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Research Article

Balancing Conditions for the Relativistic Correction Using Lorentz Acceleration

M. A. Yousef,[1] **M. I. El-Saftawy**,[1,2] **and A. Mostafa**[3]

[1]*Department of Astronomy and Space Science, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia*
[2]*National Research Institute of Astronomy and Geophysics, Helwan, Cairo, Egypt*
[3]*Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt*

Correspondence should be addressed to A. Mostafa; ahmedmostafa@sci.asu.edu.eg

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In this work, the average effects of the Lorentz acceleration on the charged spacecraft's orbit are studied encounter with relativistic correction. The relativistic correction as function of the orbital elements, and may be time, is formulated. Lagrange planetary equations are used to calculate the perturbations due to considered perturbing forces. The needed conditions to neutralize the effects of the relativistic corrections, using Lorentz acceleration, are derived. Numerical examples for different kinds of orbits are applied.

1. Introduction

The perturbing forces on the artificial bodies that can be either spacecrafts or artificial satellites are substantial topics in the field of astrodynamics. By studying these forces, the orbits of artificial bodies can be controlled differently based on the purpose of the orbits.

To determine the orbital design, six Keplerian orbital elements must be monitored. As time passes with any participating outer perturbing force on the artificial body, the orbital elements are significantly impacted. Indeed, the aim of the orbital theory is to minimize the influence of the perturbing forces as much as possible.

One of the perturbing forces is the relativistic correction (RC), which is referred to as post Newtonian effects. This disturbance perturbs the orbital motion of the earth's artificial body. Astronomers have paid attention to Einstein's prediction in 1915 for precession of a planet's perihelion around the sun which explained the discrepancy among the observed and theoretical shift of Mercury's perihelion. Later, Lense and Thirring [1] confirmed that from the aspect of general relativity. The restricted relativistic two-body problem was also discussed in detail by Bogorodskii [2]. According to Cugusi and Proverbio [3], the RC can be divided into three components addition to secular perigee shift such as the effect regard to the perigee shift, the processional term of the line of nodes, both depended on rotational earth, and a variation of the time of perigee passage. The general theory of relativity is applied in many astronomical applications. The RC, per unit inertial mass, in near-earth body motion can be outlined as being contained in the four effect components such as the Schwarzschild solution, the geodesic precession, the Lense-Thirring precession, and the earth oblateness components (represented mainly by the J_2 geopotential spherical harmonic coefficient) [4–7].

Due to perturbing forces such as RC, Lorentz acceleration (LA) is utilized to balance the perturbations on the artificial body's orbit [8]. It is important to mention that the space orientation of the orbits should be fixed for long time intervals. The family of orbits fulfills this status is called "frozen orbits." Many studies have discussed this type of orbits such as Condoleo et al. [9], El-Salam and El-Bar [10], Khattab et al. [11], and El-Salam et al. [12]. Recently, many studies using charge on the surface of the satellite are made to control one or more of the natural perturbing forces such as Mostafa et al. [13] and Yousef et al. [14]. The previous authors studied the usage of LA to balance the effects of solar radiation pressure. Fisher et al. [15] also discussed the action of the LA on the upper solar atmosphere and its interior. Additionally, the effects of the LA are investigated with artificial controlling of the collinear and triangular liberation points in the restricted three-body problem by Mostafa et al. [16] and El-Saftawy et al. [17]. The interaction between control mechanics and space environment has increased recently whether in controlling the orbits of the artificial heavenly bodies or predicting the weather, e.g., Chen et al. [18] and Sun et al. [19].

In this paper, we examine the conditions to make control on Keplerian orbital elements to avert the effects of RC by using LA with the charge per unit mass is considered a controlling parameter.

2. Formulation of the Problem

In this article, Lorentz acceleration is used to cancel the perturbations of the relativistic effects. The radial, transverse, and normal components of RC are constructed in Section 2.1. And in Section 2.2, the same components are constructed for LA. In Sections 3.1 and 3.2, the components of each force are inserted in Lagrange planetary equations to find the corresponding perturbations, while in Sections 4.1 and 4.2, the average of each perturbation is made over the fast variable ν to get only long-period and secular perturbations that may affect the orbit with time. Finally, the mean in Section 5, the conditions for controlling the shape and orientation of the orbit with respect to RC are found. Numerical and graphical illustrations are given in Section 6.

