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Research Article

Generalized Terzaghi's Effective Stress Equation for Unsaturated Soil: An Independent Phase Balance Approach That Considers a Pore Water Content Gradient

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The effective stress equation for unsaturated soil is the most important equation in unsaturated soil mechanics. It has been derived by many scholars using different methods. However, none of them considered the gradient of the pore water content, which results in unreasonable force balance equations for different constituent phases in unsaturated soil. To introduce the gradient, we propose an extended three-phase physical model that includes capillary water, air, and generalized soil skeletons. Based on this model, three balance equations for these three constituent phases are separately formulated by considering the gradient of the pore water content. Comparing the result of the superposition of these three balance equations with the total balance equation, we derive a generalized Terzaghi's effective stress equation. This equation states that the effective stress is equal to the total stress minus the neutral stress. In comparison with the classical Bishop's equation, the generalized Terzaghi's equation ensures a smooth and continuous transition from unsaturated to saturated conditions not only in mathematical expression but also in physical meaning. Furthermore, the different pressure effects of capillary water and adsorbed water, their volumetric (or areal) effects, and the transformation between them can be considered by adopting the effective saturation of the capillary water as the effective stress parameter. Therefore, the generalized Terzaghi's equation can provide a better choice for estimating the effective stress in unsaturated soils.

1. Introduction

The effective stress equation is the fundamental equation of soil mechanics and even porous media mechanics [1, 2]. It has been widely used to address the problems in the fields of geotechnical engineering [3–5], environmental engineering [6], energy engineering [7, 8], biomechanics [9], agricultural engineering [10], and materials science [11, 12], among others [13]. The effective stress equation for saturated soils was pioneered by Terzaghi [14]:

$$\sigma' = \sigma - u, \tag{1}$$

where σ' is the effective stress, σ is the total stress, and u is the neutral stress.

For unsaturated soils, Bishop [15] proposed the well-known Bishop-type effective stress equation:

$$\sigma' = \sigma - u_{\mathrm{a}} + \chi(u_{\mathrm{a}} - u_{\mathrm{w}}), \tag{2}$$

where u_{a} denotes the pore air pressure, u_{w} is the pore water pressure, and χ is the effective stress parameter, which ranges from 0 to 1. In unsaturated soil mechanics, $\sigma - u_{\mathrm{a}}$ is generally defined as the net stress and $(u_{\mathrm{a}} - u_{\mathrm{w}})$ as the matric suction.

Unfortunately, Terzaghi's effective stress equation is only an experimental true expression, and the Bishop-type equation was proposed merely based on macroscopic phenomenological intuition. This circumstance means that they lack a solid theoretical foundation, which prevents us from fully understanding the effective stress equation and properly using it to describe the mechanical behaviour of soils. Therefore, the effective stress equation should be based on not only experimental results but also rigorous principles of mechanics.

To provide a solid mechanical foundation for the above effective stress equations, tremendous research effort has been made by using a variety of approaches since the 1960s. These approaches can be roughly grouped into four categories: (1) particle-based mechanical equilibrium approaches [16, 17], (2) representative volume element- (RVE-) based mechanical equilibrium approaches [18–20], (3) thermodynamics-based approaches [21–24], and (4) differential element- (DE-) based mechanical equilibrium approaches [25–28]. For the first category, Skempton [16] formulated a balance equation that relates the total stress to pore water pressure for a saturated system of two particles in contact. Then, assuming that the effective stress is the stress that controls the changes in the volume strain or shearing strength of soils, he obtained two equivalent effective stress equations. Likos and Lu [17] analysed the equilibrium state of a free body taken from a system of two particles with a water meniscus. Then, they proposed a Bishop-type effective stress equation. Although the results of the first category can provide a mechanical explanation for the effective stress equation, they were only based on the micromechanical analysis of a system of two particles. Hence, the first category can account for neither the interaction forces between the different phases nor the water content gradient [29] in unsaturated soils. For the second category, Lu and Likos [18] introduced three types of interparticle forces to formulate mechanical balance equations for the RVEs of unsaturated soils and coin the concept of suction stress, which serves as a mechanical foundation for better understanding the effective stress equation from a microscopic particle level to an RVE level. Nevertheless, the second category also does not consider the interaction forces and the water content gradient. In addition, they neglected the stresses that act on the cross-section of soil particles and on the contact area between soil particles induced by the pore-fluid pressure in unsaturated soils. For the third category, there are two different subcategories of thermodynamic approaches. One refers to utilizing Equation (3) of the power input per unit volume of unsaturated soil to identify the work conjugate stress and strain variables:

$$\dot{W} = u_{\mathrm{a}} n(1-S)\frac{\dot{\rho}_{\mathrm{a}}}{\rho_{\mathrm{a}}} - (u_{\mathrm{a}} - u_{\mathrm{w}}) n\dot{S} + \left\{\sigma_{ij} - [S u_{\mathrm{w}} + (1-S)u_{\mathrm{a}}]\delta_{ij}\right\}\dot{\varepsilon}_{ij}, \tag{3}$$

where $\dot{W}$ is the power input per unit volume of unsaturated soil, n is the porosity, S is the degree of saturation, ρ_{a} is the pore-air density, σ_{ij} is the total stress tensor, ε_{ij} is the strain tensor, and δ_{ij} is Kronecker's delta.

According to Equation (3), a Bishop-type effective stress tensor $\sigma_{ij} - [S u_{\mathrm{w}} + (1-S)u_{\mathrm{a}}]\delta_{ij}$ conjugate to the strain tensor and a modified suction $(u_{\mathrm{a}} - u_{\mathrm{w}})n$ conjugate to the degree of saturation can be obtained. However, Equation (3) can be rearranged to give additional sets of stress and strain variables that are conjugate to each other. For instance, when Equation (3) is rewritten as Equation (4), the net stress tensor $(\sigma_{ij} - u_{\mathrm{a}}\delta_{ij})$ conjugate to the strain tensor and the suction $(u_{\mathrm{a}} - u_{\mathrm{w}})$ conjugate to the generalized strain quantity $(-n\dot{S} + S\dot{\varepsilon}_{ii})$ can also be obtained

$$\dot{W} = u_{\mathrm{a}} n(1-S)\frac{\dot{\rho}_{\mathrm{a}}}{\rho_{\mathrm{a}}} + \left(\sigma_{ij} - u_{\mathrm{a}}\delta_{ij}\right)\dot{\varepsilon}_{ij} + (u_{\mathrm{a}} - u_{\mathrm{w}})\left(-n\dot{S} + S\dot{\varepsilon}_{ii}\right) \tag{4}$$

The analysis of the principle of work conjugacy indicates that the choice of stress state variables for unsaturated soils is phenomenological and somewhat subjective. The other subcategory employed the principle of virtual work to calculate the free energy of an unsaturated soil. By evaluating the derivative of the free energy with respect to the volume, another Bishop-type effective stress equation was obtained [23]:

$$\sigma' = \sigma - u_{\mathrm{a}} + S_{\mathrm{e}}(u_{\mathrm{a}} - u_{\mathrm{w}}), \tag{5}$$

where S_{e} is the effective saturation, expressed as $S_{\mathrm{e}} = (S - S_{\mathrm{r}})/(1 - S_{\mathrm{r}})$, and S_{r} is the residual saturation.

However, some rather harsh assumptions must be made to obtain Equation (5), such as keeping the residual saturation constant and ignoring the contribution of interfaces to the free energy. Compared with the first and second categories, the third category has made an obvious improvement that the effective stress equation can be derived based on a relatively rigorous thermodynamic theory. Nevertheless, the third category still neglects the water content gradient. In comparison with the above three categories, the fourth category [25–28] was developed in recent years and has some advantages, such as the following: (1) it is based on the universally recognized concepts of mechanical equilibrium. (2) It explicitly accounts for the interaction forces between the different phases and the stresses acting on the cross-section of the soil particles and on the contact area between the soil particles induced by pore fluid pressure. (3) The formulated balance equations are based on clear physical mechanisms and models. Unfortunately, the pore water content gradient is still neglected in the fourth category.

The objective of this study is to derive a new generalized Terzaghi's effective stress equation for unsaturated soils by considering the pore water content gradient. The gradient is introduced to formulate more exact force balance equations for different phases in unsaturated soils by using the DE-based mechanical equilibrium approach. Based on the more exact force balance equations, a new generalized Terzaghi's effective stress equation is derived. The implications of this new equation are drawn out by comparing with some existing Bishop-type effective stress equations.

2. Evidence for the Pore Water Content Gradient

2.1. Vertical Spatial-Temporal Distribution of the Pore Water Content. Many scholars have explored the spatial-temporal distribution of the pore water content of unsaturated soils in a large number of different sites by various observation techniques (e.g., in situ soil-moisture sampling, ground-penetrating radar, time domain reflectometry, frequency domain reflectometry, and neutron sensor) [30–32] and analysis methods (e.g., geostatistics) [33]. These studies have indicated that the pore water content of unsaturated soils sometimes changes significantly in vertical and horizontal profiles. The vertical profile of pore water content with depth can be divided into different sublayers in terms of the coefficient of variance from geostatistical analysis [34]: the fast-changing layer, the active layer, and the relatively stable layer (as shown in Figure 1). The spatial-temporal distribution of the pore water content in these different sublayers is usually influenced by a number of factors such as rainfall infiltration, surface evapotranspiration, aboveground plant species, soil mineral components, and microtopographical features. In general, rainfall infiltration [35], surface evapotranspiration [36], and aboveground plant species [37] have a profound influence on the spatial distribution of the pore water content in the fast-changing layer. In this layer, the greater the infiltration rate is, the stronger the evapotranspiration, and the higher the root density is, the more dramatically the spatial distribution of the pore water content changes because the fast-changing layer is the interface between the active layer and the atmosphere. Soil mineral components (e.g., sand, silt, and clay fractions) [38], fine-root density [39], and microtopographical features [40] generally play a crucial role in the retention of the pore water content in the active layer. In this layer, there is a strong positive correlation between the silt and clay fractions and the pore water content, but a negative correlation between the root density and microtopographical features and pore water content. Compared with the pore water content in the above two sublayers, the pore water content in the relatively stable layer usually has a slower and smaller response to external factors.

2.2. Horizontal Spatial-Temporal Distribution of the Pore Water Content. A series of laboratory model tests have been conducted to investigate the horizontal water redistribution in a horizontal flume filled with homogeneous soils (or slightly heterogeneous soils) [29, 41]. This flume contains two parts: one part corresponds to a relatively low saturation state, namely, the dry part; the other part is the wet part, which has a higher saturation state. The dry and wet parts of the flume were originally insulated by using a very thin (0.044 mm) removable metal sheet. When the sheet was removed, the process of water redistribution was initiated. The water redistribution led to the variation in the water saturation over time and the observation positions of the flume (as seen in Figure 2). This model test reveals that the distribution of the water saturation along the flume is mainly affected by several factors such as the initial saturation conditions on both the dry and wet sides, the hysteretic water retention curves, the hydraulic conductivity of the soils, and the driving forces for the flow of pore water.

