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Research Article

Controllability Analysis of Linear Time-Varying T-H Equation with Matrix Sequence Method

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A satellite is considered to be moving relative to a nominal elliptical orbit, whose dynamics are usually described by the Tschaunner-Hempel equation (T-H equation). In this paper, we propose to transform the second-order time-varying system represented by the linear T-H equation with a second-order matrix form into a first-order time-varying system. Then, the controllability of the first-order time-varying system is investigated with the matrix sequence method including $e = 0$. Meanwhile, we study the observability of the first-order time-varying system with a specific form of measurement. The advantages of the matrix sequence method for controllability and observability analysis are tested by numerical examples, respectively. Dual theory is used to investigate the controllability and observability of the corresponding dual system of the T-H equation. The corresponding conclusions are obtained.

1. Introduction

The paper focuses on the following second-order time-varying linear satellite orbital system with measurement based on the linearized T-H equation [1]

$$\begin{aligned} M(t)\ddot{q}(t) + D(t)\dot{q}(t) + K(t)q(t) &= B(t)u(t), \\ \omega(t) &= C(t)\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \end{aligned} \tag{1}$$

where $q(t)$, $u(t)$, and $\omega(t) \in R^3$ are the state vector, control vector, and output vector, respectively, and $M(t)$, $D(t)$, $K(t)$, $B(t) \in R^{3\times3}$, and $C(t) \in R^{3\times6}$ are time-dependent matrices.

Since satellite appeared, the relative motion and attitude control of satellite have taken great interest in astronautics. Initially, the vector form of the complete relative dynamic equation contained the perturbation force and the control force of the space, which was a nonlinear time-varying system. In order to make further analysis on engineering, the linearization of the original motion equation with nonperturbed is necessary. Then, Clohessy and Wiltshire [2] proposed the first linear mathematical model in 1960, which was called the Clohessy-Wiltshire (C-W) equations. However, the C-W equations can only be applied to the description of the relative motion of satellites on a circular orbit, and the satellite relative motion is on an elliptical orbit in many practical situations [3, 4]. In 1964, Tschauner and Hempel [5] proposed linearized equations of relative motion for the elliptic reference orbit, that is, the Tschauner-Hempel (T-H) equation. Later, Yamanaka and Ankersen [6] developed a solution to the linear T-H equations. However, they studied only the case of an elliptical orbit; the solution of Yamanaka and Ankersen fails when the orbit is parabolic: $e = 1$.

The linearized T-H equation is suitable for describing the relative motion between the chaser satellite and the target satellite when the target satellite moves along the elliptical orbit. Recently, Fu et al. [1] represented the linear T-H equation as a second-order matrix form compactly. Second-order systems, which can better describe the dynamic properties of numerous natural phenomena, have played an important role in many technological advancements and have been widely applied in several engineering fields, for instance, aerospace, communications, automotive, and computer engineering. Then, controllability analysis of the second-order time-varying system represented by the linear T-H equation has attracted the attention of scholars [7–9]. In the beginning, the controllability of second-order systems was studied under time-invariant systems [10, 11]. Losse and Mehrmann [12] proposed a condition for the second-order time-invariant system. Kalenova and Morozov and Morozov and Kalenova [13, 14] presented a new analytical approach to transforming a special second-order time-varying system into a second-order time-invariant system of larger dimensions, and Mahmudov [15–18] investigated the controllability and observability of the second-order time-invariant system. However, lots of real systems are obvious time-varying; serious error will appear if we use constant models to describe them. It is well known that if the time-invariant system is controllable, the original system is also controllable. The uncontrollability of the time-invariant system may not imply the uncontrollability of the original system. So far, few results are given to study the second-order time-varying systems directly. In addition, the controllability condition of a second-order time-varying linear system is equivalent to that represented by a first-order time-varying linear system completely. Since there are lots of results with respect to first-order time-varying systems theory, many researchers study the controllability and observability of second-order time-varying systems by transforming them into first-order systems [19, 20]. Duan and Hu analyzed sufficient conditions for controllability and observability of second-order time-varying linear systems based on the linear T-H equation with matrix form. However, when $e = 0$, the target satellite moves along the circular orbit, which becomes the C-W equation, the proposed observability criterion will not be available for this situation. Therefore, the controllability and observability of the C-W equations should be investigated additionally. The matrix sequence method has been widely used in the study of controllability and observability in first-order linear time-varying systems, owe to its easy calculation and practical engineering significance [20]. Here, we propose the matrix sequence method to analyze the controllability and observability of second-order time-varying linear systems based on the linear T-H equation for $0 \le e \le 1$ by transforming them into first-order systems. The matrix sequence method is used to study the controllability and observability of the first-order system, which are tested by numerical examples. Controllability and observability of the corresponding dual system of the T-H equation are also investigated based on dual theory, which provides convenience for controllability and observability analysis of linear time-varying systems if the control vector or the observe vector is unknown.

