,

$$\begin{aligned} \chi_1 &= \frac{\chi_{-3k+1}}{-1-\chi_{-2}\chi_{-5}\cdots\chi_{-3k+1}} = \frac{a_{3k-1}}{-1-a_2a_5\ldots a_{3k-1}}, \\ \chi_2 &= \frac{\chi_{-3k+2}}{-1-\chi_{-1}\chi_{-4}\cdots\chi_{-3k+2}} = \frac{a_{3k-2}}{-1-a_1a_4\ldots a_{3k-2}}, \\ \chi_3 &= \frac{\chi_{-3k+3}}{-1-\chi_0\chi_{-3}\cdots\chi_{-3k+3}} = \frac{a_{3k-3}}{-1-a_0a_3\ldots a_{3k-3}}, \\ &\vdots \\ \chi_{3k} &= \frac{\chi_0}{-1-\chi_0\chi_3\cdots\chi_{3k-3}} = \frac{a_0}{-1-a_0a_3\ldots a_{3k-3}}, \\ \chi_{3k+1} &= \frac{\chi_1}{-1-\chi_1\chi_4\cdots\chi_{3k-2}} = a_{3k-1}, \\ \chi_{3k+2} &= \frac{\chi_2}{-1-\chi_2\chi_5\cdots\chi_{3k-1}} = a_{3k-2}, \\ \chi_{3k+3} &= \frac{\chi_3}{-1-\chi_3\chi_6\cdots\chi_{3k}} = a_{3k-3}, \\ &\vdots \\ \chi_{6k} &= \frac{\chi_{3k}}{-1-\chi_{6k-3}\chi_{6k-6}\cdots\chi_{3k}} = a_0, \end{aligned} \tag{61}$$

and the result follows by induction. □

Theorem 18. *Assume that k is odd, then equation (57) has* $\bar{\chi}_1 = 0$ *and* $\bar{\chi}_2 = -\sqrt[k]{2}$, *which are nonhyperbolic fixed points.*

Proof. The proof is similar to the proof of Theorem 7 and will be omitted. □

Theorem 19. *Assume that k is even, let* $\{\chi_s\}_{s=-3k+1}^{\infty}$ *be a solution of equation (57). Then, for* $s = 0, 1, 2, \ldots$,

$$\chi_{3ks-j} = \frac{a_j}{\left(-1-P_j\right)^{sq_j}}, \quad j = 0, 1, 2, \ldots, 3k-1, \tag{62}$$

where $P_j = \prod_{r=0}^{k-1} a_{\bmod(j,3)+3r}$, $q_j = (-1)^{[j/3]+1}$, *and* $\chi_{-j} = a_j$, *with* $P_j \neq -1$.

Proof. For $s = 0$, the result holds. Now, suppose that $s > 0$ and our assumption is verified for $s-1$. Then,

$$\chi_{3ks-3k-j} = \frac{a_j}{\left(-1-P_j\right)^{(s-1)q_j}}. \tag{63}$$

Now, it follows from equation (57) and using equation (63) that

$$\begin{aligned} \chi_{3ks-(3k-1)} &= \frac{\chi_{3ks-6k+1}}{-1-\chi_{3ks-3k-2}\chi_{3ks-3k-5}\cdots\chi_{3ks-6k+1}} \\ &= \frac{\left(a_{3k-1}/\left(-1-P_{3k-1}\right)^{s-1}\right)}{-1-\left(a_2\left(-1-P_2\right)^{s-1}\right)\left(a_5/\left(-1-P_5\right)^{s-1}\right)\ldots\left(a_{3k-1}/\left(-1-P_{3k-1}\right)^{s-1}\right)} \\ &= \frac{a_{3k-1}}{\left(-1-a_2a_5\ldots a_{3k-1}\right)^{s-1}\left(-1-a_2a_5\ldots a_{3k-1}\right)}. \end{aligned} \tag{64}$$

Hence, we have

$$\chi_{3ks-(3k-1)}=\frac{a_{3k-1}}{\left(-1-P_{3k-1}\right)^{s}}. \tag{65}$$

Also, it follows from equation (57) and using equation (63) that

$$\begin{aligned}\chi_{3ks-(3k-2)}&=\frac{\chi_{3ks-6k+2}}{-1-\chi_{3ks-3k-1}\chi_{3ks-3k-4}\cdots\chi_{3ks-6k+2}}\\&=\frac{\left(a_{3k-2}/\left(-1-P_{3k-2}\right)^{s-1}\right)}{-1-\left(a_{1}\left(-1-P_{1}\right)^{s-1}\right)\left(a_{4}/\left(-1-P_{4}\right)^{s-1}\right)\ldots\left(a_{3k-2}/\left(-1-P_{3k-2}\right)^{s-1}\right)}\\&=\frac{a_{3k-2}}{\left(-1-a_{1}a_{4}\ldots a_{3k-2}\right)^{s-1}\left(-1-a_{1}a_{4}\ldots a_{3k-2}\right)}.\end{aligned} \tag{66}$$