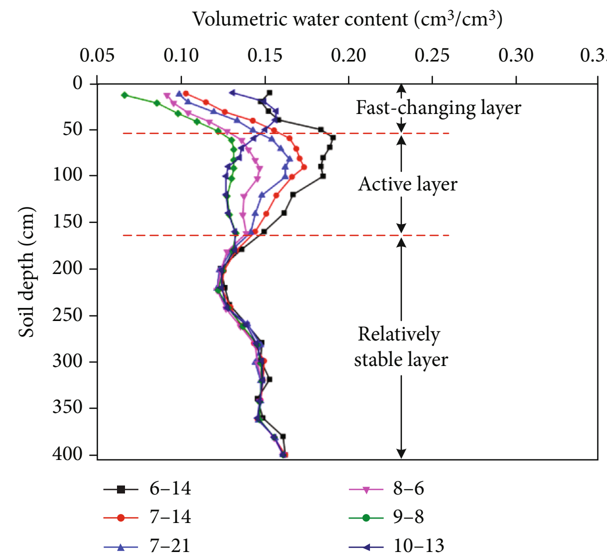

Figure 1: Dynamic change in the pore water content in the 0-400 cm soil profile of a typical site under natural rainfall conditions during various periods from June to November 2002, from Chen et al. [34].

The results from the field observations and laboratory model tests mentioned above have demonstrated that the spatial-temporal distribution of the pore water content extensively exists in unsaturated soils. This spatial-temporal distribution can give rise to a pore water content gradient that is ubiquitous in both the vertical and horizontal profiles. Since the spatial-temporal distribution of the pore water content is influenced by different factors under different conditions, the characteristics of the pore water content gradient are also influenced by the corresponding factors. Considering that the pore water content gradient plays an important role not only in the hydrological process but also in the mechanical response of unsaturated soils, it should be considered in the formulation of the force balance equations and in the derivation of the effective stress equation for unsaturated soils.

3. Interactions between Different Phases and an Extended Three-Phase Model of Unsaturated Soils

Unsaturated soil is a multiphase porous medium. How to separate this multiphase porous medium into different constituent phases is an important issue in the characterization of the stress state of unsaturated soils. Various attempts have been made to consider unsaturated soil as different porous medium models including different constituent phases, such as the three-phase porous medium model (i.e., air, water, and soil skeleton) [15], the four-phase porous medium model (i.e., air, water, contractile skin, and soil skeleton) [28], and the six-phase porous medium model (i.e., pore

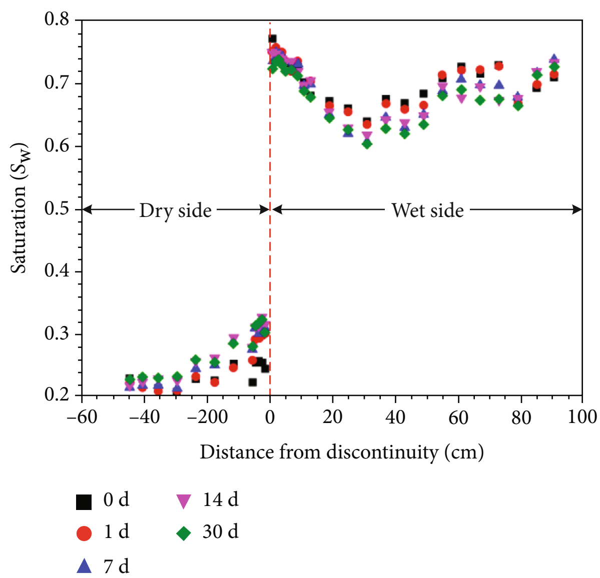

Figure 2: Measured water saturation distributions at different times along the flume from Feuring et al. [29].

air, liquid water, solid water, contractile skin, cement, and grain skeleton) [26]. The three- and four-phase models, however, are not able to consider the influences of different types of pore water on the stress state of unsaturated soils. In addition, the six-phase model would be too complex for characterizing the stress state. Therefore, it is necessary to establish an appropriate porous medium model to characterize the stress state of unsaturated soils.

Compared with saturated soils, unsaturated soils introduce pore air (see Figure 3(b)). Pore air initiates the interactions between the pore air, pore water, and soil particles, which further produce capillarity and changes in the adsorption phenomena in unsaturated soils. During the process of interaction, the capillarity and adsorption phenomena jointly transform the pore water into capillary and adsorbed water. Capillary water (see Figure 3(c)) usually refers to the water held in the pores of unsaturated soils by both the surface tension of air-water interfaces and the adsorptive forces exerted by the soil particle surface, while adsorbed water represents the water retained on the surfaces of soil particles through adsorptive forces. Although capillary water exhibits some different physical and mechanical properties at the microscopic level in comparison with the pore water in saturated soils, it can still be regarded as a fluid with macroscopically isotropic negative pore water pressure. Adsorbed water, however, manifests a series of distinct characteristics, such as greater density [42, 43], strongly coordinated structure [44, 45], higher viscosity [46], and higher shear modulus [47]. Based on these distinct characteristics, adsorbed water is believed to be a solid-like substance and can thus serve as a structure to bear and transfer loads. In addition, an important constituent related to the capillarity in unsaturated soils is contractile skin. It exhibits an appreciable surface tension in its interior to pull soil particles together and thus contributes to the shear strength and modulus of unsaturated soils [48]. Hence, contractile skin can also serve as another structure to bear and transfer loads. Furthermore, in addition to the adsorbed water and contractile skin bearing and transferring loads, the pivotal structure in unsaturated soils is the soil skeleton. In general, the soil skeleton can be described as a structure that is formed by means of the contact and cementation between soil particles. The contact is associated with the sliding and rolling friction between particles, and the cementation provides the cohesion between particles. These two different actions allow the soil skeleton to bear and transfer loads. In this case, the adsorbed water, contractile skin, and soil skeleton can be combined as a generalized phase, termed the generalized soil skeleton (see Figure 3(d)), because they can all bear and transfer loads.

According to the analysis of the interaction mechanisms between different phases and their influences on the mechanical behaviour of unsaturated soils, an extended three-phase porous medium model is presented (as shown in Figure 3). According to the extended model, we can consider the complex interactions between different phases to derive the effective stress equation.

4. Derivation of Generalized Terzaghi's Effective Stress Equation considering the Pore Water Content Gradient

4.1. Volume-Saturation Relations of the Extended Three-Phase Model. The prerequisite for deriving the effective stress equation is to define a set of clear volume-saturation relations of different phases of the extended three-phase model. Based on the definition of the extended model, the volumes related to the pore air, capillary water, and generalized soil skeleton can be denoted as V_a, V_{cw}, and V_{gs}, respectively, in an RVE of unsaturated soil. When the total volume of this RVE is assumed to be V, the following equation must be satisfied:

$$V_a + V_{cw} + V_{gs} = V. \tag{6}$$

Based on the extended model, we can combine the volume of the capillary water and that of the pore air as the volume of the effective voids:

$$V_a + V_{cw} = V_v^{eff}. \tag{7}$$

Then, the volume porosity of the effective voids can be defined as the ratio of the volume of effective voids to the total volume of the RVE:

$$n_v^{eff} = \frac{V_v^{eff}}{V} \times 100\%. \tag{8}$$

The volume porosities of the three constituent phases can be calculated as follows:

$$n_a = \frac{V_a}{V} \times 100\%, \tag{9}$$

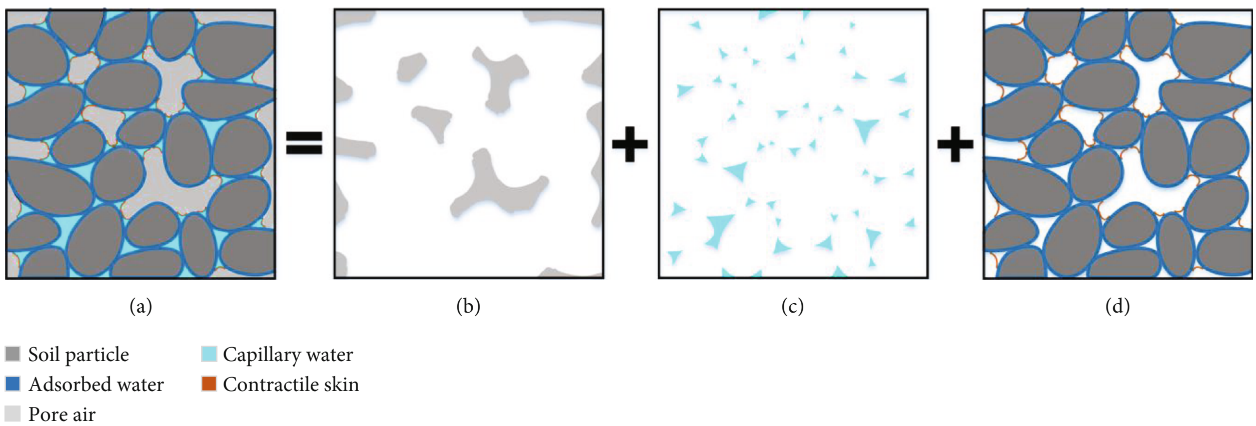

Figure 3: Schematic diagram of an extended three-phase porous medium model: (a) an RVE of unsaturated soil; (b) pore air phase; (c) capillary water phase; (d) generalized soil skeleton phase including adsorbed water, contractile skin, and soil particles.

$$n_{\mathrm{cw}} = \frac{V_{\mathrm{cw}}}{V} \times 100\%, \tag{10}$$

$$n_{\mathrm{gs}} = \frac{V_{\mathrm{gs}}}{V} \times 100\%, \tag{11}$$

where n_{a}, n_{cw}, and n_{gs} are the pore air porosity, capillary water porosity, and generalized soil skeleton porosity, respectively, as shown in Figure 4. Combining Equations (6), (9), (10), and (11) yields

$$n_{\mathrm{a}} + n_{\mathrm{cw}} + n_{\mathrm{gs}} = 1. \tag{12}$$

Similarly, the area porosity is usually defined as the ratio of the area of each phase to the total cross-sectional area of the RVE. According to the hypothesis proposed by Biot [49], it is hypothesized that the area porosity is equal to the volume porosity in a homogeneous porous medium. Additionally, the volume percentage of the capillary water within the effective voids is expressed as the effective saturation of the capillary water (Figure 4):

$$S_{\mathrm{cw}}^{\mathrm{eff}} = \frac{V_{\mathrm{cw}}}{V_{\mathrm{v}}^{\mathrm{eff}}}. \tag{13}$$

4.2. Necessity for considering the Pore Water Content Gradient. Neglecting the pore water content gradient in the derivation of the effective stress equation could be due to two causes. One cause corresponds to the confusion between the RVE of unsaturated soil and the differential element (DE) used in the mechanical equilibrium approach; the other cause refers to not realizing the existence of the pore water content gradient. To clarify the confusion and realize the importance of the pore water content gradient in the derivation of the effective stress equation, it is worthwhile to identify the conceptual differences between the RVE and DE.

