Research Article

Applying the Large Parameter Technique for Solving a Slow Rotary Motion of a Disc about a Fixed Point

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In this paper, the motion of a disk about a fixed point under the influence of a Newtonian force field and gravity one is considered. We modify the large parameter technique which is achieved by giving the body a sufficiently small angular velocity component r_0 about the fixed z -axis of the disk. The periodic solutions of motion are obtained in the neighborhood r_0 tends to 0. This case of study is excluded from the previous works because of the appearance of a singular point in the denominator of the obtained solutions. Euler-Poisson equations of motion are obtained with their first integrals. These equations are reduced to a quasilinear autonomous system of two degrees of freedom and one first integral. The periodic solutions for this system are obtained under the new initial conditions. Computerizing the obtained periodic solutions through a numerical technique for validation of results is done. Two types of analytical and numerical solutions in the new domain of the angular velocity are obtained. Geometric interpretations of motion are presented to show the orientation of the body at any instant of time t .

1. Introduction

In [1], the authors considered the limiting case for the motion of a rigid body about a fixed point in the Newtonian force field and gravity one. The small parameter technique is applied to solve this problem. The authors defined this parameter inversely proportional to the third component of the angular velocity which is assumed to be sufficiently high. The periodic solutions and their graphical representations are obtained and illustrated geometrically. In [2], the authors admitted the KBM technique for solving the problem of a rotating heavy solid about a fixed point under the influence of a gyrostatic moment. They assumed a small parameter as in [1] and found the analytical and numerical solutions for the body which moves under its gravity and a gyro moment about the minor or the major axis of the ellipsoid of inertia. In [3], Leshchenko and Ershkov presented a new type of solving procedure for Euler-Poisson equations (rigid body rotation over a fixed point) in the presence of some restricted conditions on the body angular

velocity or the applied perturbing torques. The author in [4] gave the regular precession of an asymmetric rigid body acted upon by a uniform gravity field and magnetic one. He obtained the equations of motion of the body and reduced them to a quasilinear autonomous system. He found the solution to the problem and its geometric interpretation of motion. Nayfeh [5] presented some perturbation methods such as Poincaré', the KBM, the Multiple scales, and averaging techniques for solving this kind of motion under certain conditions. The authors in [6] studied the rotating symmetric rigid body about a fixed point in the Newtonian force field in a case analogous to Kovalevskaya's problem. They described the motion of the body and derived its equations of motion and find the solution to the problem assuming Kovalevskaya's conditions.

All the previous approximated methods depend on a small parameter achieved inversely proportional to the sufficiently high angular velocity component. This study gives many applications in physics [7], gyros [8, 9], astronomy, engineering, aerospace, and other sciences.

In our problem, we assume a slow rotation (weak spin) of the body instead of a fast rotation (high spin). We introduce a large parameter inversely proportional to the weak spin about the z -axis instead of a small parameter inversely proportional to the high spin about that axis. So, we must apply the large parameter technique instead of the small one used in the previous works. The validation of our results will be given.

2. Formulation of the Problem and Construction of the Periodic Solutions

In this section, we formulate the problem of the motion of the body, deduce the equations of motion, and construct the periodic solutions:

2.1. Formulation of the Problem. Consider a rigid body of mass (M) rotates around a fixed point O . Let the two coordinate frames OX, OY, OZ and Ox, Oy, Oz are fixed in the space and the body, respectively. Initially, let the body spin with a small speed angular velocity component r_0 about the z -axis. Suppose that p_2 and γ_2 are the generalized coordinates of the reduced quasilinear autonomous system [2]. Let g be the gravity acceleration; A, B, C are the principal moments of inertia of the body for the moving coordinates; the point (x_0, y_0, z_0) is the mass center of the body, and $\underline{R}$ is the position vector of O_1 to O . Let M_1 and M_2 be the basic amplitudes for p_2 while M_3 is the basic amplitude for γ_2 . Considering $\beta_1(\mu), \beta_2(\mu)$ and $\beta_3(\mu)$ are the deviations for p_2 and γ_2 from their initial basic amplitudes M_1, M_2 , and M_3 respectively.