Hence, we have

$$\chi_{3ks-(3k-2)}=\frac{a_{3k-2}}{\left(-1-P_{3k-2}\right)^{s}}. \tag{67}$$

Also, it follows from equation (57) and using equation (63) that

$$\begin{aligned}\chi_{3ks-(3k-3)}&=\frac{\chi_{3ks-6k+3}}{-1-\chi_{3ks-3k}\chi_{3ks-3k-3}\cdots\chi_{3ks-6k+3}}\\&=\frac{\left(a_{3k-3}/\left(-1-P_{3k-3}\right)^{s-1}\right)}{-1-\left(a_{0}\left(-1-P_{0}\right)^{s-1}\right)\left(a_{3}/\left(-1-P_{3}\right)^{s-1}\right)\ldots\left(a_{3k-3}/\left(-1-P_{3k-3}\right)^{s-1}\right)}\\&=\frac{a_{3k-3}}{\left(-1-a_{0}a_{3}\ldots a_{3k-3}\right)^{s-1}\left(-1-a_{0}a_{3}\ldots a_{3k-3}\right)}.\end{aligned} \tag{68}$$

Hence, we have

$$\chi_{3ks-(3k-3)}=\frac{a_{3k-3}}{\left(-1-P_{3k-3}\right)^{s}}. \tag{69}$$

Similarly, one can easily obtain the other relations for equation (62). Hence, the proof is completed. □

Theorem 20. *Assume that k is even, then equation (57) has a unique equilibrium point $\bar{\chi}=0$, which is a nonhyperbolic equilibrium point.*

Proof. The evidence is identical to the proof of Theorem 7 and shall be neglected. □

Theorem 21. *Equation (57) is periodic of period $3k$ iff $P_j=-2$ and $j=0,1,\ldots,3k-1$ and will be take the form*

$$\chi_{3ks-j}=a_j;\quad j=0,1,\ldots,3k-1 \text{ and } s=0,1,2,\ldots. \tag{70}$$

Proof. The proof follows immediately from Theorems 17 or 19. □

Theorem 22. *Assume that $a_0,a_1,\ldots,a_{3k-1}\in(0,\infty)$, then the solution $\{\chi_s\}_{s=-3k+1}^{\infty}$ oscillates about $\bar{\chi}=0$, with positive semicycles of length $3k$ and negative semicycles of length $3k$.*

Proof. From Theorems 17 and 19, we have $\chi_1,\chi_2,\ldots,\chi_{3k}<0$ and $\chi_{3k+1},\chi_{3k+2},\ldots,\chi_{6k}>0$ and the result follows by induction. □

6. Conclusion

In this work, we get the solutions of the following difference equations:

$$\chi_{s+1}=\frac{\chi_{s-3k+1}}{\pm1\pm\Pi_{j=1}^{k}\chi_{s-3j+1}},\quad s=0,1,2,\ldots, \tag{71}$$

where the initial conditions $\chi_{-j}\in\mathbb{R}$, $j=0,1,2,\ldots,3k-1$ and $k\in\{1,2,\ldots\}$. We investigated the behavior of these solutions. Also, we used the mod function to write the solutions in a compact form for easy reading. Finally, we suggested the following future research.

Open Question 23. Discuss the global behavior of solutions of equation (21) about $\bar{\chi}=0$.

Data Availability

The data supporting the current study are available from the corresponding author upon request.

Disclosure

The abstract of this manuscript was presented orally only at the conference CMAM2021 according to the following link: https://cmam2021.sciencesconf.org/ by Ms. Lama Sh, Aljoufi, without any proofs of the results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

L. Sh. Aljoufi, A. M. Ahmed, S. Al Mohammady, H. M. Rezk, and G. AlNemer investigated the study, supervised the study, provided software analysis, and wrote the original draft. G. AlNemer, H. M. Rezk, and M. Zakarya reviewed and edited the manuscript and funded the study. All authors have read and agreed to the published version of the manuscript.

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