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Research Article

Design of Arbitrary Time Convergence Controller for *n*-th Order System

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A class of arbitrary time convergence controllers based on a special time-varying scaling function provide designers with a good choice to realize prescribed time stable systems. However, these controllers have the problem of conservative control parameter range and lack a uniform formula for systems of different order. Herein, the arbitrary time convergence controller is improved, the unified formula for n-th order system is given, and a more accurate control parameter range is obtained. By constructing an ingenious auxiliary function and a novel Lyapunov function, the arbitrary time convergence of the n-th order controller is proved, and the reasonable parameter selection range is obtained using the integrator backstepping and the mathematical induction method. The effectiveness and advantages of the proposed arbitrary time convergence controllers under saturated input constraints are illustrated through numerical simulations by comparative studies in unmanned aerial vehicle (UAV) formation control.

1. Introduction

During the design of controllers, under the premise of asymptotic stability, the convergence rate is an important factor that affects the application of the controllers. For example, the multimissile cooperative attack [1] and formation of unmanned aerial vehicles (UAV) swarm [2] and mobile robots [3] all require the controlled objects to reach specific states at a prescribed time. To meet the demand for the convergence rate, the studies of settling time have gone through the stages of asymptotic stabilization, finite-time stabilization, fixed time stabilization, and prescribed/arbitrary time stabilization.

The controller design firstly addresses the problem of asymptotic stabilization [4], where the state converges to the equilibrium point as time t goes to $+\infty$. Then, the finite-time stabilization [5] requires the convergence over a finite-time interval $T(x_0, \phi_0)$, where the settling time function $T(x_0, \phi_0)$ is a function of the initial condition (x_0) and the controller parameters (ϕ_0). Based on the Lyapunov differential inequality [5–7], Polyakov and Poznyak [8] performed a sign function-based controller, and Huang et al. [9] realized global finite-time stabilization for strict feedback systems. Based on the implicit Lyapunov function approach, Wang et al. [10] presented a finite-time stability analysis for a chain of integrators. However, the dependence of the settling time on the initial states of the system restricts the application of the finite-time convergence controllers. If the terminal time can be bounded by a constant which is irrespective of the initial conditions, the fixed time stabilization system is built by [11, 12], while they did not give the control parameter range. Dong et al. [13] designed fixed time convergence cooperative guidance laws for multiple missiles to simultaneously attack a maneuvering target at desired terminal angles. Wang et al. [14] developed a fixed time convergence error dynamic for a unified guidance law design, where the ratio between the settling time upper bound and the whole guidance time is a constant, which makes it convenient to set the parameters for different guidance scenarios. Although the existing approaches allow attaining convergence within desired time by properly choosing design parameters, the proper tuning of these

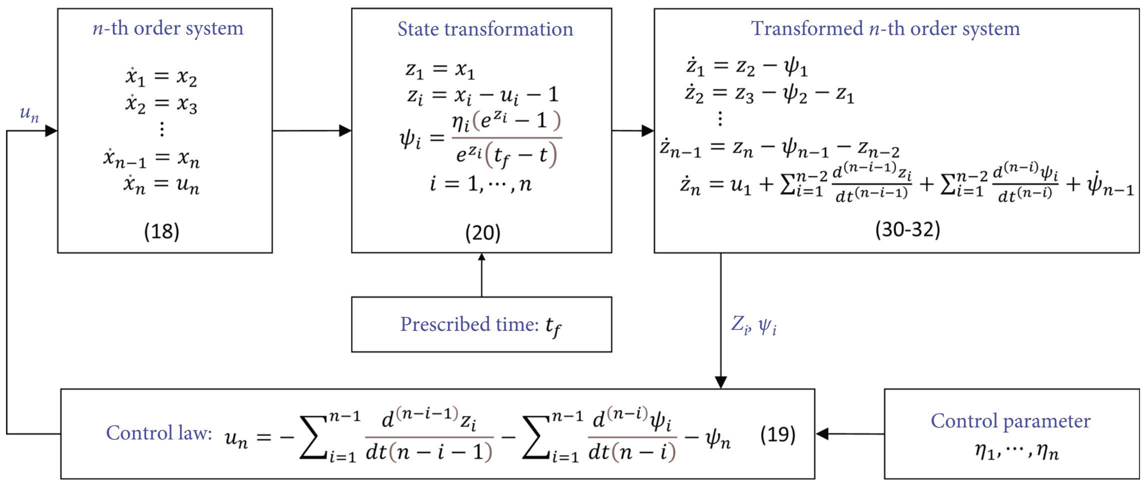


Figure 1: Control structure diagram of the proposed n-th order arbitrary time convergence controller.

parameters to attain arbitrary time convergence may not be an easy task.

The fixed time convergence controllers just partially solve the problem of desired settling time. If the upper bound of the settling time can be arbitrarily picked independent of initial conditions and any other design parameters, the controllers are called prescribed time convergence or arbitrary time convergence or having appoint time performance. Zhang et al. [15] developed appoint time controllers for quadrotors based on a new piecewise continuous funnel function. The trajectory tracking errors reach a preassigned steady state before a pregiven time, which is uncorrelated with the initial quadrotor states. Bu et al. [16–18] proposed a series of new classes of exponential decaying funnel functions which adaptively regulate the performance constraint boundary according to the sign of initial error and actuator saturation to guarantee finite-time convergence, where the convergence time can be set as needed. However, there are many parameters needed to be tuned. In [19], a prescribed time convergence controller has been proposed, but the approach also lacks simplicity due to the deployment of two different time-varying scaling functions. Pal et al. [20] proposed a novel time-varying scaling function to develop a simpler free-will arbitrary time convergence controller and applied it to the multiagent consensus control [21]. A free-will arbitrary time convergence terminal sliding mode controller [22] and prescribed time convergence controller for polytopic systems [21] were subsequently brought up. However, we find that these controllers have the problem of conservative control parameter range and lack of uniform control law forms for the system of different order, and their performance under the saturated input constraint is not clear. Based on the above analyses, the main contributions of this paper are as follows.

(1) The addressed controller eliminates the dependence of the convergence time on the initial states compared with the finite-time controllers [10]. The convergence time can be set as needed compared with the fixed time convergence controllers [13, 14]. Compared with the prescribed time/performance controllers [15–19], the addressed controller reduces the number of parameters, which is equal to the order of the system and the parameter tuning is much easier

(2) The proposed arbitrary time convergence controller has a general/consistent formula for n-th order systems, which is a deep modification of the one given in [20], where the general formula for n-th order system is not given and even the given control laws for the first- and second-order systems lack consistency

(3) A less conservative control parameter selection range is deduced by introducing an ingenious auxiliary function and a novel Lyapunov function, which extends the existing arbitrary time convergence controllers [20–23] where the Lyapunov inequalities are overly scaled during the proof of the arbitrary time stability, while such overscaling is avoided in this paper by ingeniously introducing the auxiliary function

The remainder of this paper is organized as follows. Section 2 gives the relevant definitions and theorems. In Section 3, the uniform formula and parameter range of the n-th order arbitrary time convergence controller are proposed and proved. Section 4 provides a design example for UAV formation control under the saturated input constraint, and the effectiveness and advantages are illustrated by comparative studies. Section 5 summarizes the full text.