Let the moving z -axis make an angle $\theta_0 \approx \pi/2$ with the downward fixed Z -axis. This case of a slow spinning rotary body when the natural frequency $\omega = 1$ gives the rotary motion of a disc which is excluded from the previous case $\gamma_0'' = \cos \theta_0 \approx 0$. We achieve a large parameter $0 < \mu < \infty$ inversely proportional to the sufficiently small value of r_0 . Assume that the disc rotates about the z -axis in the presence of a Newtonian force field [3] and a gravitational one. We reduce the equations of motion for this case to the following autonomous system of two degrees of freedom [4]:

$$\ddot{p}_2 + p_2 = \mu^{-2} F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu^{-1}), \quad (1)$$

$$\ddot{\gamma}_2 + \gamma_2 = \mu^{-2} \Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu^{-1}), \quad (2)$$

where

$$\begin{aligned} p_2 &= p_1 - \mu^{-1} x_0' \gamma_0'' / b, \\ \gamma_2 &= \gamma - \mu^{-1} a \gamma_0'' p_2, \end{aligned} \quad (3)$$

$$F = F_2 + \mu^{-1} F_3 + \dots, \quad (F\Phi), \mu = c \sqrt{\gamma_0''} / r_0, \quad (4)$$

$$F_2 = F_2(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2), \quad (F_2 F_3 \Phi_2 \Phi_3), \cdot = d/d\tau,$$

where

$$\begin{aligned} A_1 &= (C - B)/A, \quad (ABC), a = A/C, (ab), x_0' = x_0/\ell, (x_0' y_0'), \\ \ell^2 &= x_0'^2 + y_0'^2, \quad k = 3C/RM\ell, p_1 = p/c^2, (p_1 q_1), r_1 = r/r_0, c = \sqrt{Mg\ell/C}, \end{aligned} \quad (5)$$

where the symbols like $(ab), (abc)$, and $(abcd)$ denote omitted equations and p, q, r , and $\gamma, \gamma', \gamma''$ are the components of the angular velocity vector and the cosines direction of the unit vector Z^Δ .

The system (2) has the following integral [4]:

$$\dot{\gamma}_2^2 + \dot{p}_2^2 + 2\mu^{-1} \gamma_0'' (ap_2 \dot{\gamma}_2 - b\dot{p}_2 \dot{\gamma}_2 + s_{21}) = 1 - \gamma_0''^2, \quad (6)$$

where

$$s_{21} = a(p_{20} \gamma_{20} - p_2 \gamma_2) - b(\dot{p}_{20} \dot{\gamma}_{20} - \dot{p}_2 \dot{\gamma}_2), \quad (7)$$

where the quantities $p_{20}, \gamma_{20}, \dot{p}_{20}$, and $\dot{\gamma}_{20}$ are the initial values of the corresponding variables.

2.2. Construction of the Periodic Solutions. In this subsection, we use the large parameter technique [5] to construct the periodic solutions of system (2) when the disc rotates with slow velocity about the minor axis of the ellipsoid of inertia. Assume that

$$A + B = C, \quad a + b = 1. \quad (8)$$

Using (8) and the definition of s_{21} , we get

$$s_{21}^{(0)} = 0.5 M_3 E [\cos \varepsilon - \cos (2\tau - \varepsilon)], \quad (9)$$

where $E = \sqrt{M_1^2 + M_2^2}$, $\varepsilon = \tan^{-1}(M_2/M_1)$ and M_1 and M_2 are constants.

Substituting (9) into (3), we get

$$\begin{aligned} F_2^{(0)} &= M_1 L_{21}^* \cos \tau + M_2 L_{21}^* \sin \tau + \dots, \\ \Phi_2^{(0)} &= [N_{21}^* - 0.25a(M_1^2 - M_2^2)] M_3 \cos \tau \\ &\quad - 0.5aM_1 M_2 M_3 \sin \tau + \dots, \\ F_3^{(0)} &= [L_{31}^* M_1 + L_{32}^* (M_1^2 - M_2^2) + L_{33}^* M_3] \cos \tau \\ &\quad + [L_{31}^* M_2 + 2L_{32}^* M_1 M_2 + L_{34}^* M_3] \sin \tau + \dots, \end{aligned} \quad (10) \quad (11)$$

where

$$\begin{aligned} L_{21}^* &= 2M_3 x_0' + 0.5(a - b)(M_2^2 - M_1^2 - kM_3^2) \\ &\quad + k(\gamma_0''^2 - 0.5M_3^2), \\ N_{21}^* &= -(aM_1^2 + bM_2^2) + 2M_3 x_0' \\ &\quad + k[\gamma_0''^2 + M_3^2(b - a)], \end{aligned} \quad (12)$$