As an example, let us consider a given soil foundation comprised of a type of soil. It has dimensions of a (length) by b (width) ($a > b$) (as shown in Figure 5(a)). When characterizing the mechanical response of this foundation, we are not capable of accounting for all the microstructures of the foundation. It is necessary to introduce an RVE to solve this challenge. To better represent a discontinuous heterogeneous soil foundation, the choice of the size of the RVE must obey the following inequality constraint:

$$d \ll l \ll b, \tag{14}$$

where d is the characteristic size of the soil, representing the maximum particle size of the soil, and is usually referred to as the interior characteristic size; l is the size of the RVE; and b is the characteristic size of the foundation, commonly known as the exterior characteristic size, which depends on the minimum geometric size of the foundation and the wavelength of loading. The inequality constraint (14) indicates that the size of the RVE must be small enough relative to the minimum geometric size, b, of the foundation to describe a sufficiently close neighbourhood that encompasses the centroid of the RVE (see the "P" point in Figure 5(a)). At the same time, the size of the RVE must be large enough relative to the maximum particle size, d, of the soil to contain sufficient statistical information about the discontinuous heterogeneous soil foundation. If the appropriate size of an RVE has been defined, the microscopic values of the variables of interest are averaged over the well-defined RVE. The averaged values, referred to as the macroscopic values of the variables of interest, are assigned to the centroid of the RVE. Then, traversing the entire foundation domain with a moving RVE, we can assign the averaged values to every point of the entire foundation domain. In this way, the fields of the macroscopic variables (see Figure 5(b)) that are differentiable functions of space coordinates can be acquired to formulate the differential equations of the variables of interest [50]. On the other hand, in comparison with an RVE, a DE is defined in the continuous homogeneous foundation to characterize the variation of the macroscopic variables with the space coordinates (see Figure 5(b)). For a given macroscopic variable pertaining to the "P" point (see Figure 5(b)), if two infinitesimal distances dx and dy are gained from this point, the macroscopic variable (for example, F) will obtain two

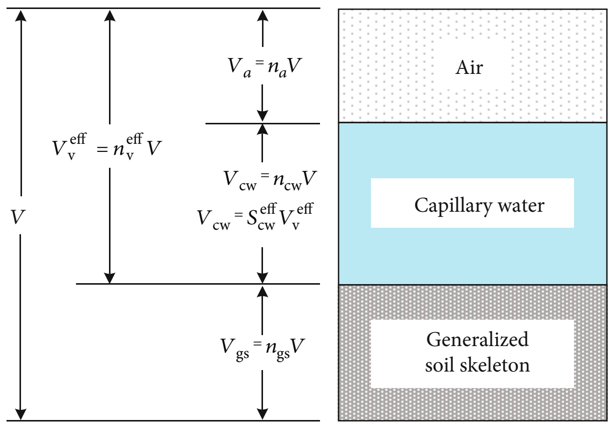

FIGURE 4: Schematic diagram of the volume-saturation relations of an extended three-phase model.

infinitesimal increments $(\partial F/\partial x)\mathrm{d}x$ and $(\partial F/\partial y)\mathrm{d}y$ along the x and y directions, respectively.

After having identified the differences between the RVE and DE, we now reexamine the analyses of the independent-phase equilibrium conducted by many investigators. Some of them did not distinguish between the RVE and DE [26, 51], and others ignored the existence of the pore water content gradient [25–28]. These two problems led these investigators to calculate the forces acting on the lower and upper surfaces of the DE of the pore water as $u_w n_w dxdz$ and $(u_w + \partial u_w/\partial y) n_w dxdz$ (see Figure 6(a)), respectively, in the y-direction, where n_w is the porosity of the pore water. However, when the pore water content gradient is introduced, the corresponding forces should be $u_w n_w dxdz$ and ($u_w + \partial u_w/\partial y)(n_w + \partial n_w/\partial y) dxdz$ (see Figure 6(b)) in the y -direction.

4.3. Derivation of Generalized Terzaghi's Effective Stress Equation. The derivation of the effective stress equation is performed by formulating the force balance equations of an unsaturated soil DE and its constituent phases DEs (i.e., capillary water, pore air, and generalized soil skeleton). There are two types of forces acting on these DEs. One corresponds to surface forces; the other is body forces. Surface forces that act on the surfaces of each DE arise from external loads. They are shown as a surface force per unit area; body forces that act through the centroid of each DE are further classified as the gravitational force and the interaction forces between different phases. If an unsaturated soil DE, with infinitesimal dimensions of dx, dy, and dz, is taken out of an unsaturated soil mass in equilibrium as the object of equilibrium analysis, the DEs of the three constituent phases are also in equilibrium. Based on the equilibrium conditions of the forces acting on all of the DEs, it is straightforward to independently formulate the force balance equations for capillary water DE, pore air DE, generalized soil skeleton DE, and unsaturated soil DE.

4.3.1. Balance Equation for Capillary Water DE. Before formulating the balance equation for the capillary water DE, it is indispensable to calculate the surface forces and gravitational and interaction forces that act on it. Each surface force is equal to the product of the capillary water pressure and the corresponding area. The gravitational force is computed as the unit gravity times the volume of the capillary water. The interaction force is expressed as the interaction force per unit volume times the corresponding volume. For simplicity, the capillary water pressures acting on the different surfaces and the gravitational and interaction forces per unit volume acting on the volume are shown only in the y -direction in Figure 7. The pore water content gradient is considered in the formulation of the balance equation for the capillary water DE. Hence, the forces acting on the lower and upper surfaces are $u_w n_{cw} dxdz$ and $(u_w + \partial u_w/\partial y)(n_{cw} + \partial n_{cw}/\partial y) dxdz$, respectively. Although the area porosity of the capillary water is equal to its volume porosity within an unsaturated soil RVE, the volume porosity of the capillary water is a variable within an unsaturated soil DE. To characterize the variation of the volume porosity of the capillary water with space coordinates, we construct a Cartesian reference frame with space coordinates (u, v, w) within the capillary water DE (as shown in Figure 7). In this way, the volume porosity of the capillary water within the DE can be characterized as $n_{cw}(u, v, w)$ (see Figure 7). Then, the gravitational force acting on the capillary water DE is calculated as $\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{cw}(u, v, w)\rho_w g du dv dw$. Analogously, the interaction force of the capillary water DE that is exerted by the generalized soil skeleton is expressed as $\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f_{gsy}^{cw}(u, v, w) du dv dw$. It should be noted that the interaction force between the capillary water and the pore air does not exist because they are separated by contractile skin in the extended three-phase model.

The equilibrium condition of the capillary water DE demands that the resultant force should vanish. Summing these forces in the y-direction yields the force balance equation for the capillary water DE:

$$
\begin{aligned}
&-\left(u_w + \frac{\partial u_w}{\partial y} dy\right)\left[\left(n_{cw} + \frac{\partial n_{cw}}{\partial y} dy\right) dxdz\right] + u_w n_{cw} dxdz \\
&\quad - \int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f_{gsy}^{cw}(u, v, w) du dv dw \\
&\quad - \int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{cw}(u, v, w)\rho_w g du dv dw = 0,
\end{aligned}
\tag{15}
$$

where n_{cw} is the area porosity of the capillary water DE; $f_{gsy}^{cw}(u, v, w)$ is the interaction force per unit volume between the capillary water and the generalized soil skeleton, which is a function of space coordinates (u, v, w) within the capillary water DE; and $n_{cw}(u, v, w)$ is the volume porosity of the capillary water within the capillary water DE. The volume porosity, $n_{cw}(u, v, w)$, will be equal to the area porosities n_{cw} of the lower and upper surfaces of the capillary water DE when point $(u,\ v,\ w)$ in the space coordinate system approaches the lower and upper surfaces, respectively; ρ_w is the water density; and g is the gravitational acceleration.

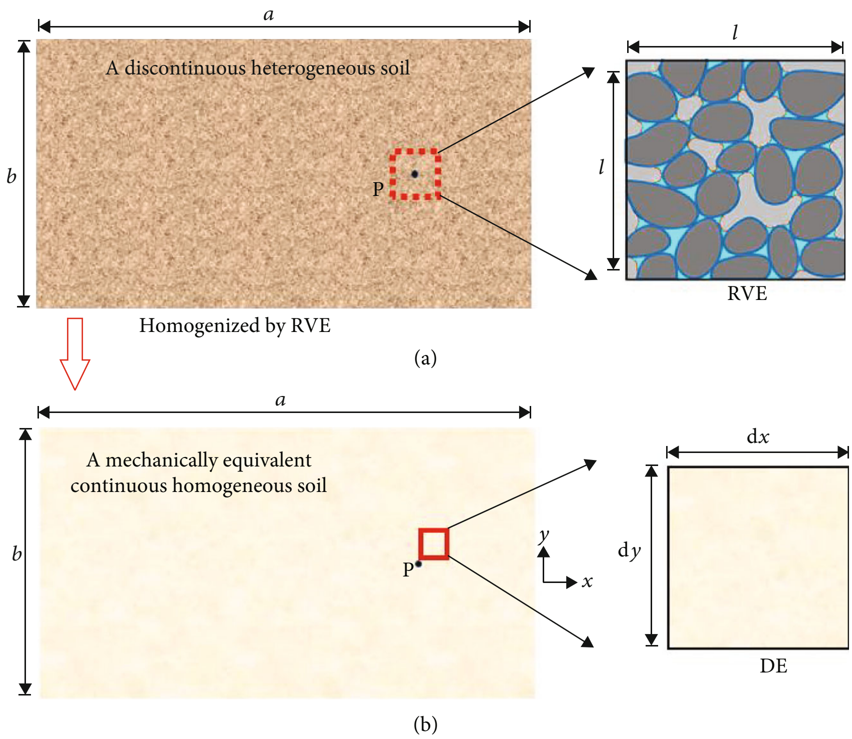

Figure 5: Illustration of the difference between the representative volume element (RVE) and the differential element (DE): (a) an RVE defined in a discontinuous heterogeneous soil foundation; (b) a DE defined in a mechanically equivalent continuous homogeneous soil foundation.