2. Preliminaries

Consider the nonautonomous nonlinear system

$$\dot{x} = f(t, x; \phi), x(t_0) = x_0, \quad (1)$$

where x is the system state, ϕ represents the constant parameters of the system, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is nonlinear

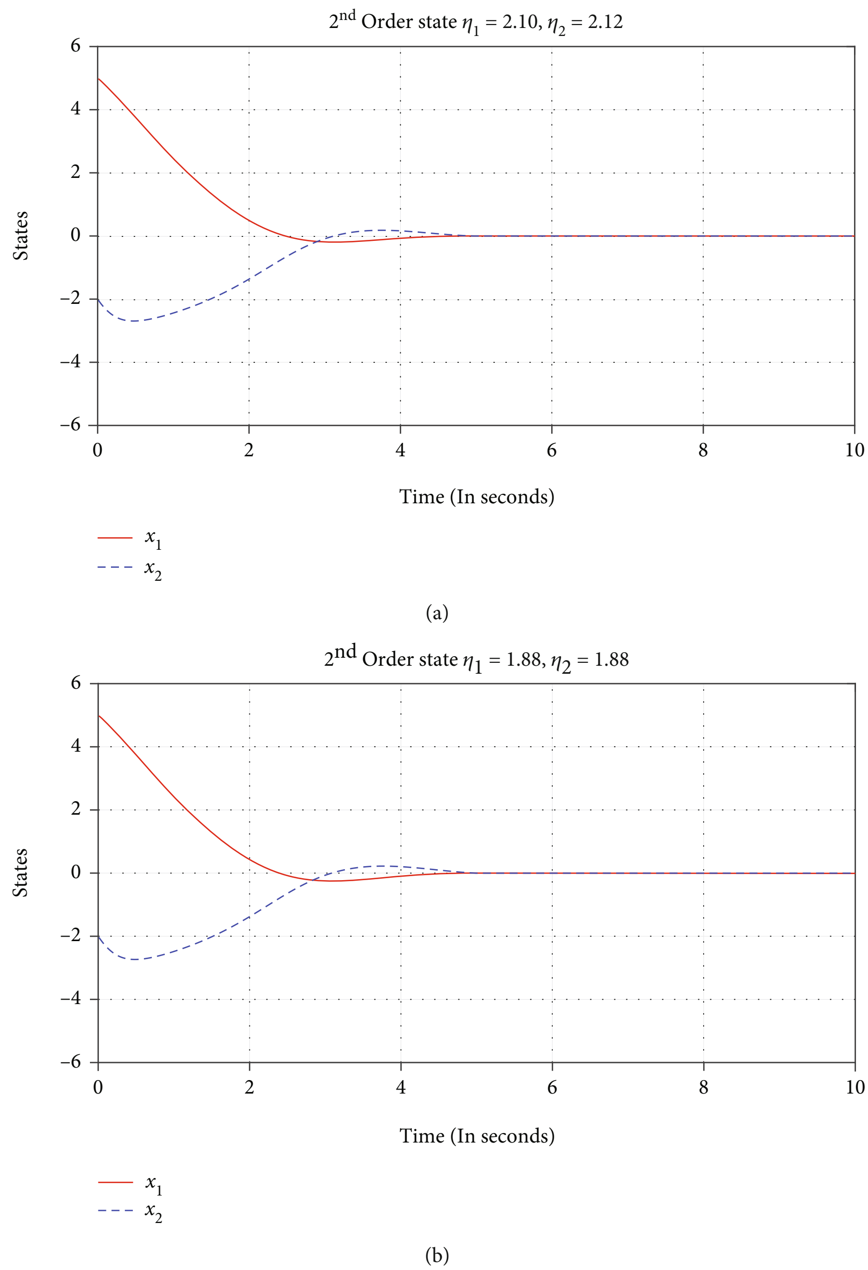


Figure 2: The states curve of the second-order system with parameters (a) $\eta_1 = \eta_2 = 2.12$ and (b) $\eta_1 = \eta_2 = 1.88$.

function, $x = 0$ is one equilibrium point, and $t_0 \in \mathbb{R}_{\geq 0}$ is the initial time.

Definition 1 (global finite-time stability [5]). The origin of system (1) is said to be globally finite-time stable if

(1) it is globally asymptotically stable

(2) any solution $x(t; t_0, x_0)$ of (1) converges to the origin at some finite time, i.e., $\forall t \geq t_0 + T(t_0, x_0)$, s.t. $x(t; t_0, x_0) = 0$, where $T : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is the settling time function

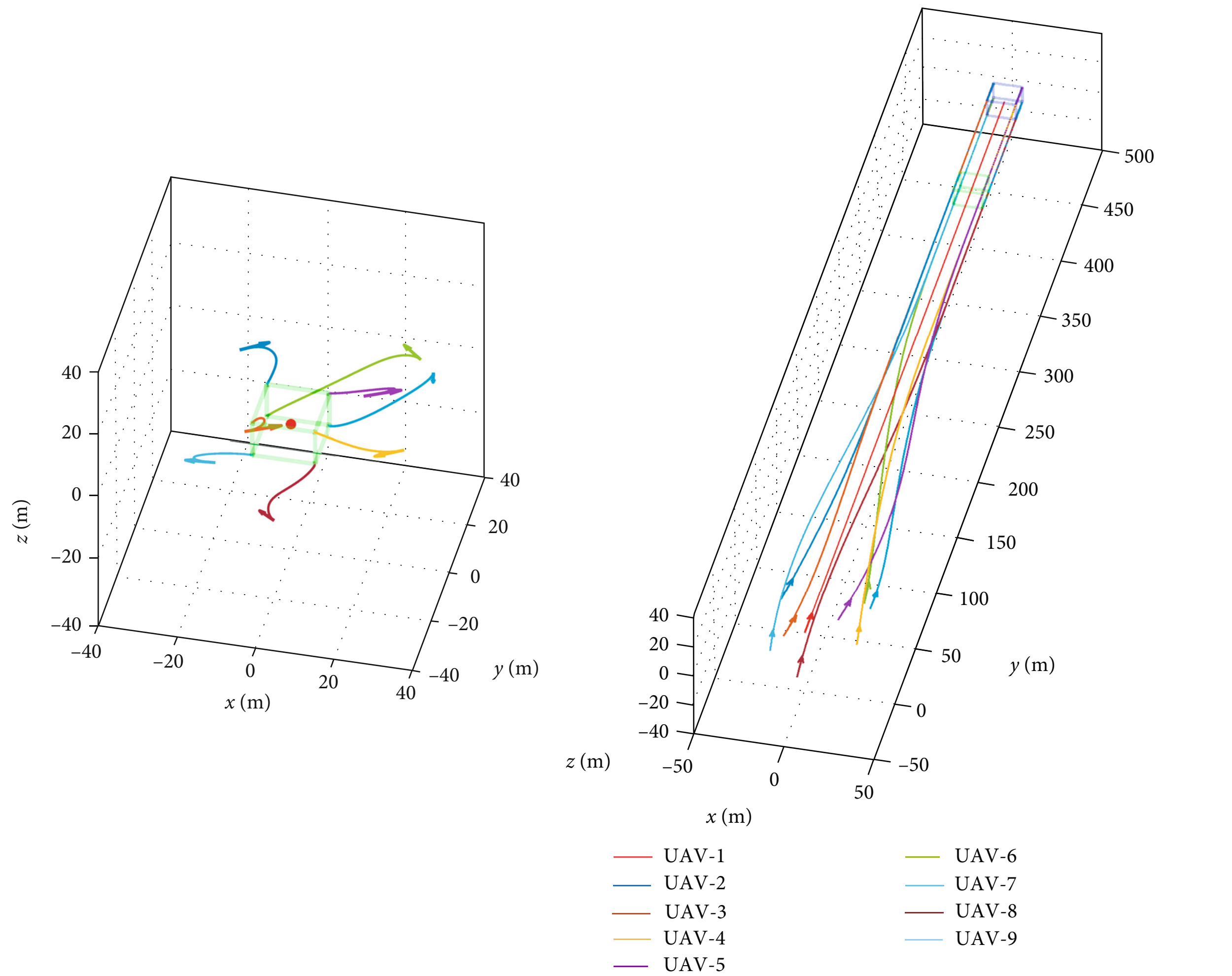
Definition 2 (fixed time stability [11]). The origin of system (1) is called fixed time stable if