$$\begin{aligned}
L_{31}^* &= 2\gamma_0'' \left[-a^2 b^{-1} x_0' M_1 + b^2 a^{-1} y_0' M_2 + a(1+b)kM_1 M_3 \right], \\
L_{32}^* &= -0.5a\gamma_0'' kM_3, \\
L_{34}^* &= 3a^{-1}b^{-1}x_0'y_0'\gamma_0''.
\end{aligned} \tag{13}$$

$$\begin{aligned}
L_{33}^* &= \gamma_0'' \left\{ -(2+ab^{-1})b^{-1}y_0'^2 + (1+a^{-1})a^{-1}y_0'^2 \right. \\
&\quad \left. + ak[0.5kM_3^2 - M_1^2 - M_2^2] \right\}.
\end{aligned} \tag{14}$$

Let

$$\begin{aligned}
g_k(\tau) &= \int_0^\tau F_k^{(0)}(t_1) \sin(\tau - t_1) dt_1, \\
h_k(\tau) &= \int_0^\tau \Phi_k^{(0)}(t_1) \sin(\tau - t_1) dt_1.
\end{aligned} \tag{15}$$

Using (15), (11), and (14), we get

$$\begin{aligned}
g_2(2\pi) &= -\pi M_2 L_{21}^*, \\
\dot{g}_2(2\pi) &= \pi M_1 L_{21}^*, \\
h_2(2\pi) &= 0.5\pi a M_1 M_2 M_3, \\
\dot{h}_2(2\pi) &= \pi [N_{21}^* - 0.25a(M_1^2 - M_2^2)] M_3, \\
g_3(2\pi) &= -\pi [L_{31}^* M_2 + 2L_{32}^* M_1 M_2 + L_{34}^* M_3], \\
\dot{g}_3(2\pi) &= -\pi [L_{31}^* M_1 + L_{32}^* (M_1^2 - M_2^2) + L_{33}^* M_3].
\end{aligned} \tag{16}$$

Using (16) and McLaren's expansion in the form [9]:

$$\begin{aligned}
U &= u + \frac{\partial u}{\partial M_1} \beta_1 + \frac{\partial u}{\partial M_2} \beta_2 + \frac{\partial u}{\partial M_3} \beta_3 \\
&\quad + \frac{1}{2} \frac{\partial^2 u}{\partial M_1^2} \beta_1^2 + \dots, \left\{ \begin{array}{l} U = G_k, H_k \\ u = g_k, h_k \end{array} \right\},
\end{aligned} \tag{17}$$

when $M_1 = M_2 = 0$, we get

$$\begin{aligned}
G_2(2\pi) &= -\pi \beta_2 \left[2\tilde{M}_3 x_0' + 0.5(a-b)(\beta_2^2 - \beta_1^2 - k\tilde{M}_3^2) \right. \\
&\quad \left. + k(\gamma_0''^2 - 0.5\tilde{M}_3^2) \right], \\
\dot{G}_2(2\pi) &= \pi \beta_1 \left[2\tilde{M}_3 x_0' + 0.5(a-b)(\beta_2^2 - \beta_1^2 - k\tilde{M}_3^2) \right. \\
&\quad \left. + k(\gamma_0''^2 - 0.5\tilde{M}_3^2) \right], \\
G_3(2\pi) &= -2\pi \beta_2 \gamma_0'' \left[b^2 a^{-1} y_0' \beta_2 - a^2 b^{-1} x_0' \beta_1 + a(1+b)k\beta_1 \tilde{M}_3 \right] \\
&\quad + \pi \tilde{M}_3 \gamma_0'' \left[ak\beta_1 \beta_2 - 3a^{-1}b^{-1}x_0'y_0' \right],
\end{aligned}$$

$$\begin{aligned}
\dot{G}_3(2\pi) &= \pi \gamma_0'' \left\{ 2\beta_1 \left[b^2 a^{-1} y_0' \beta_2 - a^2 b^{-1} x_0' \beta_1 + a(1+b)k\beta_1 \tilde{M}_3 \right] \right. \\
&\quad \left. + \tilde{M}_3 \left[a^{-2}(a+1)\gamma_0'^2 - b^{-2}(b+1)x_0'^2 \right. \right. \\
&\quad \left. \left. + 0.5ak(k\tilde{M}_3^2 - 3\beta_1^2 - \beta_2^2) \right] \right\},
\end{aligned}$$