This paper is structured as follows. In Section 2, we transform the second-order time-varying linear system represented by the linear T-H equation into a first-order time-varying linear system. Section 3 studies the controllability of the first-order linear system by the matrix sequence method, and the controllability is tested by a numerical example. In Section 4, observability analysis and numerical simulation of a first-order linear system with measurement are achieved by former transformations. In Section 5, dual theory is used to investigate the controllability and observability of the corresponding dual system of the T-H equation. We give concluding remarks in the final section.

2. Reducibility of the T-H Equation

2.1. Linearized of the T-H Equation with Matrix Form. As we know, the T-H equations [5] were derived after the linearization of the original nonperturbed equation of motion as follows:

$$\begin{aligned}
\ddot{x}(t) &= -k\omega^{3/2}x(t) + 2\omega\dot{z}(t) + \dot{\omega}z(t) + \omega^2 x(t) + a_x,\\
\ddot{y}(t) &= -k\omega^{3/2}y(t) + a_y,\\
\ddot{z}(t) &= 2k\omega^{3/2}z(t) - 2\omega\dot{x}(t) - \dot{\omega}x(t) + \omega^2 z(t) + a_z,
\end{aligned} \tag{2}$$

where ω is the angular velocity of the target orbit and $\dot{\omega}$ is the angular acceleration of target orbit.

According to [1], the linearized Equations (2) were transformed into the following linear second-order time-varying systems with measurement

$$\begin{aligned}
M(t)\ddot{q}(t) + D(t)\dot{q}(t) + K(t)q(t) &= B(t)u(t),\\
\omega(t) &= C(t)\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix},
\end{aligned} \tag{3}$$

where the state vector $q(t)$ and the control vector $u(t)$ are

$$q(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, u(t) = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \tag{4}$$

and output vector $\omega(t) \in R^3$.

The corresponding coefficient matrices are [19].

$$\begin{aligned}
&M(t) = I_3, B(t) = I_3,\\
&D(t) = \begin{bmatrix} 0 & 0 & -2\omega \\ 0 & 0 & 0 \\ 2\omega & 0 & 0 \end{bmatrix},\\
&K(t) = \begin{bmatrix} k\omega^{3/2} - \omega^2 & 0 & -\dot{\omega} \\ 0 & k\omega^{3/2} & 0 \\ -\dot{\omega} & 0 & -2k\omega^{3/2} - \omega^2 \end{bmatrix},\\
&C(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sin t & 2 & 0 & 0 & 0 & 0 \\ 3 & \cos t & -1 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{5}$$

where $\omega = k^2(1 + e\cos\theta)^2$ is the true anomaly, θ is the variable about time t, e is the eccentricity of target orbit, k is a positive constant which is equivalent to $\mu/h^{3/2}$,μ is the gravity constant, and h is the orbital angular momentum of target orbit. $x(t)$, $y(t)$, and $z(t)$ are the three chaser relative state vectors in the target orbital coordinate frame and a_x, a_y, and a_z are the three accelerations produced by the thrust in relevant directions.

2.2. Transformation of Second-Order Linear T-H Equation to a First-Order Form. So far, few theories are proposed to study second-order time-varying systems directly. The matrix sequence method is one of the most significant methods in modern control theory for controllability and observability analysis of first-order time-varying linear systems. In this part, we transform the second-order time-varying system (1) into a first-order system to investigate the controllability and observability of the second-order time-varying linear T-H equation based on the matrix sequence method.