(1) it is globally finite-time stable

(2) the settling time function is bounded, i.e., $\exists T_{\max} > 0$, s.t. $\forall x_0 \in \mathbb{R}^n$ and $\forall t_0 \in \mathbb{R}_{\geq 0}$, $T(t_0, x_0) \leq T_{\max}$

Definition 3 (free-will arbitrary time stability [20]). The origin of system (1) is called free-will arbitrary time stable if

(1) it is fixed time stable



(a) (b)

Figure 3: The trajectories in the (a) relative and (b) ground coordinates system of UAVs during rendezvous in Case 2. The arrows illustrate the initial velocities error $\dot{\mathbf{e}}_{ij,0}$ in (a) and illustrate initial velocities $\mathbf{v}_{j,0}$ in (b). The green and blue cubes depict the expected formation at the prescribed time and the terminal time, respectively. Note: the arrows in (a) are lengthened three times for clear visualization and the length of arrows in (b) equals to the speed.

(2) $\exists T_a > 0$ can be arbitrarily chosen in advance, $\forall x_0 \in \mathbb{R}^n$ and $\forall t_0 \in \mathbb{R}_{\geq 0}$, the settling time function $T(t_0, x_0) \leq T_a$

Remark 4. Theoretically, in free-will arbitrary time stable system, the settling time T_a can be arbitrarily chosen. However, it should be properly selected according to the specific application scenario because fast convergence usually requires large control input. Nevertheless, the arbitrary time stable system is irrelevant to other system parameters compared with the fixed time stable system.

Theorem 5 (Lyapunov stability criterion for free-will arbitrary time stability [20]). *Considering system (1), let $\mathscr{D} \in \mathbb{R}^n$ be a domain containing the equilibrium point $x = 0$ and let $\alpha_1(x)$ and $\alpha_2(x)$ be two continuous positive definite functions on $\mathscr{D}$. Assume that there exists a real-valued continuously differentiable function $V : [t_0, t_f] \times \mathscr{D} \longrightarrow \mathbb{R}_{\geq 0}$ and real number $\eta \geq 1$, s.t. $\forall t \in [t_0, t_f]$; if*

$$
\begin{aligned}
&\alpha_1(x) \leq V(t, x) \leq \alpha_2(x), \forall x \in \mathscr{D} \setminus \{0\}, \\
&V(t, 0) = 0, \\
&\dot{V} \leq \frac{-\eta\left(e^V - 1\right)}{e^V\left(t_f - t\right)}, \forall V \neq 0,
\end{aligned}
\tag{2}
$$

then the equilibrium point $x = 0$ is free-will arbitrary time stable; i.e., the convergence time of system (1) is $T \leq t_f - t_0$.

3. Main Results

Before giving the uniform formula of the n-th order arbitrary time convergence controller, we construct an auxiliary function and analyse its properties as a lemma and two

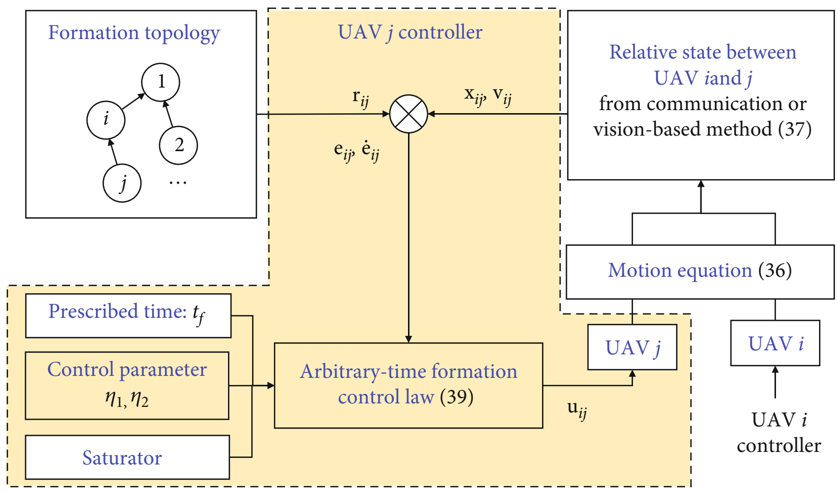


Figure 4: The diagram of the proposed arbitrary time convergence formation controller.

corollaries, which are indispensable for the stability proof and parameter range deduction of our control law.

3.1. Constructing the Auxiliary Function

Lemma 6. *Consider the function*

$$f(x) = x(1 - e^{-x}) \quad (3)$$

in the real domain. The following inequality always holds.

$$F(x_1, x_2) = f(x_1) + f(x_2) - f\left(\sqrt{x_1^2 + x_2^2}\right) > 0, \forall x_1, x_2 > 0. \quad (4)$$

Proof. Because

$$f'(x) = (1 - e^{-x}) + xe^{-x} > 0, \forall x > 0, \quad (5)$$

and $f(0) = 0$, and $f'(0) = 0$, $f(x) > 0$ is strictly monotonically increasing in $x \in (0, +\infty)$. Let $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \tan^{-1}(x_2/x_1)$; then, $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, $r \in (0, +\infty)$, $\theta \in (0, \pi/2)$. Substituting x_1, x_2 by r, θ, the function $F(x_1, x_2)$ becomes

$$\begin{aligned} G(r,\theta) &= f(r\cos\theta) + f(r\sin\theta) - f(r) \\ &= r\left[\cos\theta\left(1 - e^{-r\cos\theta}\right) + \sin\theta\left(1 - e^{-r\sin\theta}\right) - (1 - e^{-r})\right]. \end{aligned} \quad (6)$$

Let $w = e^{-r}$ so that $w \in (0, 1)$. Define the function

$$\begin{aligned} H(w,\theta) &\triangleq \frac{G(r,\theta)}{r} = \cos\theta\left(1 - w^{\cos\theta}\right) + \sin\theta\left(1 - w^{\sin\theta}\right) \\ &\quad - (1 - w) = (\cos\theta + \sin\theta - 1) \\ &\quad + \left(w - \cos\theta w^{\cos\theta} - \sin\theta w^{\sin\theta}\right). \end{aligned} \quad (7)$$

Hence, the partial derivative of function H with respect to w is

$$\frac{\partial H}{\partial w} = 1 - \cos^2\theta w^{(\cos\theta - 1)} - \sin^2\theta w^{(\sin\theta - 1)}. \quad (8)$$

Noticing that $\forall w \in (0, 1), \theta \in (0, \pi/2) \Rightarrow \cos\theta - 1 \in (-1, 0), \sin\theta - 1 \in (-1, 0)$, it is easy to get $w^{\cos\theta - 1} > 1$ and $w^{\sin\theta - 1} > 1$. Thus,

$$\frac{\partial H}{\partial w} < 1 - \left(\cos^2\theta + \sin^2\theta\right) = 0, \quad (9)$$

which means the function H strictly monotonically decreases with respect to variable w in the domain $(w, \theta) \in (0, 1) \times (0, \pi/2)$. Then, the minimum value of function H is

$$H(w,\theta)|_{w=1} = (\cos\theta + \sin\theta - 1) + (1 - \cos\theta - \sin\theta) = 0, \quad (10)$$

so that $H(w,\theta)|_{0<w<1} > H(w,\theta)|_{w=1} = 0, \forall\theta \in (0, \pi/2)$.