$$H_2(2\pi) = 0.5a\pi\beta_1\beta_2\tilde{M}_3,$$

$$\begin{aligned}
\dot{H}_2(2\pi) &= \pi \tilde{M}_3 \left[2x_0' \tilde{M}_3 - (a\beta_1^2 + b\beta_2^2) - 0.25a(\beta_1^2 - \beta_2^2) \right. \\
&\quad \left. + k\gamma_0''^2 + k\tilde{M}_3^2(b-a) \right],
\end{aligned} \tag{18}$$

where $\tilde{M}_i = M_i + \beta_i$, $i = 1, 2, 3$.

Using (18), the independent periodicity conditions give [10]

$$\begin{aligned}
v_0 \beta_1 &= -\mu^{-1} v_1 + \dots, \\
v_0 \beta_2 &= -\mu^{-1} v_2 + \dots,
\end{aligned} \tag{19}$$

$$\alpha(\mu^{-1}) = \mu^{-2} \pi \left[2M_3 x_0' + k(\gamma_0''^2 + C_1 M_3^2) \right] + \dots, \tag{20}$$

where

$$\begin{aligned}
v_0 &= a_1 \beta_1^2 + a_2 \beta_2^2 - 0.5k(C_1 + 1)M_3^2, \\
a_1 &= 0.5(b + 0.67a), \\
a_2 &= 0.5(b + 0.5a), \\
v_1 &= M_3 \gamma_0'' \left[2a^{-2}(a+1)\gamma_0'^2 - 2b^{-2}(b+1)x_0'^2 + ak^2 M_3^2 \right], \\
v_2 &= -6M_3 \gamma_0'' a^{-1} b^{-1} x_0' y_0'.
\end{aligned} \tag{21}$$

The expansions β_1, β_2 are put in power series form as follows:

$$\begin{aligned}
\beta_1 &= \sum_{k=1}^3 \mu^{-k} \ell_k + O(\mu^{-4}), \\
\beta_2 &= \sum_{k=1}^3 \mu^{-k} m_k + O(\mu^{-4}).
\end{aligned} \tag{22}$$

Substituting (22) into (19) and equating the coefficients of μ^{-1} in both sides, we get

$$\begin{aligned}
\ell_1 &= 2v_1 [k(C_1 + 1)M_3^2]^{-1}, \\
m_1 &= 2v_2 [k(C_1 + 1)M_3^2]^{-1}.
\end{aligned} \tag{23}$$

From [1] we get

$$\begin{aligned} q_1 &= -\dot{p}_1 + \mu^{-1} a^{-1} \left(\gamma_0' \gamma_0'' - z_0' \gamma' + a k \gamma' \gamma_0'' \right) + \dots, \\ \gamma' &= \dot{\gamma} - \mu^{-1} \gamma_0'' \dot{p}_1 + \dots. \end{aligned} \quad (24)$$

The following quantities are derived:

$$\begin{aligned} M_3 &= 1, \\ \beta_3 &= -a\Gamma\mu^{-2}\tilde{M}_1 - 0.5\mu^{-2}\Gamma^2 + \dots. \end{aligned} \quad (25)$$

Making use of (18), (23), (8), (24), (3), and (25), we obtain the following periodic solutions forms:

$$\begin{aligned} p_1 &= v_1 k^* \cos \tau + v_2 k^* \sin \tau + \mu^{-1} \left[x_0' b^{-1} \gamma_0'' \right] + \dots, \\ q_1 &= v_1 k^* \sin \tau - v_2 k^* \cos \tau + \mu^{-1} \left[\gamma_0' a^{-1} \gamma_0'' - k \gamma_0'' \sin \tau \right] + \dots, \\ r_1 &= 1 - 0.25\mu^{-2} \left[kC_1 + 4x_0'(1 - \cos \tau) + 4\gamma_0' \sin \tau \right. \\ &\quad \left. - kC_1 \cos 2\tau \right] + \dots, \\ \gamma &= \cos \tau + \mu^{-1} a \gamma_0'' [v_1 k^* \cos \tau + v_2 k^* \sin \tau] + \dots, \\ \gamma' &= -\sin \tau + \mu^{-1} b \gamma_0'' [v_1 k^* \sin \tau - v_2 k^* \cos \tau] + \dots, \\ \gamma'' &= \gamma_0'' + \mu^{-2} \gamma_0'' b^{-1} x_0' (1 - \cos \tau) + a^{-1} \gamma_0' \sin \tau \\ &\quad - 0.25k(1 - \cos 2\tau) + \dots \end{aligned} \quad (26)$$