Theorem 1. *The second-order linear system (1) is reduced to first-order time-varying linear system as follows:*

$$\begin{aligned} \tilde{q}(t) &= \tilde{A}(t)\tilde{q}(t) + \tilde{B}(t)u(t), \\ \omega(t) &= C(t)\tilde{q}(t), \end{aligned} \tag{6}$$

where $\tilde{q}(t) \in R^6$, $\omega(t) \in R^3$, and $u(t) \in R^3$ show the state variable, the output variable, and the control variable, respectively, the coefficient matrices are

$$\begin{aligned} \tilde{A}(t) &= \begin{bmatrix} 0 & I \\ -M^{-1}(t)K(t) & -M^{-1}(t)D(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k\omega^{3/2} + \omega^2 & 0 & \dot{\omega} & 0 & 0 & 2\omega \\ 0 & -k\omega^{3/2} & 0 & 0 & 0 & 0 \\ \dot{\omega} & 0 & 2k\omega^{3/2} + \omega^2 & -2\omega & 0 & 0 \end{bmatrix}, \\ \tilde{B}(t) &= \begin{bmatrix} 0 \\ M^{-1}(t)B(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

$$C(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sin t & 2 & 0 & 0 & 0 & 0 \\ 3 & \cos t & -1 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

Proof. Because of $M(t) = I_3$, matrix $M(t)$ is nonsingular for all $t \geq 0$. Then, Equation (1) is equivalent to the following equation:

$$\begin{aligned} \ddot{q}(t) + M^{-1}(t)D(t)\dot{q}(t) + M(t)^{-1}K(t)q(t) &= M(t)^{-1}B(t)u(t), \\ \omega(t) &= C(t)\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}. \end{aligned} \tag{8}$$

Sorting out Equation (8), we have

$$\begin{aligned} \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -M^{-1}(t)K(t) & -M^{-1}(t)D(t) \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}(t)B(t) \end{bmatrix} u(t), \\ \omega(t) &= C(t)\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}. \end{aligned} \tag{9}$$

Let

$$\tilde{q}(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, \tag{10}$$

then, the second-order linear system (1) is reduced to first-order time-varying linear system (6). □

3. Controllability of the Linear T-H Equation

3.1. Controllability Analysis. In order to investigate the controllability of system (6), we present matrix sequence method for the general first-order system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \end{aligned} \tag{11}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $y(t) \in R^r$ are the state vector, control vector, and output vector, respectively. The coefficient matrices $A(t) \in R^{n\times n}$, $(t) \in R^{n\times m}$, and $C(t) \in R^{r\times n}$ are time-dependent.

The definition and main theorem of controllability of linear time-varying system (11) can be found in [21].

Definition 2. System (11) is called controllable on $[t_0, t_f]$ if for given any initial $x_0 = x(t_0)$ and desired final state value x_1 at $t = t_f$, there exists a continuous control $u(t)$ defined on $[t_0, t_f]$ such that the corresponding solution of (11) satisfies $x(t_f) = x_1$.

Proposition 3. *If $A(t) \in R^{n\times n}$ is a differentiable matrix function of k-2 order and $B(t) \in R^{n\times m}$ is a differentiable matrix function of k-1 order when $t \geq t_0$, then we can define a matrix sequence $\{M_j(t)\} \in R^{n\times m}$ satisfying*

$$\begin{aligned} M_0(t) &= B(t), \\ M_{j+1}(t) &= -A(t)M_j(t) + \frac{d}{dt}M_j(t), \end{aligned} \tag{12}$$

where $j = 0, 1, \cdots, k-2, k \geq 2$.

Theorem 4. *If $A(t)$ has $k-2$ derivatives, $B(t)$ has $k-1$ derivatives when $t \geq t_0$, there exists $t \in [t_0, t_1]$ for every $t_1 > t_0$ satisfying*

$$\operatorname{rank}\left[M_0(t), M_1(t), \cdots, M_{k-1}(t)\right] = n, \tag{13}$$

then system (11) is controllable.