According to (7),

$$G(r,\theta) = rH(w,\theta)|_{0<w<1} > 0, \forall w \in (0, 1), \theta \in \left(0, \frac{\pi}{2}\right), \quad (11)$$

which means inequality (4) constantly holds. □

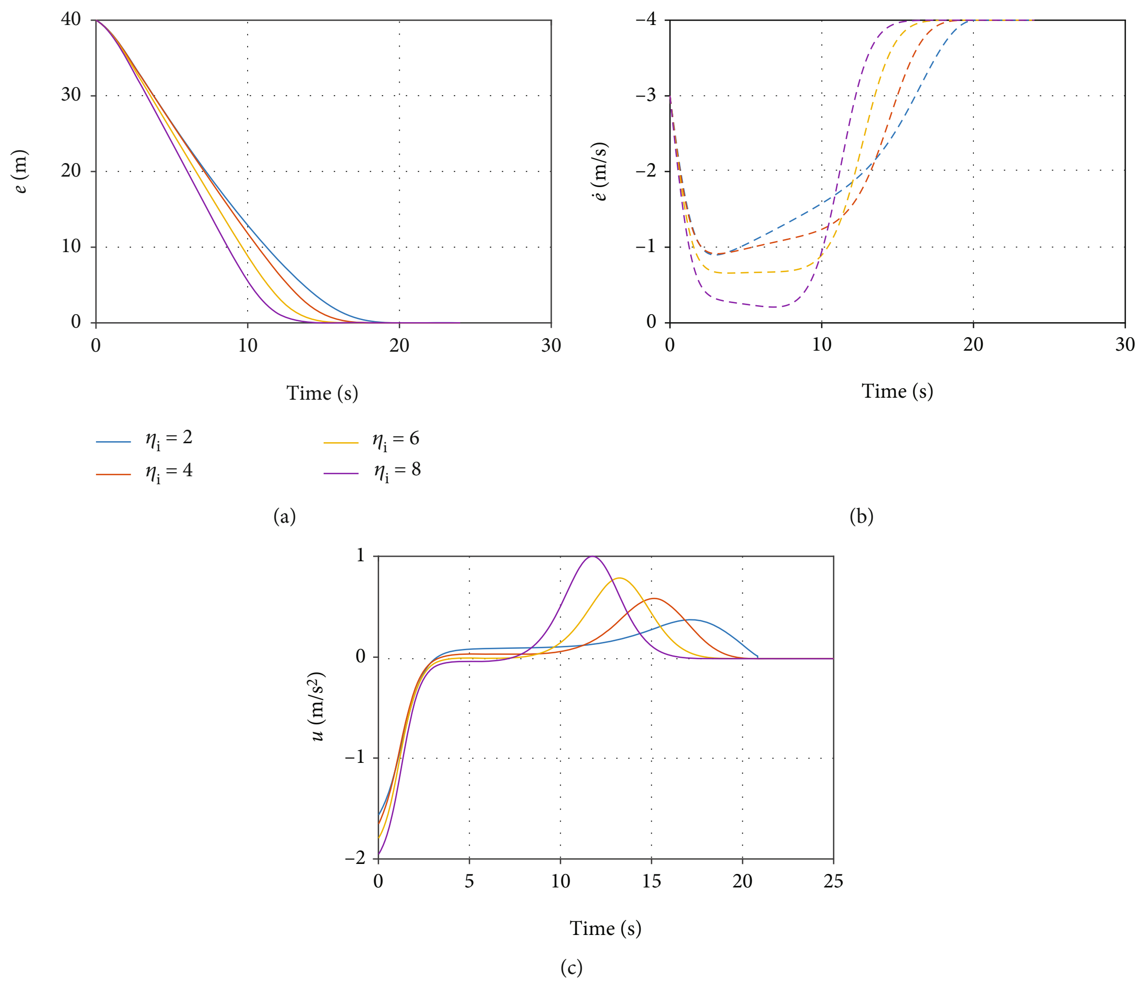


FIGURE 5: The state error (a) e, (b) $\dot{e}$, and (c) control output u curves of the second-order system with $e_0 = 40$, $\dot{e}_0 = -1$, and $t_f = 20$ s. The performance of $\eta_1 = \eta_2 = 2, 4, 6, 8$ are compared.

Corollary 7. *Consider function (3) in the real domain. The following inequality always holds.*

$$\eta_1 f(x_1) + \eta_2 f(x_2) \geq \eta f\left(\sqrt{x_1^2 + x_2^2}\right) > 0, \forall x_1, x_2 > 0, \forall \eta_1, \eta_2 > 0, \tag{12}$$

where $\eta = \min\{\eta_1, \eta_2\}$.

Proof. According to Lemma 6, $f(x_1) + f(x_2) > f(\sqrt{x_1^2 + x_2^2})$. As mentioned in (5), $f(x) > 0, \forall x > 0$; therefore,

$$\eta_1 f(x_1) + \eta_2 f(x_2) \geq \eta[f(x_1) + f(x_2)] > \eta f\left(\sqrt{x_1^2 + x_2^2}\right) > 0. \tag{13}$$

□

Corollary 8. *Consider function (3) in the real domain.* $\forall n \in \mathbb{N}$ *and* $n \geq 2$*, the following inequalities always hold.*

$$F(x_1, \cdots, x_n) = \sum_{i=1}^{n} f(x_i) - f\left(\sqrt{\sum_{i=1}^{n} f(x_i^2)}\right) > 0, \forall x_i > 0, i = 1, \cdots, n, \tag{14}$$

$$\sum_{i=1}^{n} \eta_i f(x_i) \geq \eta \sum_{i=1}^{n} f(x_i) \geq \eta f\left(\sqrt{\sum_{i=1}^{n} f(x_i^2)}\right) > 0, \forall x_i > 0, i = 1, \cdots, n, \tag{15}$$

where $\eta = \min\{\eta_1, \cdots, \eta_n\}$.