The correction of the period is:

$$\begin{aligned} \alpha(\mu^{-1}) &= \mu^{-2} \pi \left[2x_0' + \left(\gamma_0''^2 + C_1 \right) / bk^* \right] + \dots, \\ k &= 1/bk^*, \tau = r_o^{-1} t. \end{aligned} \quad (27)$$

3. Geometric Interpretation of the Motion

In this section, we search the description of the motion of the disk at any instant of the time using Euler's angles θ , ψ , ϕ [11, 12]:

$$\begin{aligned} \theta &= \theta_0 - \mu^{-2} \cot \theta_0 [\theta_2(t + t_0) - \theta_2(t_0)] + \dots, \\ r_0(\psi - \psi_0) &= 0.5 c \operatorname{cosec} \theta_0 \left\{ c \cot \theta_0 k^{*2} (v_1^2 + v_2^2) t \right. \\ &\quad \left. + \operatorname{cosec} \theta_0 \left[k \gamma_0'' r_0 t + \psi_1(t + t_0) - \psi_1(t_0) \right] \right. \\ &\quad \left. + 2\mu^{-1} \cot \theta_0 [\psi_2(t + t_0) - \psi_2(t_0)] \right\} + \dots, \\ \phi - \phi_0 &= \left[r_0 - 0.5 c \cot^2 \theta_0 (k - c r_0^{-1} k^{*2} (v_1^2 + v_2^2)) \right] t \\ &\quad - \mu^{-1} \cot \theta_0 \operatorname{cosec} \theta_0 [\phi_1(t + t_0) - \phi_1(t_0)] + \dots, \end{aligned} \quad (28)$$

where

$$\begin{aligned} \theta_2(t) &= -b^{-1} x_0' \cos r_0 t + a^{-1} \gamma_0' \sin r_0 t + 0.25k \cos 2r_0 t, \\ \psi_1(t) &= 2a^{-1} \gamma_0' \gamma_0'' \cos r_0 t - v_2 k^* \cos 2r_0 t \\ &\quad + \left(v_1 k^* - 0.5k \gamma_0'' \right) \sin 2r_0 t, \\ \psi_2(t) &= x_0' (1 - a)^{-1} \sin r_0 t + 0.5C_1 k^{*2} v_1 v_2 \cos 2r_0 t \\ &\quad - 0.25C_1 k^{*2} (v_1^2 - v_2^2) \sin 2r_0 t, \\ \phi_1(t) &= a^{-1} \gamma_0' \gamma_0'' \cos r_0 t - 0.5v_2 k^* \cos 2r_0 t \\ &\quad + 0.5 \left(v_1 k^* - 0.5k \gamma_0'' \right) \sin 2r_0 t. \end{aligned} \quad (29)$$

These formulas depend on four arbitrary constants θ_0 , ψ_0 , ϕ_0 , and r_0 , where r_0 is sufficiently small. The obtained analytical periodic solutions are considered a generalization of the ones obtained in [13, 14].

4. Numerical Solutions

In this section, we use a smooth and suitable numerical method for obtaining the approximated numerical solutions for the autonomous system (2). Such a method is named the fourth-order Runge-Kutta method [15] which is used through a computerized program to find the numerical solutions of the considered problem. In the other side, we computerize the obtained analytical solutions and compare them with the numerical ones aiming to find the errors between them:

4.1. The Analytical Solutions. Rewriting the resulted analytical solutions in the form:

$$\begin{aligned} p_2 &= v_1 k^* \cos(h) + v_2 k^* \sin(h), \\ x &= dp_2/dt = -v_1 k^* \sin(h) + v_2 k^* \cos(h), \\ \gamma_2 &= \cos(h), \quad y = d\gamma_2/dt = -\sin(h). \end{aligned} \quad (30)$$

Let the disk parameters are

$$\begin{aligned} A &= 9, \\ B &= 11, \\ C &= 20, \\ x_0 &= 1, \\ \gamma_0 &= 6, \\ R &= 2000, \\ M &= 200, \\ \gamma_0'' &= 0.000007, \\ \lambda &= 0.6, \end{aligned}$$

$$\begin{aligned} r_0 &= 0.000013, \\ T &= 18.78775142. \end{aligned} \quad (31)$$