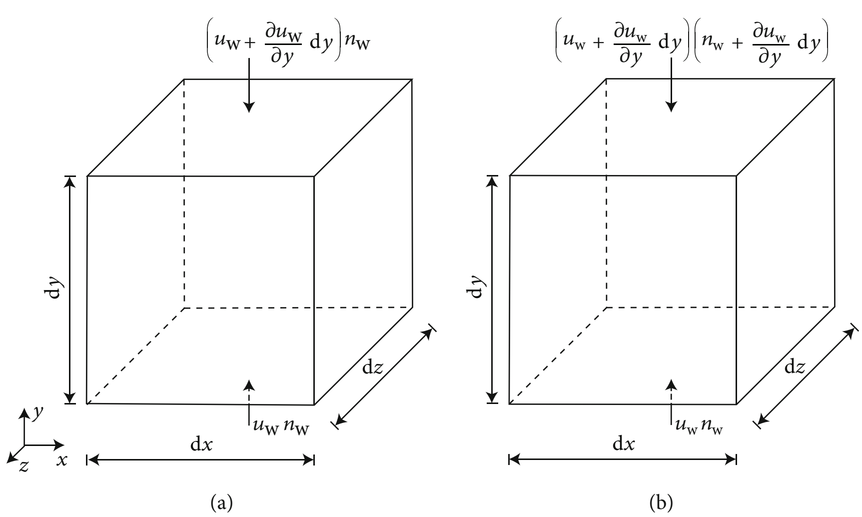

Figure 6: Pore water content and water pressure of the water phase in the y-direction: (a) without consideration of the pore water content gradient; (b) consideration of the pore water content gradient.

Simplifying Equation (15), we can obtain the balance differential equation for the capillary water DE:

$$\frac{\partial (n_{cw} u_w)}{\partial y} + \frac{1}{dxdydz} \int_x^{x+dx} \int_y^{y+dy} \int_z^{z+dz} f_{gsy}^{cw}(u, v, w) dudvdw + \frac{1}{dxdydz} \int_x^{x+dx} \int_y^{y+dy} \int_z^{z+dz} n_{cw}(u, v, w) \rho_w g dudvdw = 0. \quad (16)$$

Detailed steps of the mathematical operation for Equation (16) are given in Appendix A for completeness.

4.3.2. Balance Equation for Pore Air DE. The capillary water and pore air jointly occupy the pore space of an unsaturated soil DE, which means that the pore air content varies with the capillary water content. Since the capillary water content gradient has been considered, the pore air content gradient should also be considered in the analysis of the pore air

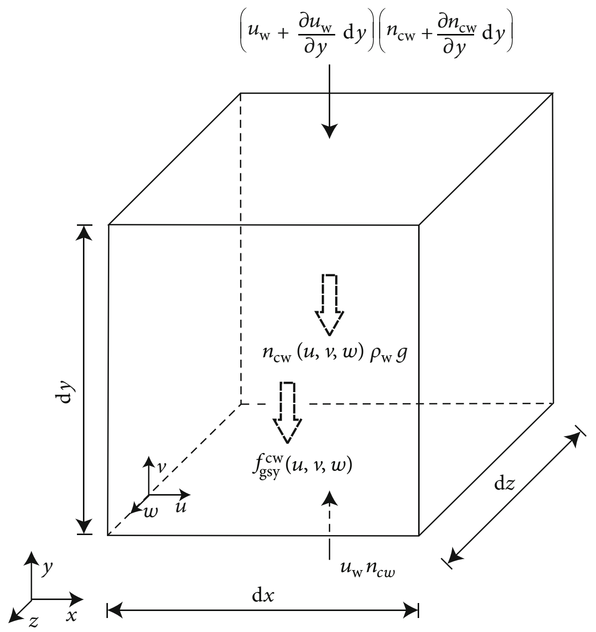

Figure 7: Components for the force equilibrium of capillary water DE in the y-direction.

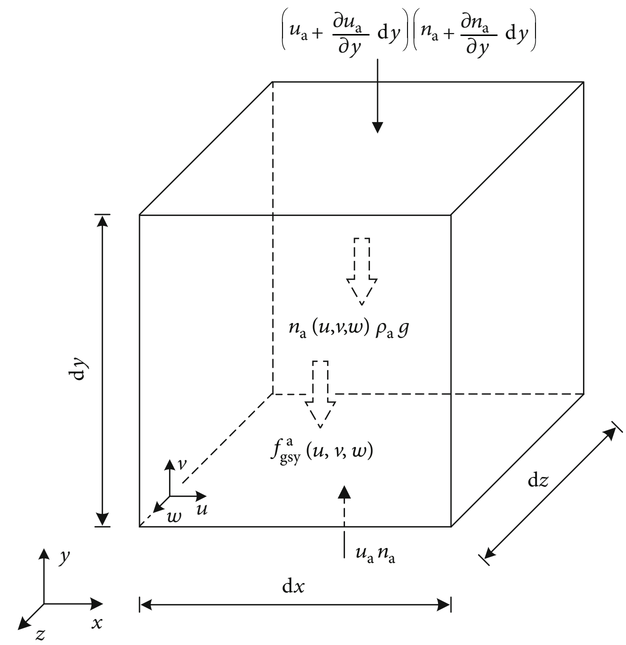

Figure 8: Components for the force equilibrium of the pore air DE in the y-direction.

equilibrium state. Figure 8 depicts the pore air pressure and the gravitational and interaction forces per unit volume in the y-direction.

Based on the equilibrium condition of the pore air DE, the balance differential equation is given in the y-direction:

$$\frac{\partial(n_a u_a)}{\partial y} + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f^a_{gsy}(u,v,w)dudvdw + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_a(u,v,w)\rho_a g dudvdw = 0, \quad (17)$$

where n_a is the area porosity of the pore air DE; $f^a_{gsy}(u, v, w)$ is the interaction force per unit volume between the pore air and the generalized soil skeleton, which is a function of space coordinates (u, v, w) within the air DE, and $n_a(u, v, w)$ is the volume porosity of the pore air within the pore air DE. The volume porosity, $n_a(u, v, w)$, will be equal to the area porosities n_a of the lower and upper surfaces of the pore air DE when the point (u, v, w) in the space coordinate system approaches the lower and upper surfaces, respectively; ρ_a is the air density. Some detailed mathematical steps for Equation (17) are given in Appendix B.

4.3.3. Balance Equation for Generalized Soil Skeleton DE. To formulate the force balance equation of the generalized soil skeleton, we must in advance determine the forces acting on the generalized soil skeleton DE. Compared with the capillary water DE and pore air DE, the generalized soil skeleton DE is subjected to more complex forces. These forces can also be classified into body and surface forces. Body force refers to gravitational and interaction forces. The interaction forces are further subdivided into the forces exerted by the capillary water and by the pore air. Surface forces are generated not only from the adjacent generalized soil skeleton DE but also from the capillary water pressure and the pore air pressure. Of all the forces acting on the generalized soil skeleton DE, the forces generated from the capillary water pressure and the pore air pressure are the most difficult to determine. This difficulty stems from two aspects: how to determine the magnitudes of the stresses acting on the surfaces of the generalized soil skeleton DE and how to identify the corresponding areas on which these stresses act.

To overcome the difficulty mentioned above, we shall first resort to a simple case of saturated soil. Specifically, we consider a system of two idealized spherical soil particles under saturated conditions without external loads in hydrostatic equilibrium [15, 25] (as shown in Figure 9). In this system, we can use the equilibrium condition of the cross-section 1-1 (see Figure 9(b)) to obtain the following equation:

$$\sigma^w_{sp} = \frac{u_w A_{sp}}{A_{sp}} = u_w, \quad (18)$$

where σ^w_{sp} represents the stress acting on the cross-section of the soil particles induced by the water pressure and A_{sp} is the cross-sectional area of the soil particles.

Similarly, using the equilibrium condition of the cross-section 2-2 (see Figure 9(c)), we can obtain:

$$\sigma^w_c = \frac{u_w A_c}{A_c} = u_w, \quad (19)$$

where σ^w_c denotes the stress acting on the contact area

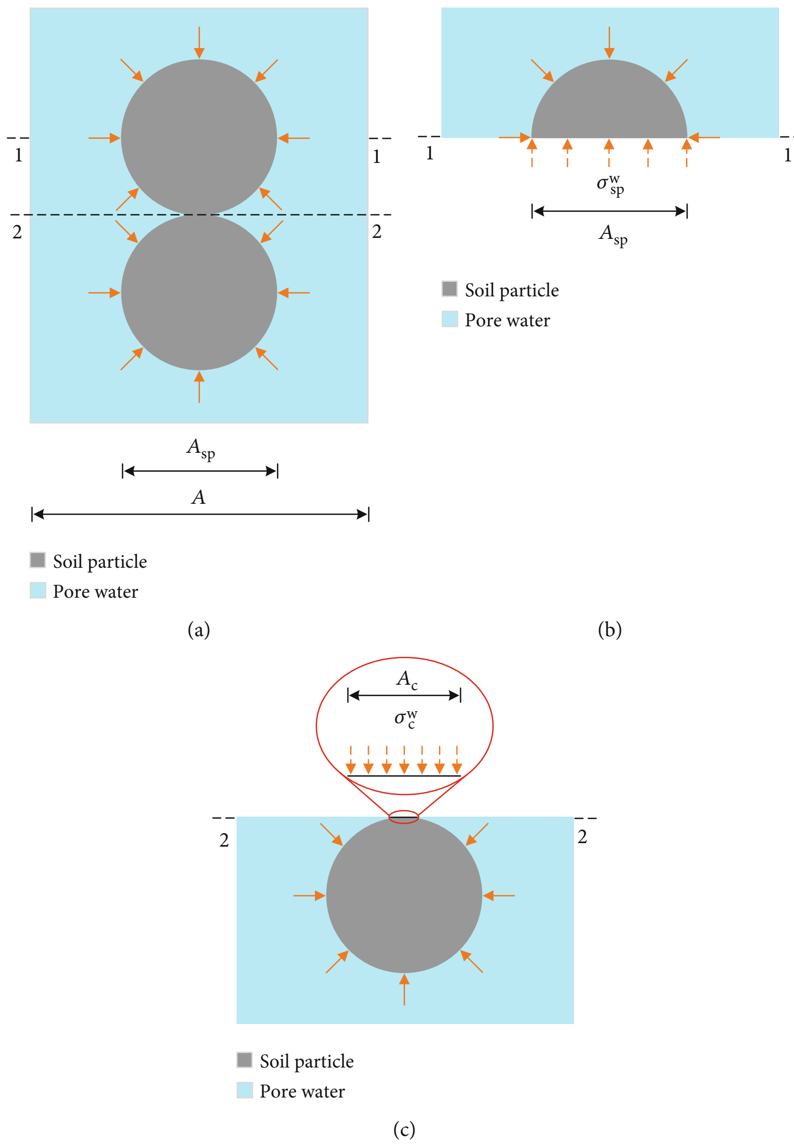

FIGURE 9: Stresses induced by the water pressure: (a) a system of two idealized spherical soil particles in hydrostatic equilibrium, where A is the cross-sectional area of this system; (b) stress acting on the cross-section of soil particles; (c) stress acting on the contact area between soil particles.

between two idealized spherical soil particles induced by the water pressure and A_c is the contact area between soil particles.