Proof. The proof of (14) can be established by mathematical induction. Obviously, when $n = 2$, inequality (14) is correct, i.e., Lemma 6. Suppose $F(x_1, \cdots, x_{n-1}) > 0, \forall x_i > 0, i = 1, \cdots,$

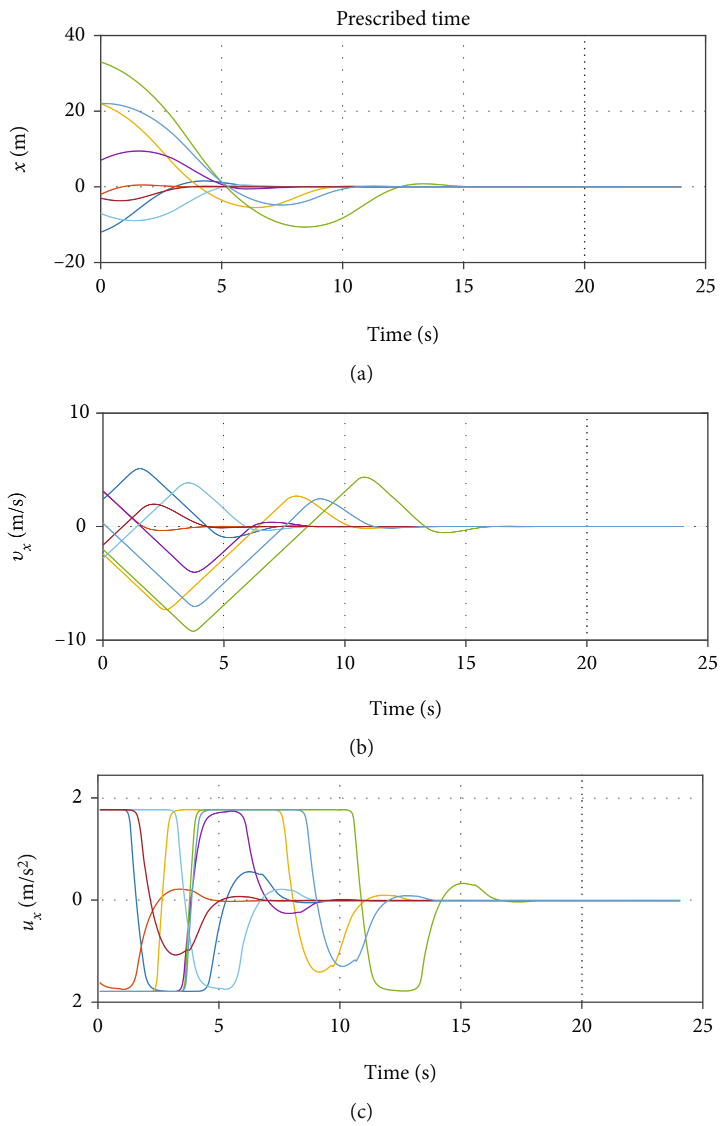


FIGURE 6: The x-axis (a) position error, (b) velocity error, and (c) control input curves of the finite-time convergence controller [25].

$n-1$, which means

$$f(x_1)+\cdots+f(x_{n-1}) > f\left(\sqrt{x_1^2+\cdots+x_{n-1}^2}\right). \tag{16}$$

Adding $f(x_n)$ to both sides of the above equation and let $y_{n-1}=\sqrt{x_1^2+\cdots+x_{n-1}^2}>0$, it comes

$$f(x_1)+\cdots+f(x_{n-1})+f(x_n) > f(y_{n-1})+f(x_n). \tag{17}$$

According to Lemma 6, $f(y_{n-1})+f(x_n)>f(\sqrt{y_{n-1}^2+x_{n-1}^2}) = f(\sqrt{x_1^2+\cdots+x_{n-1}^2})$.

Therefore, equation (14) is proved. Then, the process of proving (15) is the same as that of Corollary 7. □

3.2. n-th Order Arbitrary Time Convergence Controller. Consider the n-th order system

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \cdots, \dot{x}_{n-1} = x_n, \dot{x}_n = u_n, \tag{18}$$

where $x_i \in \mathbb{R}$, u_n is the control input, and $\forall i = 1, \cdots, n, x_i = 0$ is one equilibrium point.

The block diagram of the proposed n-th order arbitrary time convergence controller is depicted in Figure 1. By taking the prescribed time t_f as variable of time-varying scaling function ψ_i, the original n-th order system with states x_i, $i = 1, \cdots, n$, can be transformed into a new n-th order system with states z_i. Then, the control law (Theorem 9) with parameters η_i is designed to ensure the transformed states z_i converge to zeros before the prescribed time t_f and thus the original states x_i also converge to zeros in the same manner, which will be rigorously proved by backstepping method, Lyapunov differential inequality, and mathematical induction.

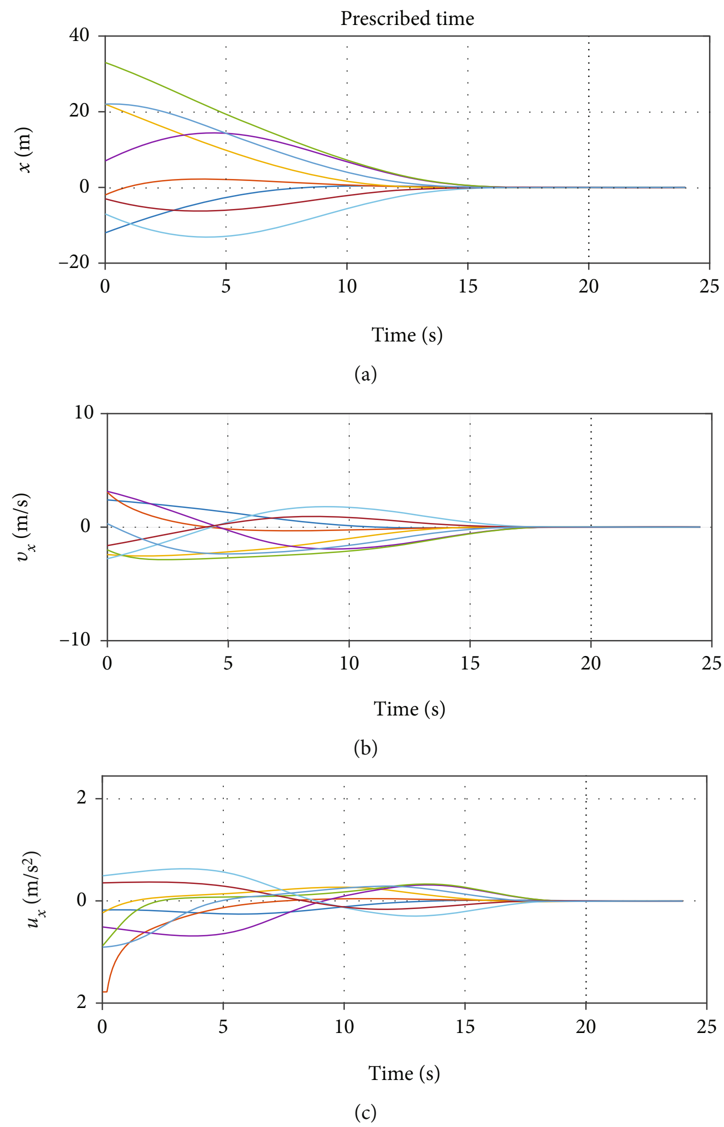


FIGURE 7: The x-axis (a) position error, (b) velocity error, and (c) control input curves of the proposed controller with the prescribed time $t_f = 20$ s.