Based on the matrix sequence method of first-order time-varying linear systems, we analyze the controllability of system (6) and we can get the following results.

Definition 5. System (6) is called controllable on $[t_0, t_f]$ if for given any initial $\tilde{q}_0 = \tilde{q}(t_0)$ and desired final state value $\tilde{q}_1$ at $t = t_f$, there exists a continuous control $u(t)$ defined on $[t_0, t_f]$ such that the corresponding solution of (6) satisfies $\tilde{q}(t_f) = \tilde{q}_1$.

Proposition 6. *If $\tilde{A}(t) \in R^{6\times 6}$ is a differentiable matrix function of k-2 order and $\tilde{B}(t) \in R^{6\times 3}$ is a differentiable matrix function of k-1 order when $t \geq t_0$, then we can define a matrix sequence $\{M_j(t)\} \in R^{6\times 3}$ satisfying*

$$\begin{aligned} M_0(t) &= \tilde{B}(t), \\ M_{j+1}(t) &= -\tilde{A}(t)M_j(t) + \frac{d}{dt}M_j(t), \end{aligned} \tag{14}$$

where $j = 0, 1, \cdots, k-2, k \geq 2$.

Theorem 7. *If $\tilde{A}(t)$ has $k-2$ derivatives, $\tilde{B}(t)$ has $k-1$ derivatives when $t \geq t_0$, there exists $t \in [t_0, t_1]$ for every $t_1 > t_0$ satisfying*

$$\operatorname{rank}\left[M_0(t), M_1(t), \cdots, M_{k-1}(t)\right] = 6, \tag{15}$$

then system (6) is controllable.

Proof. To system (6), we can get the following matrix sequence:

$$\begin{aligned} M_0(t) &= \tilde{B}(t) = \begin{bmatrix} 0 \\ M^{-1}(t)B(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \frac{d}{dt}M_0(t) = O^{6\times 3}, \\ \tilde{A}(t) &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k\omega^{3/2} + \omega^2 & 0 & \dot{\omega} & 0 & 0 & 2\omega \\ 0 & -k\omega^{3/2} & 0 & 0 & 0 & 0 \\ \dot{\omega} & 0 & 2k\omega^{3/2} + \omega^2 & -2\omega & 0 & 0 \end{bmatrix}, \\ M_1(t) &= -\tilde{A}(t)M_0(t) + \frac{d}{dt}M_0(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -2\omega \\ 0 & 0 & 0 \\ 2\omega & 0 & 0 \end{bmatrix}, \end{aligned} \tag{16}$$

then

$$\det\left(M_0(t), M_1(t)\right) = 1 \neq 0, \tag{17}$$

that is,

$$\operatorname{rank}\left[M_0(t), M_1(t)\right] = 6. \tag{18}$$

Therefore, according to Theorem 4, system (6) is controllable. □

3.2. Numerical Calculation. In [19], Duan and Hu proposed that system (1) is controllable when $M(t)$ and $B(t)$ are nonsingular for all $t \geq 0$, which holds obviously under Equation (5). However, for some situations, coefficient matrix $B(t)$ is not always nonsingular; then, the proposed theory cannot be used to study the controllability. At these moments, the matrix sequence method can still analyze the controllability of the second-order time-varying system by transforming it into a first-order time-varying system. According to Theorem 7,

$$\det\left(M_0(t), M_1(t)\right) = \det\left(\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -2\omega \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2\omega & 0 & 0 \end{bmatrix}\right) = 1 \neq 0. \quad (19)$$

In fact, $\det\left(M_0(t), M_1(t)\right) = 1 \neq 0$ for arbitrary ω, that is, rank $[M_0(t), M_1(t)] = 6$. Therefore, system (6) is controllable. Based on system (6) is equal to system (1) when $\det M(t) \neq 0$.

Particularly, when $e = 0$, the controllability criterion proposed in [19] is not available. Whereas, our results will still apply.