Driven by analogy with the above case under saturated conditions, we now consider a system of two idealized spherical soil particles under unsaturated conditions without external loads in equilibrium (as shown in Figure 10). In this figure, both the contractile skin and the adsorbed water for simplicity are not shown because we only analyse the stresses generated by the capillary water pressure and the pore air pressure in this section. Due to the capillary water and the pore air together filling the pore space of this system, the approach that is used to compute the stresses acting on the cross-section of the soil particles and on the contact area between the soil particles under saturated conditions cannot be directly applied to compute the corresponding stresses under unsaturated conditions. To solve this problem, an assumption that is similar to the idea of the theory of mixtures [52] has been made here. It is assumed that the capillary water and the pore air independently fill the total pore space of this system in terms of their respective volume porosities. In this case, the system of two idealized spherical soil particles under unsaturated conditions can be regarded as two subsystems (see Figures 10(b) and 10(c)). One subsystem corresponds to a system in which the capillary water, having a homogenized capillary water pressure $[n_{cw}/(n_{cw} + n_a)]u_w$, completely occupies the pore space; the other subsystem corresponds to a system where the pore space is filled with the pore air with a homogenized air pressure $[n_a/(n_{cw} + n_a)]u_a$. Based on the results under saturated conditions (see Figure 9), we can readily calculate the stresses acting on the cross-section of the soil particles and on the contact

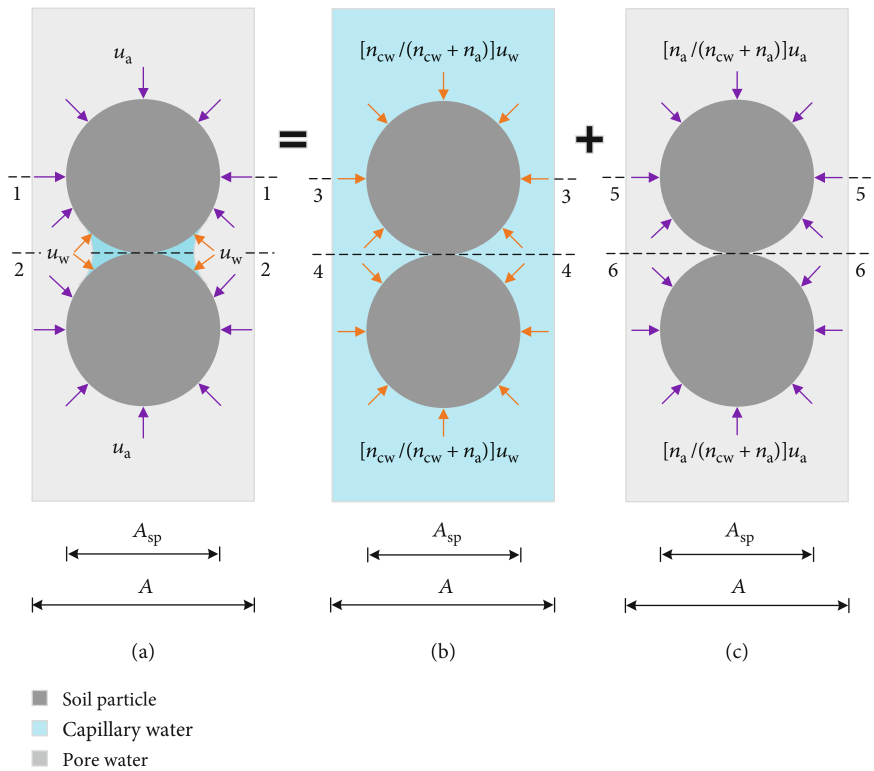

Figure 10: A system of two idealized spherical soil particles under unsaturated conditions including two subsystems: (a) a system where the pore space is filled with capillary water and pore air; (b) a subsystem where the pore space is filled with capillary water with a homogenized capillary water pressure; and (c) a subsystem where the pore space is filled with pore air with a homogenized pore air pressure.

area between the soil particles under unsaturated conditions. From Figure 10(b), on the basis of the equilibrium condition of the cross-section 3-3, the following equation holds:

$$\sigma_{sp}^{cw} = \frac{(n_{cw}/(n_{cw}+n_a))u_w A_{sp}}{A_{sp}} = \frac{n_{cw}}{n_{cw}+n_a}u_w, \quad (20)$$

where σ_{sp}^{cw} represents the stress acting on the cross-section of the soil particles generated by the homogenized capillary water pressure.

Likewise, using the equilibrium condition of the cross-section 4-4 (see Figure 10(b)), we have:

$$\sigma_{c}^{cw} = \frac{(n_{cw}/(n_{cw}+n_a))u_w A_{c}}{A_{c}} = \frac{n_{cw}}{n_{cw}+n_a}u_w, \quad (21)$$

where σ_{c}^{cw} denotes the stress acting on the contact area between the soil particles generated by the homogenized capillary water pressure.

Based on the equilibrium condition of the cross-section 5-5 in Figure 10(c), the following equation holds:

$$\sigma_{sp}^{a} = \frac{(n_{a}/(n_{cw}+n_a))u_a A_{sp}}{A_{sp}} = \frac{n_{a}}{n_{cw}+n_a}u_a, \quad (22)$$

where σ_{sp}^{a} represents the stress acting on the cross-section of the soil particles generated by the homogenized air pressure.

Similarly, on the basis of the equilibrium condition of the cross-section 6-6 (see Figure 10(c)), we have:

$$\sigma_{c}^{a} = \frac{(n_{a}/(n_{cw}+n_a))u_a A_{c}}{A_{c}} = \frac{n_{a}}{n_{cw}+n_a}u_a, \quad (23)$$

where σ_{c}^{a} denotes the stress acting on the contact area between the soil particles generated by the homogenized air pressure.

Next, we extend the above analysis results of a system of two idealized spherical soil particles under unsaturated conditions to an unsaturated soil DE. It is therefore uncomplicated to determine the magnitudes and the corresponding areas of the stresses that are generated by the homogenized capillary water pressure and by the homogenized air pressure in an unsaturated soil DE. These stresses acting on the surfaces of the generalized soil skeleton DE are shown in Figure 11. In addition, Figure 11 also shows the stress fields and the gravitational and interaction forces per unit volume of the generalized soil skeleton DE in the y-direction.

Using the equilibrium condition of the generalized soil skeleton DE, we can obtain the balance differential equation

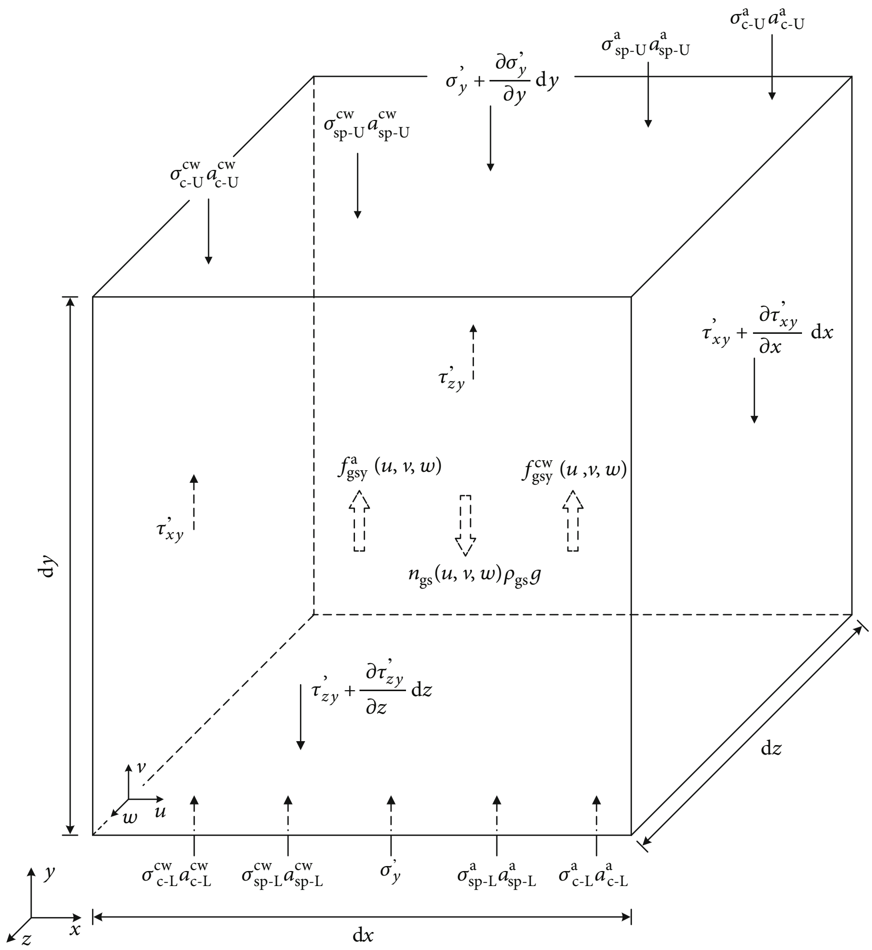

FIGURE 11: Components for the force equilibrium of the generalized soil skeleton DE in the y-direction.

in the y-direction:

$$\begin{aligned}
&\frac{n_{gs}}{(n_{cw}+n_a)}\frac{\partial(n_{cw}u_w)}{\partial y} + \frac{n_{gs}}{(n_{cw}+n_a)}\frac{\partial(n_a u_a)}{\partial y} \\
&\quad - \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f^{cw}_{gsy}(u,v,w)\,dudvdw \\
&\quad - \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f^{a}_{gsy}(u,v,w)\,dudvdw \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{gs}(u,v,w)\rho_{gs}g\,dudvdw \\
&\quad + \frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + \frac{\partial \tau'_{zy}}{\partial z} = 0,
\end{aligned} \tag{24}$$

where n_{gs} is the area porosity of the generalized soil skeleton DE; $n_{gs}(u, v, w)$ is the volume porosity of the generalized soil skeleton DE, which is a function of space coordinates (u, v, w) within the generalized soil skeleton DE. The volume porosity, $n_{gs}(u, v, w)$, will be equal to the area porosities n_{gs} of the lower and upper surfaces of the generalized soil skeleton DE when the point (u, v, w) in the space coordinate system approaches the lower and upper surfaces, respectively; ρ_{gs} is the density of the generalized soil skeleton; τ'_{xy} is the shear stress acting on the x-plane in the y-direction; σ'_y is the normal stress acting on the y-plane; and τ'_{zy} is the shear stress acting on the z-plane in the y-direction. Some detailed mathematical steps for Equation (24) are given in Appendix C.