Theorem 9. *With the controller*

$$u_n = \begin{cases} -\sum_{i=1}^{n-1} \frac{d^{(n-i-1)} z_i}{dt(n-i-1)} - \sum_{i=1}^{n-1} \frac{d^{(n-i)} \psi_i}{dt(n-i)} - \psi_n, & \text{if } t_0 \le t < t_f, \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

where $z_1 = x_1$, $z_i = x_i - u_i - 1$, *and*

$$\psi_i = \frac{\eta_i (e^{z_i} - 1)}{e^{z_i} (t_f - t)}. \tag{20}$$

If $\min \{\eta_1, \cdots, \eta_n\} \ge 1$, *system (18) is free-will arbitrary time stable.*

Proof. The convergence within arbitrary time will be proved using mathematical induction and backstepping as chosen by [20]. First, the free-will arbitrary time stable of the control law in the first- and second-order system is proved. Then, we will prove that if the n-th $(n \ge 2)$ order system is free-will arbitrary time stable, the $(n+1)$-th system also satisfies free-will arbitrary time stability. □

Step 1 (first-order system). For the first-order system $\dot{x}_1 = u_1$, the controller according to (19) is

$$u_1 = \begin{cases} -\frac{\eta (e^{x_1} - 1)}{e^{x_1} (t_f - t)}, & \text{if } t_0 \le t < t_f, \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

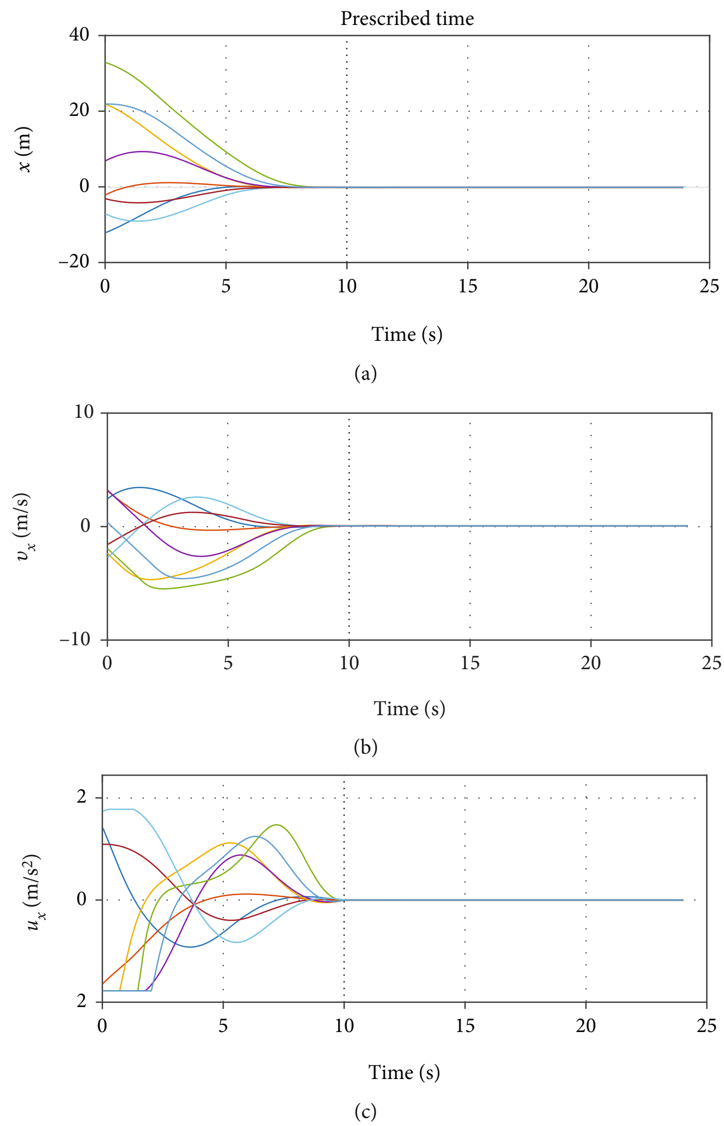


Figure 8: The x-axis (a) position error, (b) velocity error, and (c) control input curves of the proposed controller with the prescribed time $t_f = 10\,\mathrm{s}$.

Select the Lyapunov function different from [20] as follows.

$$V_1 = \sqrt{x_1^2} = |x_1| \geq 0. \tag{22}$$

The time derivative of the Lyapunov function is

$$\begin{aligned} \dot{V}_1 &= \frac{x_1 \dot{x}_1}{\sqrt{x_1^2}} = \frac{x_1 \dot{x}_1}{V_1} = -\frac{1}{V_1} \frac{\eta_1 x_1 \left(e^{x_1} - 1\right)}{e^{x_1}\left(t_f - t\right)} \\ &\leq -\frac{1}{V_1} \frac{\eta_1 |x_1| \left(e^{|x_1|} - 1\right)}{e^{|x_1|}\left(t_f - t\right)} = -\frac{\eta_1 \left(e^{V_1} - 1\right)}{e^{V_1}\left(t_f - t\right)}. \end{aligned} \tag{23}$$

According to the condition of Theorem 5, as long as $\eta_1 \geq 1$, controller (23) makes the first-order system be free-will arbitrary time stable.

Step 2 (second-order system). For the second-order system $\dot{x}_1 = x_2, \dot{x}_2 = u_2$, the controller according to (19) is

$$u_2 = \begin{cases} -x_1 - \dot{\psi}_1 - \dfrac{\eta\left(e^{z_2} - 1\right)}{e^{z_2}\left(t_f - t\right)}, & \text{if } t_0 \leq t < t_f, \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

where $z_2 = x_2 - u_1$. Similar to [20], according to the integrator backstepping [24], when z_2 closes to zero, the rate of x_1 closes to (21).

The transformed system is

$$\begin{aligned} \dot{z}_1 &= z_2 - \psi_1, \\ \dot{z}_2 &= u_2 + \dot{\psi}_1. \end{aligned} \tag{25}$$

Select the Lyapunov function as

$$V_2 = \sqrt{x_1^2 + z_2^2} \ge 0, \tag{26}$$

and its time derivative is

$$\begin{aligned}\dot{V}_2 &= \frac{x_1\dot{x}_1 + z_2\dot{z}_2}{\sqrt{x_1^2 + z_2^2}} = \frac{1}{\sqrt{V_2}}(x_1\dot{x}_1 + z_2\dot{z}_2)\\ &= \frac{1}{V_2}[x_1(z_2 - \psi_1) + z_2(u_2 + \dot{\psi}_1)]\\ &= \frac{1}{V_2}[-x_1\psi_1 + z_2(x_1 + u_2 + \dot{\psi}_1)].\end{aligned} \tag{27}$$

The term $-x_1 - \dot{\psi}_1$ in (24) compensates the extra parts in (27) and the term $-\eta(e^{z_2} - 1)/e^{z_2}(t_f - t)$ will make z_2 converge to zero. Substituting the control u_2 into the derivative (27), one can get

$$\begin{aligned}\dot{V}_2 &= \frac{1}{V_2}\left[-x_1\frac{\eta_1(e^{x_1} - 1)}{e^{x_1}(t_f - t)} - z_2\frac{\eta_2(e^{z_2} - 1)}{e^{z_2}(t_f - t)}\right]\\ &= -\frac{1}{V_2(t_f - t)}[\eta_1 x_1(1 - e^{-x_1}) + \eta_2 z_2(1 - e^{-z_2})]\\ &\le -\frac{1}{V_2(t_f - t)}\left[\eta_1|x_1|\left(1 - e^{-|x_1|}\right) + \eta_2|z_2|\left(1 - e^{-|z_2|}\right)\right].\end{aligned} \tag{28}$$

Referring to Corollary 7,

$$\begin{aligned}\dot{V}_2 &\le -\frac{1}{V_2(t_f - t)}[\eta_1 f(|x_1|) + \eta_2 f(|z_2|)]\\ &\le -\frac{\eta f(V_2)}{V_2(t_f - t)} = -\frac{\eta(e^{V_2} - 1)}{e^{V_2(t_f - t)}},\end{aligned} \tag{29}$$

where $\eta = \min\{\eta_1, \eta_2\}$. Therefore, according to Theorem 5, as long as $\eta_1, \eta_2 \ge 1$, controller (24) makes the second-order system be free-will arbitrary time stable.