We obtain the following parameters of the motion:

$$\begin{aligned} C_1 &= 1.0E-01, \\ \ell &= 6.082763, \\ x'_0 &= 1.643990E-01, \\ y'_0 &= 9.863939E-01, \\ g &= 1.500000E-07, \\ c &= 3.020620E-03, \\ \mu &= 2.323554E06, \\ a &= 4.5E-01, \\ b &= 5.5E-01, \\ k &= 2.465985E-05, \\ \chi_2 &= -1.726189E-10. \end{aligned} \quad (32)$$

Making use of (30), (31), and (32) through a computer program, we obtain the values of the analytical solutions (see Table 1).

4.2. The Numerical Solutions. The system (30) is rewritten in the form:

$$\begin{aligned} \frac{dp_2}{dt} &= x, \\ \frac{d\gamma_2}{dt} &= y, \\ \frac{dx}{dt} &= -p_2 + \mu^2 f_1, \\ \frac{dy}{dt} &= -\gamma_2 + \mu^2 g_1, \end{aligned} \quad (33)$$

where $f_1 = F_2$, $g_1 = \Phi_2$.

Using (33), (31), (32), the fourth-order Runge-Kutta method, and the same initial values in the Table 1, we find the results of the numerical solutions in Table 2. Table 1 is in good agreement with Table 2; that is, the analytical solutions are in full approximation with the numerical ones.

5. Conclusions

In this paper, we studied the singular value of the natural frequency problem $\omega = 1$ in a limiting case when $\gamma_o'' = \cos \theta_o \approx 0$. In this case, the body rotates about the z -axis which should coincide with the minor axis of the ellipsoid of inertia. This problem corresponds to the disc motion

TABLE 1: The values of the analytical solutions in the interval $t \in [0, 300]$.

t	p_{2a}	γ_{2a}	$x_a = dp_{2a}/dt$	$y_a = d\gamma_{2a}/dt$
0	7.048539	1	-2.028946	0
10	4.52171	0.8102261	-5.775176	-0.5861173
20	2.79E-01	0.3129329	-7.329452	-0.9497752
30	-4.070127	-0.3031333	-6.101851	-0.9529482
40	-6.874124	-0.804146	-2.558307	-0.5944318
50	-7.069064	-0.999947	1.956236	-0.01030037
60	-4.580957	-0.8162205	5.728294	0.5777406
70	-3.54E-01	-0.3226997	7.326192	0.9465014
80	4.007059	0.2933016	6.143451	0.95602
90	6.847407	0.7979805	2.628977	0.6026833
100	7.088839	0.9997878	-1.883318	0.02059964
110	4.639718	0.8221281	-5.680805	-0.5693026
120	4.30E-01	0.3324316	-7.322155	-0.9431273
130	-3.943567	-0.2834384	-6.1844	-0.9589905
140	-6.819966	-0.7917299	-2.699373	-0.6108714
150	-7.107862	-0.9995226	1.810199	-0.03089697
160	-4.697986	-0.8279484	5.632714	0.5608044
170	-5.05E-01	-0.3421288	7.317342	0.9396531
180	3.879655	0.2735457	6.224691	0.961859
190	6.791799	0.7853958	2.769477	0.6189939
200	7.12613	0.9991513	-1.736891	0.04119054
210	4.755757	0.8336813	-5.584021	-0.5522458
220	5.80E-01	0.3517892	-7.311752	-0.9360793
230	-3.815332	-0.263623	-6.264326	-0.9646258
240	-6.762911	-0.7789784	-2.839286	-0.6270508
250	-7.143644	-0.998674	1.663392	-0.05148069
260	-4.813022	-0.8393253	5.534739	0.5436295
270	-6.56E-01	-0.3614123	7.305387	0.9324061
280	3.750605	0.2536723	6.303295	0.9672902
290	6.733307	0.7724778	2.908801	0.6350418
300	7.160398	0.9980907	-1.589723	0.06176443