4.3.4. Total Balance Equation for Unsaturated Soil DE. The total balance equation refers to the force balance equation of an unsaturated soil DE. For simplicity, only the total stress fields and the gravitational force per unit volume of the unsaturated soil DE in the y-direction are illustrated in Figure 12.

Based on the equilibrium condition of the unsaturated soil DE, the balance differential equation related to Figure 12 can be formulated:

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}\rho(u,v,w)g\,dudvdw = 0, \tag{25}$$

where τ_{xy} is the total shear stress acting on the x-plane in the

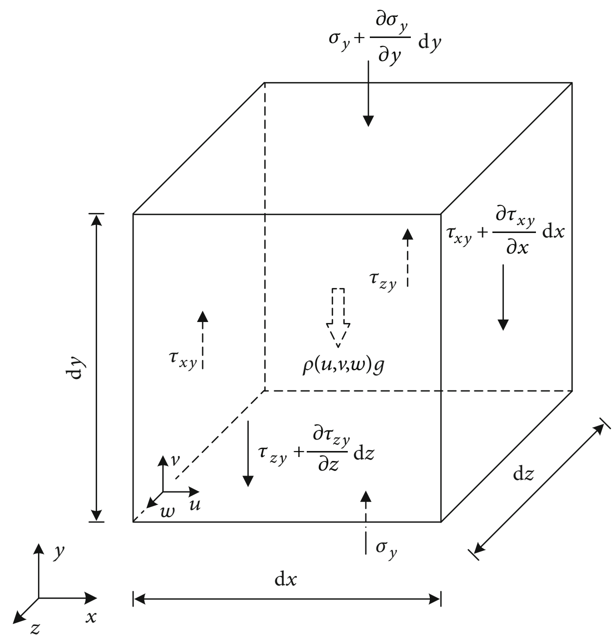

Figure 12: Components for the force equilibrium of unsaturated soil DE in the y-direction.

y-direction; σ_y is the total normal stress acting on the y-plane; τ_{zy} is the total shear stress acting on the z-plane in the y-direction; and $\rho(u, v, w)$ is the density of the unsaturated soil.

It should be noted that $\rho(u, v, w)$ can be calculated by using the following equation:

$$\rho(u, v, w) = n_{\mathrm{cw}}(u, v, w)\rho_{\mathrm{w}} + n_{\mathrm{a}}(u, v, w)\rho_{\mathrm{a}} + n_{\mathrm{gs}}(u, v, w)\rho_{\mathrm{gs}}. \tag{26}$$

Substituting Equation (26) into Equation (25), we can obtain the total balance differential equation for unsaturated soil DE in the y-direction:

$$\begin{aligned}
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{cw}}(u, v, w)\rho_{\mathrm{w}} g du dv dw \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{a}}(u, v, w)\rho_{\mathrm{a}} g du dv dw \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{gs}}(u, v, w)\rho_{\mathrm{gs}} g du dv dw = 0.
\end{aligned} \tag{27}$$

4.3.5. Generalized Terzaghi's Effective Stress Equation for Unsaturated Soils. If an unsaturated soil DE is in equilibrium, the DEs of its three constituent phases are also in equilibrium. Furthermore, each phase is assumed to behave like an independent, continuous, linear, and coincident stress field in each direction [28]. Hence, the independent balance differential Equations (16), (17) and (24) for these three constituent phases can be superimposed by the principle of superposition to form the total balance differential equation for the unsaturated soil DE:

$$\begin{aligned}
&\frac{\partial \tau'_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + \frac{1}{(n_{\mathrm{cw}} + n_{\mathrm{a}})}\frac{\partial (n_{\mathrm{cw}} u_{\mathrm{w}})}{\partial y} + \frac{1}{(n_{\mathrm{cw}} + n_{\mathrm{a}})}\frac{\partial (n_{\mathrm{a}} u_{\mathrm{a}})}{\partial y} \\
&\quad + \frac{\partial \tau'_{zy}}{\partial z} + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{cw}}(u, v, w)\rho_{\mathrm{w}} g du dv dw \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{a}}(u, v, w)\rho_{\mathrm{a}} g du dv dw \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{gs}}(u, v, w)\rho_{\mathrm{gs}} g du dv dw = 0.
\end{aligned} \tag{28}$$

On the one hand, the identity $(n_{\mathrm{cw}} + n_{\mathrm{a}}) + n_{\mathrm{gs}} = 1$ holds; on the other hand, the assumption has been made in Appendix C that the value of $\partial n_{\mathrm{gs}}/\partial y$ is equal to zero. Therefore, $\partial(n_{\mathrm{cw}} + n_{\mathrm{a}})/\partial y = 0$ is valid. Additionally, both capillary water and pore air are usually considered to be unable to resist shear stress, and thus, $\tau'_{xy} = \tau_{xy}$ and $\tau'_{zy} = \tau_{zy}$. Then, Equation (28) can be rewritten as:

$$\begin{aligned}
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma'_y}{\partial y} + \frac{\partial ((n_{\mathrm{cw}} u_{\mathrm{w}} + n_{\mathrm{a}} u_{\mathrm{a}})/(n_{\mathrm{cw}} + n_{\mathrm{a}}))}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{cw}}(u, v, w)\rho_{\mathrm{w}} g du dv dw \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{a}}(u, v, w)\rho_{\mathrm{a}} g du dv dw \\
&\quad + \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{gs}}(u, v, w)\rho_{\mathrm{gs}} g du dv dw = 0.
\end{aligned} \tag{29}$$

Comparing Equation (29) with Equation (27) and extending the result in the y-direction to the x- and z-directions, we can easily obtain the following equation:

$$\sigma = \sigma' + \frac{n_{\mathrm{cw}}}{n_{\mathrm{cw}} + n_{\mathrm{a}}} u_{\mathrm{w}} + \frac{n_{\mathrm{a}}}{n_{\mathrm{cw}} + n_{\mathrm{a}}} u_{\mathrm{a}}. \tag{30}$$

Substituting Equations (7), (9), (10), and (13) into Equation (30), the effective stress equation for unsaturated soils can be obtained:

$$\sigma' = \sigma - \left[S_{\mathrm{cw}}^{\mathrm{eff}} u_{\mathrm{w}} + \left(1 - S_{\mathrm{cw}}^{\mathrm{eff}}\right) u_{\mathrm{a}}\right], \tag{31}$$

where σ' is the effective normal stress, usually referred to as the effective stress; σ is the total normal stress acting on the unsaturated soil, generally referred to as the total stress; $[S_{\mathrm{cw}}^{\mathrm{eff}} u_{\mathrm{w}} + (1 - S_{\mathrm{cw}}^{\mathrm{eff}}) u_{\mathrm{a}}]$ is the neutral stress under unsaturated conditions, which is similar to the neutral stress in the effective stress equation proposed by Terzaghi [14] under saturated conditions; and $S_{\mathrm{cw}}^{\mathrm{eff}}$ is the effective saturation of the capillary water, which is imposed to vary from 0 (corresponding to the state where the capillary water disappears) to 1 (corresponding to the state where the capillary water reaches its maximum, i.e., water-saturated state). Under a water-saturated state, Equation (31) can be reduced to Terzaghi's effective stress equation.

Considering that the form of Equation (31) for unsaturated soils is the same as that of Terzaghi's effective stress equation (Equation (1)) for saturated soils (i.e., the effective stress is equal to the total stress minus the neutral stress), Equation (31) is called the generalized Terzaghi's effective stress equation in this paper.

5. Implications of the Generalized Terzaghi's Effective Stress Equation

There are two fundamental differences between the Bishop-type effective stress equation (Equation (2)) and the generalized Terzaghi's effective stress equation (Equation (31)). First, they are derived based on different physical models. The Bishop-type equation is directly built on the soil skeleton of unsaturated soils. Although this fact is not explicitly stated, it can be discerned by carefully exploring the process of the derivation of the Bishop-type equation. During this process, investigating the macroscopic shear strength and deformation behaviour of unsaturated soils, many researchers [17, 20, 53] argued that the stresses contributing to the shear strength and deformation acted on the soil skeleton. Then, they defined the sum of these stresses as the effective stress. These stresses are classified into two groups: microscopic interparticle stresses and macroscopic stresses [18]. Unfortunately, the contribution mechanisms of these two groups of stresses on the shear strength and deformation are not well distinguished. In fact, because some of the microscopic interparticle stresses (such as those induced by physicochemical forces, cementation forces, and surface tension forces) are not transmitted from one soil particle to another through the soil skeleton, their effects should be associated with the strength and deformation parameters rather than the effective stress. However, the generalized Terzaghi's effective stress equation (Equation (31)) is built on the extended three-phase model. Under the framework of the extended model, the effects of the stresses induced by physicochemical forces, cementation forces, and surface tension forces can be viewed as the change in the strength and deformation parameters instead of the change in the effective stress. This viewpoint is more consistent with Terzaghi's soil mechanics and is more convenient to describe the strength and deformation behaviour of unsaturated soils. Furthermore, under this framework, the interaction forces can be taken into account for the solution of hydromechanical coupling problems by means of the balance equations for these three constituent phases. The second difference consists in the mathematical forms of Equations (2) and (31) and the physical meanings of the stresses in them. Equation (2) is the sum of the net stress and the product of the effective stress parameter χ and matric suction. The net stress represents the stress acting on the soil skeleton induced by external loads. The product was first termed the suction stress by Lu and Likos [18], which was widely used to quantify the contributions of the microscopic interparticle stresses and matric suction to the effective stress [23, 54–56]. In contrast, Equation (31) is the difference between the total stress and the neutral stress. The total stress acts on unsaturated soil containing three constituent phases. The neutral stress is concurrently present in the pore fluid, on the cross-section of the soil particles, and on the contact area between soil particles (as shown in Figures 9 and 10). When the transition of the soil from an unsaturated state to a saturated state occurs, the net stress $(\sigma - u_a)$ and suction stress $\chi (u_a - u_w)$ in Equation (2) are reduced to the total stress σ and pore water pressure u_w (only present in the pore water), respectively, whereas the total stress and neutral stress in Equation (31) can be consistently reduced to the total stress and neutral stress in Equation (1), respectively. Although the neutral stress in Equation (1) is numerically equal to the pore water pressure u_w, their physical meanings, as stated above, are entirely different. From the contrast, it is clear that Equation (2) can only achieve a smooth and continuous transition in the mathematical form of the effective stress equation between unsaturated and saturated conditions, while Equation (31) can do a smooth and continuous transition not only in the mathematical form but also in the physical meanings of the stresses in the effective stress equation. Under Equation (31), almost all classical soil mechanics theories for saturated conditions, such as the Mohr-Coulomb failure criterion, critical state soil mechanics, and consolidation theory, can therefore be directly applied to unsaturated conditions. In addition, the effect of the neutral stress on the mechanical behaviour of unsaturated soils can be considered under Equation (31), but it cannot be considered under Equation (2).