Step 3 (n-th order to $(n+1)$-th order system). Supposed that the control law (19) for the n-th order system (18) satisfies free-will arbitrary time stability; in the $(n+1)$-th order system, let the desired state of $x_{n+1,d}$ be the controller u_n of the n-th order system according to the integrator backstepping. Define another new variable

$$z_{n+1} = x_{n+1} - x_{n+1,d} = x_{n+1} + \sum_{i=1}^{n-1}\frac{d^{(n-i-1)}z_i}{dt^{(n-i-1)}} + \sum_{i=1}^{n-1}\frac{d^{(n-i)}\psi_i}{dt^{(n-i)}} + \psi_n. \tag{30}$$

Therefore, for $k = 2, \cdots, n$, the time derivatives of all newly defined state variables z_k are

$$\begin{aligned}\dot{z}_k &= \frac{dz_k}{dt} = \frac{d}{dt}\left[x_k + \sum_{i=1}^{k-2}\frac{d^{(k-i-2)}z_i}{dt^{(k-i-2)}} + \sum_{i=1}^{k-2}\frac{d^{(k-i-1)}\psi_i}{dt^{(k-i-1)}} + \psi_{k-1}\right]\\ &= \dot{x}_k + \sum_{i=1}^{k-2}\frac{d^{(k-i-1)}z_i}{dt^{(k-i-1)}} + \sum_{i=1}^{k-2}\frac{d^{(k-i)}\psi_i}{dt^{(k-i)}} + \dot{\psi}_{k-1}\\ &= x_{k+1} + \sum_{i=1}^{k-1}\frac{d^{(k-i)}\psi_i}{dt^{(k-i)}} + \sum_{i=1}^{k-1}\frac{d^{(k-i-1)}z_i}{dt^{(k-i-1)}} - z_{k-1}\\ &= z_{k+1} - \psi_k - z_{k-1}.\end{aligned} \tag{31}$$

The transformed $(n+1)$-th order system becomes

$$\begin{aligned}\dot{z}_1 &= z_2 - \psi_1,\\ \dot{z}_2 &= z_3 - \psi_2 - z_1,\\ \dot{z}_3 &= z_4 - \psi_3 - z_2,\\ &\vdots\\ \dot{z}_n &= z_{n+1} - \psi_n - z_{n-1},\\ \dot{z}_{n+1} &= u_{n+1} + \sum_{i=1}^{n-1}\frac{d^{(n-i)}z_i}{dt^{(n-i)}} + \sum_{i=1}^{n-1}\frac{d^{(n-i+1)}\psi_i}{dt^{(n-i+1)}} + \dot{\psi}_n.\end{aligned} \tag{32}$$

Select the Lyapunov function

$$V_{n+1} = \sqrt{\sum_{i=1}^{n+1} z_i^2} \ge 0, \tag{33}$$

and its time derivative is

$$\begin{aligned}\dot{V}_{n+1} &= \frac{1}{V_{n+1}}\sum_{i=1}^{n} x_i\dot{x}_i\\ &= \frac{1}{V_{n+1}}\left[x_1(z_2 - \psi_1) + \sum_{i=2}^{n} z_i(z_{i+1} - \psi_i - z_{i-1})\right.\\ &\quad\left. + z_{n+1}\left(u_{n+1} + \sum_{i=1}^{n-1}\frac{d^{(n-i)}z_i}{dt^{(n-i)}} + \sum_{i=1}^{n-1}\frac{d^{(n-i+1)}\psi_i}{dt^{(n-i+1)}} + \dot{\psi}_n\right)\right]\\ &= \frac{1}{V_{n+1}}\left[-\sum_{i=1}^{n} z_i\psi_i + z_{n+1}\left(z_n + u_{n+1} + \sum_{i=1}^{n-1}\frac{d^{(n-i)}z_i}{dt^{(n-i)}}\right.\right.\\ &\quad\left.\left. + \sum_{i=1}^{n-1}\frac{d^{(n-i+1)}\psi_i}{dt^{(n-i+1)}} + \dot{\psi}_n\right)\right] = \frac{1}{V_{n+1}}\sum_{i=1}^{n+1} z_i\psi_i\\ &= -\frac{1}{V_{n+1}(t_f - t)}\sum_{i=1}^{n}\eta_i z_i(1 - e^{-z_i})\\ &\le -\frac{1}{V_{n+1}(t_f - t)}\sum_{i=1}^{n}\eta_i|z_i|\left(1 - e^{-|z_i|}\right).\end{aligned} \tag{34}$$

According to Corollary 8,

$$\dot{V}_{n+1} = -\frac{1}{V_{n+1}(t_f - t)} \sum_{i=1}^{n} \eta_i f(|z_i|) \le -\frac{\eta f(V_{n+1})}{V_{n+1}(t_f - t)} = -\frac{\eta(e^{V_{n+1}} - 1)}{e^{V_{n+1}}(t_f - t)}, \tag{35}$$

where $\eta = \min\{\eta_1, \cdots, \eta_{n+1}\}$. Therefore, according to Theorem 5, if $\eta_1, \cdots, \eta_{n+1} \ge 1$, the $(n+1)$-th order system is free-will arbitrary time stable.

To sum up, Theorem 9 is established.

Remark 10. The number of the parameters is equal to the order of the system. For example, our method only needs two parameters η_1 and η_2 for the second-order system which is less than some existing methods [25]. The simplicity is exactly one of the advantages of the proposed method.

Remark 11. When $n = 1, 2, 3$, the controllers degrade into those of [20] which requires a more conservative parameter range that $\eta_1 \ge 1, \eta_2 \ge 2$ for the second-order system and $\eta_1 \ge 1, \eta_2 \ge 2, \eta_3 \ge 3$ for third-order system. In contrast, our controllers indicate a less conservative parameter range that $\min\{\eta_1, \cdots, \eta_n\} \ge 1$. Taking the second-order controller as example, using the same simulation as [20] that the initial state is $x_0 = [5, 2]^\top$, $t_0 = 0$ and $t_f = 5$. Figures 2(a) and 2(b) illustrate the state curves under $\eta_1 = \eta_2 = 2.12$ and $\eta_1 = \eta_2 = 1.88$, respectively. When $\eta_1 = \eta_2 = 2.12$, the requirements of [20] are satisfied and the system converges to the equilibrium point before the prescribed time. However, although the $\eta_1 = \eta_2 = 1.88$ does not meet the convergence condition of [20], it also allows prescribed time convergence according to our less conservative parameter range. The mathematical explanation is that [20] overly scales the Lyapunov inequality during the proof of arbitrary time stability while we avoid such overscaling by ingeniously introducing the auxiliary function.