Specifically, if $e = 0$, then $\omega = 1$ and the time-varying system (1) can be reduced to the following second-order time-invariant system:

$$A_0\ddot{x}(t) + A_1\dot{x}(t) + A_2x(t) = B_0u(t), \quad (20)$$

and the coefficient matrices are constant as follows:

$$\begin{aligned} &A_0 = I_3, B_0 = I_3, \\ &A_1 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \\ &A_2 = \begin{bmatrix} k-1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & -2k-1 \end{bmatrix}, \end{aligned} \quad (21)$$

where $k = 2.267 \times 10^{-2} s^{-1/2}$ [1].

For the controllability of system (20), it is necessary and sufficient that the condition [11]

$$\text{rank}\left[A_0\lambda^2 + A_1\lambda + A_2, B_0\right] = 3 \quad (22)$$

is satisfied for any eigenvalue λ, that is, $\det\left(A_0\lambda^2 + A_1\lambda + A_2\right) = 0$.

Substituting (21), we get

$$\begin{aligned} \det\left(A_0\lambda^2 + A_1\lambda + A_2\right) &= \det\left(\begin{bmatrix} \lambda^2 + k - 1 & 0 & -2\lambda \\ 0 & \lambda^2 + k & 0 \\ 2\lambda & 0 & \lambda^2 - 2k - 1 \end{bmatrix}\right) \\ &= \left(\lambda^2 + k\right)\left(\lambda^4 + (2-k)\lambda^2 + \left(-2k^2 + k + 1\right)\right) = 0, \end{aligned} \quad (23)$$

the solutions of Equation (23) are $\lambda = \sqrt{k}i, -\sqrt{k}i$, and λ ^2 = $(k - 2 + \sqrt{(8k - 9k\text{^}2)}\, i)/2$, $(k - 2 - \sqrt{(8k - 9k\text{^}2)}\, i)/2$, respectively. Here, we choose $\lambda = \sqrt{k}i$ and other solutions can calculate similarly.

$$\begin{aligned} &\text{rank}\left[A_0\lambda^2 + A_1\lambda + A_2, B_0\right] \\ &= \text{rank}\left[\begin{bmatrix} -1 & 0 & -2\sqrt{k}i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 2\sqrt{k}i & 0 & -3k-1 & 0 & 0 & 1 \end{bmatrix}\right] \\ &= 3. \end{aligned} \quad (24)$$

Therefore, system (20) is controllable, which means that the time-varying system (1) is controllable when $e = 0$. As we can see, the computing of λ is a very complex process, whereas our controllable condition can be used directly and verified easily.

4. Observability of the Linear T-H Equation

4.1. Observability Analysis. In order to investigate the observability of system (6), we also present matrix sequence method for general of first-order time-varying linear system (11). The definition and main theorem of observability for linear time-varying system (11) can be found in [20].

Definition 8. System (11) is called observable on $[t_0, t_f]$ if for given any continuous control $u(t)$ defined on $[t_0, t_f]$, initial state $x_0 = x(t_0)$ can be determined from the output $y(t)$ uniquely.

Proposition 9. *If $A(t) \in R^{n\times n}$ is a differentiable matrix function of k-2 order and $C(t) \in R^{r\times n}$ is a differentiable matrix function of k-1 order when $t \geq t_0$, then we can define a matrix sequence $\{N_j(t)\} \in R^{r\times n}$ satisfying*

$$\begin{aligned} &N_0(t) = C(t), \\ &N_{j+1}(t) = \frac{d}{dt}N_j(t) + N_j(t)A(t), \end{aligned} \quad (25)$$

where $j = 0, 1, \cdots, k-2, k \geq 2$.

Theorem 10. *If $A(t)$ has $k-2$ derivatives, $C(t)$ has $k-1$ derivatives when $t \geq t_0$, there exists a $t \in [t_0, t_1]$ for every $t_1 > t_0$ satisfying*

$$\text{rank}\left[N_0^T(t), N_1^T(t), \cdots, N_{k-1}^T(t)\right] = n, \quad (26)$$

then system (11) is observable.

By analogy way in controllability, based on the matrix sequence method of first-order time-varying linear systems, we analyze the observability of system (6) and we can get the following results.