Accurately quantifying the magnitude of the effective stress requires reasonable identification of the effects of different types of pore water in unsaturated soils. It is well recognized that pore water can be divided into two types: capillary water and adsorbed water [56, 57]. The effects of capillary water and adsorbed water on the magnitude of the effective stress lie mainly in three aspects. The first refers to the pressure effects of these two types of pore water. The capillary water held in the pores of unsaturated soils can be regarded as a fluid with macroscopically isotropic negative pore water pressure. A change in the negative water pressure can lead to a change in the neutral stress in Equation (31) or a change in the suction stress in Equation (2) and thus influence the effective stress. However, the adsorbed water wraps the surfaces of the soil particles and the pressure of the adsorbed water mainly induces the volumetric deformation of the soil particles themselves. The pressure of the adsorbed water thus does not influence the effective stress. The second represents the volumetric (or areal) effects of capillary water and adsorbed water. When adsorbed water is considered as a component of the generalized soil skeleton, the effective volume (or area) of the voids of unsaturated soils will decrease with an increase in the volume (or area) of the adsorbed water. In this case, the contribution of the volumetric (or areal) effect of capillary water to the effective stress will increase [23, 25]. When the adsorbed water is ignored, the effective volume (or area) of the voids of unsaturated soils will be equal to the volume (or area) of the voids of unsaturated soils. In this case, the contribution of the volumetric (or areal) effect of the capillary water to the effective stress will decrease [58]. The last represents a transformation between capillary water and adsorbed water. This transformation can develop as the degree of saturation of unsaturated soils changes, which also has an important influence on the magnitude of the effective saturation of the capillary water or the effective stress parameter, χ, in Equation (2) and thus influences the effective stress. The above three effects of capillary water and adsorbed water have been taken into account in the proposed equation (Equation (31)) by using the effective saturation of the capillary water, $S_{\mathrm{cw}}^{\mathrm{eff}}$.

Comparing Equation (31) with Equation (2), we could determine that there are three disadvantages of Equation (2). First, Equation (2) with the effective stress parameter, χ, equal to the degree of saturation, S, [18, 59] assumes that all of the pore water pressures contribute to the effective stress and thus overestimates the effective stress of unsaturated soils, especially for fine-grained soils with low saturation [60]. Second, Equation (2), which uses the effective saturation, S_{e}, as the effective stress parameter, distinguishes the pressure and volumetric (or areal) effects of capillary water from those of adsorbed water and thus somewhat improves the estimation of the effective stress [23, 25, 61]. However, in the process of defining the effective saturation, the residual saturation used to quantify the volume of water adsorbed on the surfaces of soil particles or trapped in the micropores is generally supposed to be independent of the degree of saturation of unsaturated soils. This supposition of constant residual saturation neglects the transformation between the capillary water and adsorbed water and thus still overestimates the effective stress. Last, Equation (2), which adopts the capillary component of the degree of saturation [58], accounts for both the different pressure effects of the capillary water and adsorbed water and the transformation between them and hence makes it possible to better estimate the effective stress, especially for fine-grained soils with low saturation. However, in the definition of the capillary component of the degree of saturation, the volumetric (or areal) effect of the adsorbed water is ignored, which could lead to an underestimation of the effective stress. It should be noted that an opposite conclusion can be drawn from Zhou et al. [58] that Equation (2) with the effective stress parameter equal to the capillary component of the degree of saturation overestimates the effective stress for fine-grained soils with low saturation, but the overestimation is mainly attributed to adopting an unreasonable soil water retention model. According to the above comparisons between Equations (2) and (31), it is obvious that Equation (31) can account for the different pressure effects of capillary water and adsorbed water, their different volumetric (or areal) effects, and the transformation between the two. When the shear strength parameters of unsaturated soils are assumed to be the same as those of saturated soils, Equation (31) is preliminarily validated by considering the shear strength data in the literature [57]. Therefore, the generalized Terzaghi's effective stress equation (Equation (31)) is a better choice for estimating the effective stress.

6. Conclusions and Outlook

An extended three-phase porous medium model is presented, which consists of capillary water, pore air, and a generalized soil skeleton. Based on the extended model, the balance equations for these three constituent phases are formulated by introducing the water content gradient, which was entirely neglected in previous studies. Comparing the result of the superposition of these three balance equations with the total balance equation, a generalized Terzaghi's effective stress equation is derived. On the one hand, the formulated balance equations can consider the interaction forces between different phases and the stresses acting on the cross-section of soil particles and on the contact area between soil particles induced by the capillary water pressure and pore air pressure. On the other hand, the introduction of the water content gradient ensures more rigorous mathematical formulations, which can consider the effect of the water content gradient on the balance equations for these three constituent phases. Therefore, both the extended model and the formulated balance equations provide a solid mechanical foundation for the generalized Terzaghi's effective stress equation proposed in this work.

The generalized Terzaghi's effective stress equation (Equation (31)) states that the effective stress is equal to the total stress minus the neutral stress. This statement means that Equation (31) can achieve a smooth and continuous transition from unsaturated to saturated states not only in the mathematical form of the effective stress equation but also in the physical meanings of the stresses. Under Equation (31), classical soil mechanics theories for saturated conditions can therefore be directly extended to unsaturated conditions. In addition, the effects of neutral stress on the mechanical behaviour of unsaturated soils can also be considered under Equation (31). Compared with the Bishop-type equation with the effective stress parameter equal to the degree of saturation [59], the effective saturation [23], or the capillary component of the degree of saturation [58], Equation (31) using the effective saturation of the capillary water as the effective stress parameter can account for the different pressure effects of the capillary water and adsorbed water, their volumetric (or areal) effects, and the transformation between the two. Therefore, Equation (31) can provide a better choice for estimating the effective stress of unsaturated soils. This statement is preliminarily validated by considering the shear strength data in the literature [57] when the shear strength parameters are assumed to be independent of the degree of saturation. In the future, it is worthwhile to perform a full experimental validation of Equation (31) by considering the variation of the strength parameters with the degree of saturation, especially for fine-grained soils with low saturation.

Appendix

A. Balance Equation for Capillary Water DE Equation Section 1

Expanding and rearranging Equation (15) in Section 4.3.1, we have

$$\begin{aligned}
&-u_{\mathrm{w}}\frac{\partial n_{\mathrm{cw}}}{\partial y}dxdydz - n_{\mathrm{cw}}\frac{\partial u_{\mathrm{w}}}{\partial y}dxdydz - \frac{\partial u_{\mathrm{w}}}{\partial y}\frac{\partial n_{\mathrm{cw}}}{\partial y}dx(dy)^2dz \\
&\quad - \int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f_{\mathrm{gsy}}^{\mathrm{cw}}(u,v,w)\,dudvdw \\
&\quad - \int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{\mathrm{cw}}(u,v,w)\rho_{\mathrm{w}} g\,dudvdw = 0.
\end{aligned} \tag{A.1}$$

Omitting the higher-order term including $(dy)^2$ and dividing by $dxdydz$, we obtain

$$u_{\mathrm{w}}\frac{\partial n_{\mathrm{cw}}}{\partial y}+n_{\mathrm{cw}}\frac{\partial u_{\mathrm{w}}}{\partial y}+\frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}f_{\mathrm{gs}y}^{\mathrm{cw}}(u,v,w)dudvdw+\frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}n_{\mathrm{cw}}(u,v,w)\rho_{\mathrm{w}}gdudvdw=0. \tag{A.2}$$

Combining the first and second terms on the left-hand side of Equation (A.2), we are finally led to the balance differential equation for the capillary water DE (see Equation (16)).

B. Balance Equation for Pore Air DE Equation Section 2

The equilibrium condition of the pore air DE, as shown in Figure 8, demands that the resultant force should vanish. The summation of the forces in the y-direction leads to the force balance equation for the pore air DE:

$$-\left(u_{\mathrm{a}}+\frac{\partial u_{\mathrm{a}}}{\partial y}dy\right)\left[\left(n_{\mathrm{a}}+\frac{\partial n_{\mathrm{a}}}{\partial y}dy\right)dxdz\right]+u_{\mathrm{a}}n_{\mathrm{a}}dxdz-\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}f_{\mathrm{gs}y}^{\mathrm{a}}(u,v,w)dudvdw-\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}n_{\mathrm{a}}(u,v,w)\rho_{\mathrm{a}}gdudvdw=0. \tag{B.1}$$

Expanding and rearranging Equation (B.1), we have

$$-u_{\mathrm{a}}\frac{\partial n_{\mathrm{a}}}{\partial y}dxdydz-n_{\mathrm{a}}\frac{\partial u_{\mathrm{a}}}{\partial y}dxdydz-\frac{\partial u_{\mathrm{a}}}{\partial y}\frac{\partial n_{\mathrm{a}}}{\partial y}dx(dy)^2dz-\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}f_{\mathrm{gs}y}^{\mathrm{a}}(u,v,w)dudvdw-\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}n_{\mathrm{a}}(u,v,w)\rho_{\mathrm{a}}gdudvdw=0. \tag{B.2}$$

Eliminating the higher-order term including $(dy)^2$ and dividing by $dxdydz$, we obtain

$$u_{\mathrm{a}}\frac{\partial n_{\mathrm{a}}}{\partial y}+n_{\mathrm{a}}\frac{\partial u_{\mathrm{a}}}{\partial y}+\frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}f_{\mathrm{gs}y}^{\mathrm{a}}(u,v,w)dudvdw+\frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}n_{\mathrm{a}}(u,v,w)\rho_{\mathrm{a}}gdudvdw=0. \tag{B.3}$$

Combining the first and second terms on the left-hand side of Equation (B.3), we can finally obtain the balance differential equation for the pore air DE (see Equation (17)).