4. Design Example for UAV Formation Control

In the rendezvous of UAV formation control, due to the difficulty to accomplish synchronous take-off, the UAVs usually have unaligned initial velocities and irregular initial position distribution. Thus, during the design of formation controller, the initial states should be carefully handled without the prescribed time convergence controller, which undoubtedly increases the complexity of the design as the number of UAVs increases. Thus, the arbitrary time convergence controller is very applicable. In this section, three formation control cases will be given to illustrate the effectiveness and advantages of the proposed arbitrary time convergence controller.

Assume that there are N UAVs with directed graph topology. In the ground coordinate system, the kinematics of each UAV is

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = \mathbf{a}_i, \end{cases} \tag{36}$$

where $\mathbf{x}_i, \mathbf{v}_i, \mathbf{a}_i$ are the position, velocity, and acceleration of UAV $i, i = 1, 2, \cdots, N$. Without loss of generality, taking UAV i and UAV j as the leader and the follower, the relative error kinematics is established as follows.

$$\begin{cases} \dot{\mathbf{e}}_{ij} = \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i = \mathbf{v}_j - \mathbf{v}_i, \\ \ddot{\mathbf{e}}_{ij} = \dot{\mathbf{v}}_j - \dot{\mathbf{v}}_i = \mathbf{a}_j - \mathbf{a}_i \triangleq \mathbf{u}_j, \end{cases} \tag{37}$$

where $\mathbf{e}_{ij}(t) = \mathbf{x}_j(t) - \mathbf{x}_i(t) - \mathbf{r}_{ij}$ is the relative position error and $\mathbf{r}_{ij}$ is the expected relative position. Assume that the relative position and their change rates can be observed by wireless communication or vision-based method. The rendezvous of nine UAVs is simulated, where the initial positions and velocities are randomly generated as shown in Figure 3. The initial error states of UAV j is $\mathbf{e}_{ij} = \mathbf{e}_{ij,0}$ and $\dot{\mathbf{e}}_{ij} = \dot{\mathbf{e}}_{ij,0}$. The expected formation is a cube with a side length $L = 20$ m, where UAV 1 is in the center and the remaining UAVs are located in the vertices. The UAV1 flies in constant velocity $[0, 20, 0]^\top$, and the remaining UAVs use the same controllers. The initial time is t_0 and the prescribed rendezvous time is t_f.

The structure of the formation controller is shown in Figure 4. Given the formation topology, the controllers take the state errors $\mathbf{e}_{ij}, \dot{\mathbf{e}}_{ij}$ as input and generate acceleration command $\mathbf{u}_{ij}$, where the parameters for all UAVs can be simply set the same thanks to the arbitrary time convergence controller's irrelevance to the initial state or the control parameters. To show the advantages of the arbitrary time convergence controller over the finite-time convergence controller and prove that the convergence time can be set arbitrarily, three cases of simulation are provided with the following three controllers.

Case 1. A second-order finite-time convergence formation controller with input saturation is assumed [25]. We rewrite controller of [25] in error dynamics

$$u_{ij} = -\overline{\mathrm{sat}}_{(Q,M)}\left[k_1 \operatorname{sgn}(\mathbf{e}_{ij})|\mathbf{e}_{ij}|^{\gamma_1} + k_2 \operatorname{sgn}(\dot{\mathbf{e}}_{ij})|\dot{\mathbf{e}}_{ij}|^{\gamma_2}\right], \tag{38}$$

where

$$\overline{\mathrm{sat}}_{Q,M}(x) = \begin{cases} x, & |x| \le Q, \\ -(M-Q)e^{-(x-Q)/(M-Q)} + M, & x > Q, \\ (M-Q)e^{(x+Q)/(M-Q)} - M, & x < -Q, \end{cases} \tag{39}$$

and $0 < Q < M < \sigma, k_1 > 0, k_2 > 0, 0 < \gamma_1 < 1, \gamma_2 = 2\gamma_1/1 + \gamma_1$. The upper bound of the control input is M. For the sake of fairness, we choose the parameters same as the authors [25], i.e., $Q = 1.5, M = 2, k_1 = 2, k_2 = 2, \gamma_1 = 0.8, \gamma_2 = 8/9$, which means the saturated control input is $u_{\max} = 2\mathrm{m/s}^2$.

Case 2. The proposed controller with convergence time $t_f = 20$ s is given. According to (18), during $t \in [t_0, t_f)$, the formation controller of UAV j can be designed as

$$\mathbf{u}_{ij} = \text{sat}_\text{M}\left(\text{sgn}\left(\mathbf{e}_{ij,0}\right)\left[-\mathbf{y}_{ij} - \frac{\eta_1}{t_{go}^2}\left(1 - e^{-\mathbf{y}_{ij}}\right) - \frac{\eta_1}{t_{go}}\dot{\mathbf{y}}_{ij}e^{-\mathbf{y}_{ij}} - \frac{\eta_2}{t_{go}}\left(1 - \exp\left(-\dot{\mathbf{y}}_{ij} - \frac{\eta_1}{t_{go}}\left(1 - e^{-\mathbf{y}_{ij}}\right)\right)\right)\right]\right), \tag{40}$$

where $\mathbf{y}_{ij} = \text{sgn}\,(\mathbf{e}_{ij,0})\mathbf{e}$ is the transformed error state to ensure symmetrical performance, $t_{go} = t_f - t$ is the residual settling time, and $\text{sat}_\text{M}(x) = \text{sgn}\,(x)\min\{|x|, M\}$ imposes input constraint. When the saturated input constraints are considered, there should be a minimum convergence time for any controllers as well as the arbitrary time convergence controller. As shown in Figure 5, the proposed method is not sensitive to parameters thanks to the adaptive manner of the time-varying scaling function. In general, larger parameters mean quicker convergence but larger control output. The parameters are set to $\eta_1 = \eta_2 = 3$.

Case 3. The proposed controller with convergence time $t_f = 10$ s is given. Benefiting from the property that the convergence time of the arbitrary time convergence controller is not dependent on the initial state or the control parameters, the only changes of Case 3 to Case 2 are the convergence time, i.e., $t_f = 10$ s. The parameters are still $\eta_1 = \eta_2 = 3$.

Due to space limitations, only the x-axis position errors, velocity errors and control input curves in the three cases are shown in Figures 6–8, respectively. In Case 1, although the state errors of finite-time convergence controller can converge to zeros before 20 s, the control inputs change rapidly from their lower bound to upper bound which means more control efforts are needed. On the contrary, in Case 2, the proposed controller makes the formation stable before 20 s with more moderate control input. The convergence time of the finite-time convergence controller [25] cannot be readjusted without retuning the control parameters. On the contrary, in Case 3 (Figure 8), the convergence time of the proposed controller can be easily reset to 10 s without any parameter retuning, and the formation becomes stable before the 10 s by saturated input at the beginning of the control.

5. Conclusions

In this paper, a general formula of n-th order arbitrary time convergence controller is proposed. By introducing an auxiliary function and proving its properties, the arbitrary time stability and a less conservative parameter range are proved. Considering the saturated control input, a symmetrical arbitrary time convergence formation controller is designed and verified by simulation. This paper improves the theory and practicability of the arbitrary time convergence controllers, which provides a valuable reference to solving control problems where the convergence time is important. The future work includes theoretical analyses of the lower bound of the convergence time under the saturated control input and consideration on perturbation and time delay.

Data Availability

All data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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