Proposition 11. *If $\tilde{A}(t) \in R^{6\times 6}$ is a differentiable matrix function of k-2 order and $C(t) \in R^{3\times 6}$ in system (6) is a differentiable matrix function of k-1 order when $t \geq t_0$, then we can define a matrix sequence $\{N_j(t)\} \in R^{3\times 6}$ satisfying*

$$\begin{aligned} N_0(t) &= C(t), \\ N_{j+1}(t) &= \frac{d}{dt}N_j(t) + N_j(t)\tilde{A}(t), \end{aligned} \tag{27}$$

where $j = 0, 1, \cdots, k-2, k \geq 2$.

Theorem 12. *If $\tilde{A}(t) \in R^{6\times 6}$ is a differentiable matrix function of k-2 order and $C(t)$ in system (6) has $k-1$ derivatives when $t \geq t_0$, there exists $t \in [t_0, t_1]$ for every $t_1 > t_0$ satisfying*

$$\operatorname{rank}\left[N_0^T(t), N_1^T(t), \cdots, N_{k-1}^T(t)\right] = 6, \tag{28}$$

then system (6) is observable.

Proof. To system (6), we can get the following matrix sequence

$$\begin{aligned} N_0(t) = C(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sin t & 2 & 0 & 0 & 0 & 0 \\ 3 & \cos t & -1 & 0 & 0 & 0 \end{bmatrix}, \\ N_1(t) &= \frac{d}{dt}N_0(t) + N_0(t)\tilde{A}(t) \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \cos t & 0 & 0 & \sin t & 2 & 0 \\ 0 & -\sin t & 0 & 3 & \cos t & -1 \end{bmatrix}, \\ N_2(t) &= \frac{d}{dt}N_1(t) + N_1(t)\tilde{A}(t) \\ &= \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & 0 & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix}, \end{aligned} \tag{29}$$

where

$$\begin{aligned} & a_{11} = \omega^2 - k\omega^{3/2}, a_{13} = \dot{\omega}, a_{16} = 2\omega, \\ & a_{21} = -\left(k\omega^{3/2} - \omega^2 + 1\right), a_{22} = -2\omega^{3/2}, \\ & a_{23} = \dot{\omega}\sin t, a_{24} = 2\cos t, a_{26} = 2\omega\sin t, \\ & a_{31} = 3\omega^2 - 3k\omega^{3/2} - \dot{\omega}, \\ & a_{32} = -\cos t - kk\omega^{3/2}\cos t, \\ & a_{33} = 3\dot{\omega} - 2k\omega^{3/2} - \omega^2, \\ & a_{34} = 2\omega, a_{35} = -2\sin t, a_{36} = 6\omega, \\ & \det\left(N_0^T(t), N_1^T(t)\right) = 4 \neq 0, \end{aligned} \tag{30}$$

then

$$\operatorname{rank}\left[N_0^T(t), N_1^T(t)\right] = 6. \tag{31}$$

Therefore, according to Theorem 10, system (6) is observable. □

4.2. Numerical Calculation. In [19], Duan and Hu proposed that system (6) is observable when $1 + e\cos\theta \neq 3/2 - \sqrt{1/4 + 4/3e^2}$,with

$$C(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sin t & 2 & 0 \\ 0 & 0 & 0 & 3 & \cos t & -1 \end{bmatrix}. \tag{32}$$

In this situation, substituting $e = 0.73074, k = 2.267 \times 10^{-2}s^{-1/2}$ [1], we have $\omega = 1.385 \times 10^{-4}\text{rad}/s$. According to Theorem 12,