C. Balance Equation for Generalized Soil Skeleton DE Equation Section 3

The equilibrium condition of the generalized soil skeleton DE, as shown in Figure 11, demands that the resultant force should vanish. Summing the forces in the y-direction yields the force balance equation for the generalized soil skeleton DE:

$$\begin{aligned}&-\left(\sigma_y'+\frac{\partial\sigma_y'}{\partial y}dy\right)dxdz+\sigma_y'dxdz-\left(\tau_{xy}'+\frac{\partial\tau_{xy}'}{\partial x}dx\right)dydz+\tau_{xy}'dydz\\&-\left(\tau_{zy}'+\frac{\partial\tau_{zy}'}{\partial z}dz\right)dxdy+\tau_{zy}'dxdy-\sigma_{\text{sp-U}}^{\mathrm{cw}}a_{\text{sp-U}}^{\mathrm{cw}}dxdz\\&+\sigma_{\text{sp-L}}^{\mathrm{cw}}a_{\text{sp-L}}^{\mathrm{cw}}dxdz-\sigma_{\text{c-U}}^{\mathrm{cw}}a_{\text{c-U}}^{\mathrm{cw}}dxdz+\sigma_{\text{c-L}}^{\mathrm{cw}}a_{\text{c-L}}^{\mathrm{cw}}dxdz-\sigma_{\text{sp-U}}^{\mathrm{a}}a_{\text{sp-U}}^{\mathrm{a}}dxdz\\&+\sigma_{\text{sp-L}}^{\mathrm{a}}a_{\text{sp-L}}^{\mathrm{a}}dxdz-\sigma_{\text{c-U}}^{\mathrm{a}}a_{\text{c-U}}^{\mathrm{a}}dxdz+\sigma_{\text{c-L}}^{\mathrm{a}}a_{\text{c-L}}^{\mathrm{a}}dxdz\\&+\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}f_{\mathrm{gs}y}^{\mathrm{cw}}(u,v,w)dudvdw\\&+\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}f_{\mathrm{gs}y}^{\mathrm{a}}(u,v,w)dudvdw\\&-\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz}n_{\mathrm{gs}}(u,v,w)\rho_{\mathrm{gs}}gdudvdw=0,\end{aligned} \tag{C.1}$$

where $\sigma_{\text{sp-U}}^{\mathrm{cw}}$ and $\sigma_{\text{c-U}}^{\mathrm{cw}}$ denote the stresses acting on the cross-sections of the soil particles and on the contact areas between the soil particles of the upper surface, respectively, which are induced by the homogenized capillary water pressure of the upper surface; $a_{\text{sp-U}}^{\mathrm{cw}}$ and $a_{\text{c-U}}^{\mathrm{cw}}$ represent the corresponding areas of $\sigma_{\text{sp-U}}^{\mathrm{cw}}$ and $\sigma_{\text{c-U}}^{\mathrm{cw}}$; $\sigma_{\text{sp-L}}^{\mathrm{cw}}$ and $\sigma_{\text{c-L}}^{\mathrm{cw}}$ denote the stresses acting on the cross-sections of the soil particles and on the contact areas between the soil particles of the lower surface, respectively, which are induced by the homogenized capillary water pressure of the lower surface; $a_{\text{sp-L}}^{\mathrm{cw}}$ and $a_{\text{c-L}}^{\mathrm{cw}}$ represent the corresponding areas of $\sigma_{\text{sp-L}}^{\mathrm{cw}}$ and $\sigma_{\text{c-L}}^{\mathrm{cw}}$. $\sigma_{\text{sp-U}}^{\mathrm{a}}$ and $\sigma_{\text{c-U}}^{\mathrm{a}}$ denote the stresses acting on the cross-sections of soil particles and on the contact areas between soil particles of the upper surface, respectively, which are induced by the homogenized air pressure of the upper surface; $a_{\text{sp-U}}^{\mathrm{a}}$ and $a_{\text{c-U}}^{\mathrm{a}}$ represent the corresponding areas of $\sigma_{\text{sp-U}}^{\mathrm{a}}$ and $\sigma_{\text{c-U}}^{\mathrm{a}}$. $\sigma_{\text{sp-L}}^{\mathrm{a}}$ and $\sigma_{\text{c-L}}^{\mathrm{a}}$ denote the stresses acting on the cross-sections of soil particles and on the contact areas between soil particles of the lower surface, respectively, which are induced by the homogenized air pressure of the lower surface; $a_{\text{sp-L}}^{\mathrm{a}}$ and $a_{\text{c-L}}^{\mathrm{a}}$ represent the corresponding areas of $\sigma_{\text{sp-L}}^{\mathrm{a}}$ and $\sigma_{\text{c-L}}^{\mathrm{a}}$.

Based on the analysis of Figure 10, we can obtain

$$\sigma_{\text{sp-U}}^{\mathrm{cw}}=\frac{(n_{\mathrm{cw}}+(\partial n_{\mathrm{cw}}/\partial y)dy)}{(n_{\mathrm{cw}}+(\partial n_{\mathrm{cw}}/\partial y)dy)+(n_{\mathrm{a}}+(\partial n_{\mathrm{a}}/\partial y)dy)}\left(u_{\mathrm{w}}+\frac{\partial u_{\mathrm{w}}}{\partial y}dy\right);\sigma_{\text{sp-L}}^{\mathrm{cw}}=\frac{n_{\mathrm{cw}}}{n_{\mathrm{cw}}+n_{\mathrm{a}}}u_{\mathrm{w}},$$

$$\sigma_{\text{c-U}}^{\mathrm{cw}}=\frac{(n_{\mathrm{cw}}+(\partial n_{\mathrm{cw}}/\partial y)dy)}{(n_{\mathrm{cw}}+(\partial n_{\mathrm{cw}}/\partial y)dy)+(n_{\mathrm{a}}+(\partial n_{\mathrm{a}}/\partial y)dy)}\left(u_{\mathrm{w}}+\frac{\partial u_{\mathrm{w}}}{\partial y}dy\right);\sigma_{\text{c-L}}^{\mathrm{cw}}=\frac{n_{\mathrm{cw}}}{n_{\mathrm{cw}}+n_{\mathrm{a}}}u_{\mathrm{w}},$$

$$\sigma^{a}_{\text{sp-U}} = \frac{(n_a + (\partial n_a/\partial y)dy)}{(n_{cw} + (\partial n_{cw}/\partial y)dy) + (n_a + (\partial n_a/\partial y)dy)}\left(u_a + \frac{\partial u_a}{\partial y}dy\right); \sigma^{a}_{\text{sp-L}} = \frac{n_a}{n_{cw} + n_a}u_a,$$

$$\sigma^{a}_{\text{c-U}} = \frac{(n_a + (\partial n_a/\partial y)dy)}{(n_{cw} + (\partial n_{cw}/\partial y)dy) + (n_a + (\partial n_a/\partial y)dy)}\left(u_a + \frac{\partial u_a}{\partial y}dy\right); \sigma^{a}_{\text{c-L}} = \frac{n_a}{n_{cw} + n_a}u_a,$$

$$a^{cw}_{\text{sp-U}} + a^{cw}_{\text{c-U}} = \left[1 - \left(n_{cw} + \frac{\partial n_{cw}}{\partial y}dy\right) - \left(n_a + \frac{\partial n_a}{\partial y}dy\right)\right]; a^{cw}_{\text{sp-L}} + a^{cw}_{\text{c-L}} = (1 - n_{cw} - n_a),$$

$$a^{a}_{\text{sp-U}} + a^{a}_{\text{c-U}} = \left[1 - \left(n_{cw} + \frac{\partial n_{cw}}{\partial y}dy\right) - \left(n_a + \frac{\partial n_a}{\partial y}dy\right)\right]; a^{a}_{\text{sp-L}} + a^{a}_{\text{c-L}} = (1 - n_{cw} - n_a). \tag{C.2}$$

Substituting Equation (C.2) into Equation (C.1), expanding Equation (C.1), and eliminating the higher-order term including $(dy)^2$, we have

$$\begin{aligned}
&\frac{(1 - n_{cw} - n_a)(\partial(n_{cw}u_w)/\partial y)dxdydz - n_{cw}u_w(\partial(n_{cw} + n_a)/\partial y)dxdydz}{(n_{cw} + n_a) + (\partial(n_{cw} + n_a)/\partial y)dy} \\
&+ \frac{(1 - n_{cw} - n_a)n_{cw}u_w dxdz}{(n_{cw} + n_a) + (\partial(n_{cw} + n_a)/\partial y)dy} - \frac{(1 - n_{cw} - n_a)n_{cw}u_w dxdz}{(n_{cw} + n_a)} \\
&+ \frac{(1 - n_{cw} - n_a)(\partial(n_a u_a)/\partial y)dxdydz - n_a u_a(\partial(n_{cw} + n_a)/\partial y)dxdydz}{(n_{cw} + n_a) + (\partial(n_{cw} + n_a)/\partial y)dy} \\
&+ \frac{(1 - n_{cw} - n_a)n_a u_a dxdz}{(n_{cw} + n_a) + (\partial(n_{cw} + n_a)/\partial y)dy} - \frac{(1 - n_{cw} - n_a)n_a u_a dxdz}{(n_{cw} + n_a)} \\
&- \int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f^{cw}_{gsy}(u, v, w)dudvdw - \int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f^{a}_{gsy}(u, v, w)dudvdw \\
&+ \int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{gs}(u, v, w)\rho_{gs} g dudvdw + \frac{\partial\sigma'_y}{\partial y}dxdydz + \frac{\partial\tau'_{xy}}{\partial x}dxdydz \\
&+ \frac{\partial\tau'_{zy}}{\partial z}dxdydz = 0.
\end{aligned} \tag{C.3}$$

Substituting the identity, $n_{gs} = (1 - n_{cw} - n_a)$, and dividing by $dxdydz$, we obtain

$$\begin{aligned}
&\frac{n_{gs}(\partial(n_{cw}u_w)/\partial y) + n_{cw}u_w\left(\partial n_{gs}/\partial y\right)}{(n_{cw} + n_a) - \left(\partial n_{gs}/\partial y\right)dy} + \frac{n_{gs}(\partial(n_a u_a)/\partial y) + n_a u_a\left(\partial n_{gs}/\partial y\right)}{(n_{cw} + n_a) - \left(\partial n_{gs}/\partial y\right)dy} \\
&- \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f^{cw}_{gsy}(u, v, w)dudvdw \\
&- \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} f^{a}_{gsy}(u, v, w)dudvdw \\
&+ \frac{1}{dxdydz}\int_x^{x+dx}\int_y^{y+dy}\int_z^{z+dz} n_{gs}(u, v, w)\rho_{gs} g dudvdw + \frac{\partial\sigma'_y}{\partial y} \\
&+ \frac{\partial\tau'_{xy}}{\partial x} + \frac{\partial\tau'_{zy}}{\partial z} = 0.
\end{aligned} \tag{C.4}$$

In classical soil mechanics, the soil skeleton is generally defined as an assembly of soil particles [62], and its porosity under homogeneous soil conditions can be assumed to be independent of space coordinates [63, 64]. Since the adsorbed water is controlled by the adsorptive forces exerted by the surfaces of soil particles, the porosity of the adsorbed water can also be assumed to be constant under homogeneous soil conditions when space coordinates change. In addition, compared with the porosity of the soil skeleton, the porosity of the contractile skin can be ignored. In this case, the value of $\partial n_{gs}/\partial y$ can be assumed to be zero; therefore, Equation (C.4) can be simplified as Equation (24).

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors state that they do not have any financial or nonfinancial conflict of interests.

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