$A + B = C(a + b = 1)$. The motion takes a slow spin rotation about the symmetric z -axis of the disc. The equations of motion and their first integrals are derived and reduced to two quasilinear differential equations of the second order and one first integral [16–19]. The periodic solutions for this problem are constructed applying the periodicity conditions and assuming a large parameter [20] proportional to $1/r_0$. We used here the large parameter technique instead of the small one well-known in [21–26]. The advantage of this technique comes from the saving of the high initial energy which is given for the body to start the motion, and the solving of the problem in a new domain of the motion $F(t, \mu \rightarrow \infty, r_o \rightarrow 0)$ and under new considerations. A geometric interpretation using Euler's angles of motion of the body as a function of time is presented. This interpretation shows the orientation of the disc at any instant of time t . There are numerical considerations of the problem and the solutions using the fourth-order Runge-Kutta method which is a

TABLE 2: The values of numerical solutions in the interval $t \in [0, 300]$.

t	p_{2n}	γ_{2n}	$x_n = dp_{2n}/dt$	$y_n = d\gamma_{2n}/dt$
0	7.048539	1	-2.028946	0
10	4.521935	0.8102441	-5.774878	-0.5860706
20	0.2793855	0.3130168	-7.329234	-0.9497204
30	-4.069078	-0.3029831	-6.102208	-0.9529552
40	-6.87327	-0.8039892	-2.559511	-0.5945567
50	-7.06908	-0.9998797	1.954393	-0.01054169
60	-4.582269	-0.8163247	5.726515	0.5774587
70	-0.3566143	-0.3229907	7.325408	0.9463065
80	4.004261	0.292901	6.144369	0.9560346
90	6.845462	0.7976249	2.63166	0.6029608
100	7.088834	0.9996485	-1.879637	0.02108086
110	4.642086	0.8223141	-5.677522	-0.5687839
120	0.4337946	0.3329276	-7.320768	-0.9427877
130	-3.939007	-0.2827875	-6.185843	-0.9590072
140	-6.816898	-0.7911727	-2.703507	-0.6112964
150	-7.107798	-0.9993057	1.804683	-0.0316163
160	-4.701379	-0.8282109	5.627904	0.5600467
170	-0.5109161	-0.3428261	7.315316	0.9391646
180	3.873325	0.2726439	6.226625	0.9618729
190	6.78758	0.7846335	2.775045	0.6195633
200	7.125968	0.9988525	-1.729537	0.04214693
210	4.760143	0.8340154	-5.577667	-0.5512487
220	0.5879716	0.3526854	-7.309052	-0.9354378
230	-3.807219	-0.2624713	-6.266709	-0.9646314
240	-6.75751	-0.7780079	-2.846264	-0.6277602
250	-7.143345	-0.998288	1.654209	-0.05267154
260	-4.818369	-0.8397263	5.526817	0.5423905
270	-0.6649504	-0.3625042	7.301974	0.9316074
280	3.7407	0.2522707	6.306089	0.9672824
290	6.726693	0.7712965	2.917158	0.6358864
300	7.159923	0.9976127	-1.578707	0.063189

specialized and smooth method for finding the approximated periodic solutions of the nonlinear differential equations. Computerized programs are given as a validation of the technique and the results. These programs are carried in a closed interval of time $t \in [0, 300]$ and showed that a full agreement between the analytical solutions and the numerical ones. The agreement of the numerical results with the analytical ones proves that the accuracy (validation) of the resulted solutions and studied techniques (see Tables 1 and 2). From the above tables, we find that the errors between the analytical and the numerical solutions are very small and can be neglected. To study the behavior of the body, we investigate a geometric interpretation of the motion using Euler's angles. We obtain an arbitrary initial angle of nutation θ_0 , precession ψ_0 , and pure rotation ϕ_0 . We note also that the expressions for the Euler's angles depend on four arbitrary constants θ_0 , ψ_0 , ϕ_0 , and r_0 (where r_0 is sufficiently small). Moreover, we note that

the disc spins slowly about the minor axis of the ellipsoid of inertia (that is a case of weak oscillations is obtained). In the first approximation, the case of a pseudoregular precession about the vertical axis is attained. As an example, the case of a regular Precession of a slowly spinning Lagrange gyroscope ($A = B$, $x_0 = y_0 = 0$) is obtained as a special case of this motion. There are many applications of these results in both military and civil life. This study is important for the satellite motion which has the correspondence of inertia moments, the antennas, the navigations, the solar collectors, and aerospace dynamics.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that he has no competing interests.

Authors' Contributions

I am the individual author of the manuscript.

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