$$\begin{aligned} \det\left(N_0^T(t), N_1^T(t)\right) &= \det\left(\begin{bmatrix} 0 & 0 & 0 & -k\omega^{3/2} + \omega^2 & \sin t\left(-k\omega^{3/2} + \omega^2\right) & 3\left(-k\omega^{3/2} + \omega^2\right) - \dot{\omega} \\ 0 & 0 & 0 & 0 & -2 - k\omega^{3/2} & -2k - k\omega^{3/2} \\ 0 & 0 & 0 & \dot{\omega} & \dot{\omega}\sin t & 3\dot{\omega} - \left(2k\omega^{3/2} + \omega^2\right) \\ 1 & \sin t & 3 & 0 & \cos t & 2\omega \\ 0 & 2 & \cos t & 0 & 0 & -\sin t \\ 0 & 0 & -1 & 2\omega & 2\sin t\omega & 6\omega \end{bmatrix}\right) \\ &= \det\begin{bmatrix} 0 & 0 & 0 & -1.777 \times 10^{-8} & -1.777 \times 10^{-8}\sin t & -5.331 \times 10^{-8} \\ 0 & 0 & 0 & 0 & -5.695 \times 10^{-8} & -4.534 \times 10^{-2} - 3.695 \times 10^{-8} \\ 0 & 0 & 0 & 0 & 0 & -18.377 \times 10^{-8} \\ 1 & \sin t & 3 & 0 & \cos t & 2.77 \times 10^{-4} \\ 0 & 2 & \cos t & 0 & 0 & -\sin t \\ 0 & 0 & -1 & 2.77 \times 10^{-4} & 2.77 \times 10^{-4}\sin t & 8.31 \times 10^{-4} \end{bmatrix} \\ &= 3.7195 \times 10^{-22} \neq 0, \end{aligned} \tag{33}$$

which is equal to rank $(N_0^T(t), N_1^T(t)) = 6$. Therefore, system (6) is observable even if at $1 + e\cos\theta = 3/2 - \sqrt{1/4 + 4/3e^2}$. By analogy, the observability of system (6) with

$$C(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sin t & 2 & 0 & 0 & 0 & 0 \\ 3 & \cos t & -1 & 0 & 0 & 0 \end{bmatrix}, \tag{34}$$

can be analyzed by

$$\det\left(N_0^T(t), N_1^T(t)\right) = \det\left(\begin{bmatrix} 1 & \sin t & 3 & 0 & \cos t & 0 \\ 0 & 2 & \cos t & 0 & 0 & -\sin t \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sin t & 3 \\ 0 & 0 & 0 & 0 & 2 & \cos t \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}\right) = 4 \neq 0. \tag{35}$$

The results are consistent with Duan and Hu.

Particularly, when $e = 0$, the matrix $K(t)$ will be singular on the entire time domain, and the observability criterion proposed in [19] is not available. Here, we choose (34) and the other situation can calculate similarly. At this moment, we can represent the measurement matrix in the following form

$$C(t) = \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3, \tag{36}$$

where

$$\alpha_1 = 1, \alpha_2 = \sin t, \alpha_3 = \cos t,$$
$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

Introducing $\alpha^T = [\alpha_1, \alpha_2, \alpha_3] = [1, \sin t, \cos t]$ and using transformation $z(t) = \zeta(t)x(t), \zeta(t) = \alpha \otimes E_3$. (The symbol $\otimes$ denotes the Kronecker matrix product.) Then, system (20) with measurement (34) can be reduced into the higher order time-invariant system

$$\begin{aligned} \tilde{A}_0\ddot{z}(t) + \tilde{A}_1\dot{z}(t) + \tilde{A}_2 z(t) &= \tilde{B}_0 u(t), \\ \delta(t) &= \Gamma z(t), \end{aligned} \tag{38}$$

where

$$\tilde{A}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\tilde{A}_1 = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\tilde{A}_2 = \begin{bmatrix} k-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2k-1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{39}$$

However, the observability condition in [13] is appropriate for time-independent systems with square matrix coefficients, which cannot be utilized in analyzing the observability of system (38) with (34). In addition, this transformation and the change of variables extend the state space, which may cause calculation complexity increase. Therefore, our matrix sequence method is more effective on a broader scale.

5. Dual Principle of the Linear T-H Equation

In this section, we present dual system of first-order linear T-H equation (6). Then, we can get the controllability and observability of the corresponding dual system. The following lemma reveals the relation between the dual system and first-order linear T-H equation.

Definition 13 (see [19]). For the general first-order time-varying linear system (11), let us introduce the following dual system"

$$\begin{aligned} \dot{x}^*(t) &= -A^T(t)x^*(t) + C^T(t)v(t), \\ h(t) &= B^T(t)x^*(t), \end{aligned} \tag{40}$$

where $x^*(t) \in R^n$, $v(t) \in R^m$, and $h(t) \in R^r$ are the state vector, the control vector, and the output vector of system.

The following lemma reveals the relation between a system and its dual system.

Lemma 14 (see [22]).

(1) *System (11) is controllable at t_0 if and only if system (40) is observable at t_0*

(2) *System (11) is observable at t_0 if and only if system (40) is controllable at t_0.*

According to Definition 13, we get the dual system of system (6) as follows:

$$\begin{aligned} \tilde{q}^*(t) &= -\tilde{A}^T(t)\tilde{q}^*(t) + C^T(t)v(t), \\ h(t) &= \tilde{B}^T(t)\tilde{q}^*(t). \end{aligned} \tag{41}$$

Then, to system (41), we can get the following matrices based on matrix sequence method:

$$M_0(t) = C^T(t) = \begin{bmatrix} 1 & \sin t & 3 \\ 0 & 2 & \cos t \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{A}^T(t) = \begin{bmatrix} 0 & 0 & 0 & -k\omega^{3/2} + \omega^2 & 0 & \dot{\omega} \\ 0 & 0 & 0 & 0 & -k\omega^{3/2} & 0 \\ 0 & 0 & 0 & \dot{\omega} & 0 & 2k\omega^{3/2} + \omega^2 \\ 1 & 0 & 0 & 0 & 0 & -2\omega \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2\omega & 0 & 0 \end{bmatrix},$$

$$M_1(t) = \tilde{A}^T(t)M_0(t) + \frac{d}{dt}M_0(t) = \begin{bmatrix} 0 & \cos t & 0 \\ 0 & 0 & -\sin t \\ 0 & 0 & 0 \\ 1 & \sin t & 3 \\ 0 & 2 & \cos t \\ 0 & 0 & -1 \end{bmatrix},$$

$$\det\left(M_0(t), M_1(t)\right) = 4 \neq 0, \tag{42}$$

then

$$\operatorname{rank}\left[M_0(t), M_1(t)\right] = 6. \tag{43}$$

Therefore, according to Theorem 4, system (41) is controllable, and then system (6) is observable, based on Lemma 14. In the same way, we can prove the controllability of system (6) by proving its dual system (41) is observable. To system (6), we can also get the following matrix sequence:

$$N_0(t) = B^T(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{aligned} N_1(t) &= \frac{d}{dt}N_0(t) - N_0(t)\tilde{A}^T(t) \\ &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 2\omega \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2\omega & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\det\left(N_0^T(t), N_1^T(t)\right) = 1 \neq 0, \tag{44}$$

then

$$\operatorname{rank}\left[N_0^T(t), N_1^T(t)\right] = 6. \tag{45}$$

By analyzing the controllability and observability of the dual system of the first-order linear system (6), we can also obtain the controllability and observability of linear T-H equation (1).

6. Conclusion

In this paper, the second-order time-varying system with a second-order matrix form represented by the linear T-H equation is transformed into a first-order time-varying system. Then, the matrix sequence method is presented to analyze the controllability and observability of reduced T-H equations. The controllability of the first-order time-varying system is investigated with the matrix sequence method including $e = 0$. Meanwhile, we study the observability of the first-order time-varying linear system converted by the second-order time-varying system with measurement. The controllability and observability of the corresponding dual system of the T-H equation are also investigated based on dual theory, which provides convenience for the controllability and observability analysis of linear time-varying systems if the control vector or measurement is unknown. Research results and numerical calculations show that transforming second-order time-varying systems into first-order systems is superior, and utilizing the matrix sequence method for first-order time-varying linear system is simple and effective. At the same time, the available range of the matrix sequence method is more widely, which is free from complex computation and extension of state space. In the future, we recommend applying the matrix sequence method in analyzing the controllability and observability of many other systems in engineering fields, such as mechanics, communications, and geographic measurement.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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