2.1. Relativistic Correction. Because of the small order of the relativistic corrections found by Albert Einstein in his theory of general relativity and because of the complexity of relativistic equations compared to the classical Newtonian mechanics, the RC are treated as a perturbation to the original problem solved by Newtonian mechanics. Let R_{RC}, T_{RC}, and W_{RC} be the radial, transverse, and normal components of the RC, affecting a body with distance r from the center of spherical planet with R_0, m, and ω_0 which are the radius, relativistic mass, and rotating angular speed, respectively. Bogorodskii [20] drove these components as function of the orbital elements,

$$R_{\mathrm{RC}} = \frac{c^2 m^2}{r^2}\left(\frac{2}{r} + \frac{1-e^2+4e^2\,\mathrm{Sin}^2\,\nu}{p}\right) + \frac{4c\,p^{1/2}\,m^{3/2}\,\omega_0\,R_0^2\,C}{5r^4} - \frac{2m\,\omega_0^2\,R_0^2}{5r^2} + \frac{3m\,\omega_0^2\,R_0^4}{35r^4}\left[3\,\mathrm{Sin}^2(\omega+\nu)\,S^2 - 1\right],$$

$$T_{\mathrm{RC}} = \frac{4c^2m^2}{r^3}\,e\,\mathrm{Sin}\nu - \frac{4\,c}{5r^3}p^{-1/2}\,m^{3/2}\,\omega_0\,R_0^2\,e\,C\,\mathrm{Sin}\nu - \frac{6}{35r^4}\,m\,\omega_0^2\,R_0^4\,S^2\,\mathrm{Sin}(\omega+\nu)\,\mathrm{Cos}(\omega+\nu),$$

$$W_{\mathrm{RC}} = \frac{4\,c}{5\,r^3}\,p^{-1/2}m^{3/2}\,\omega_0\,R_0^2\,S[-e\,\mathrm{Sin}\omega + (2+3e\,\mathrm{Cos}\nu)\mathrm{Sin}(\omega+\nu)] - \frac{6\,m}{35\,r^4}\,\omega_0^2\,R_0^4\,S\,C\,\mathrm{Sin}(\omega+\nu), \quad (1)$$

where c and p are the speed of light and semiparameter of the orbit, respectively. m, S, and C are related to the mass of the planet, M, and inclination of the orbit through

$$m = \frac{G}{c^2}M, \quad S = \mathrm{Sin}I, \quad C = \mathrm{Cos}I. \quad (2)$$

The Keplerian orbital elements, used, a, e, I, ω, Ω, and ν are the semimajor axis, eccentricity, inclination, argument of preside, argument of ascending node, and true anomaly, respectively.

The previous equations can be written in compact form as

$$R_{\mathrm{RC}} = \sum_{j=0}^{1}\sum_{i=2}^{4} \Phi^i\left(A_{j,i}^C\,\mathrm{Cos}2j\nu + A_{j,i}^S\,\mathrm{Sin}2j\nu\right), \quad (3)$$

$$T_{\mathrm{RC}} = \sum_{i=1}^{2} \Phi^{2+i}\left(B_i^C\,\mathrm{Cos}i\nu + B_i^S\,\mathrm{Sin}i\nu\right), \quad (4)$$

$$W_{\mathrm{RC}} = \Phi^3\sum_{i=0}^{2}\left[C_i^C\,\mathrm{Cos}i\nu + C_i^S\,\mathrm{Sin}i\nu\right] + \Phi^4\left(C_3^C\mathrm{Cos}\nu + C_3^S\mathrm{Sin}\nu\right), \quad (5)$$

where $\Phi = (a/r)$,

$$A_{1,3}^C = A_{1,2}^S = A_{1,3}^S = 0,$$

$$A_{0,2}^C = \frac{c^2m^2}{a^3}\left(1+\frac{2ae^2}{p}\right) - \frac{2m\,\omega_0^2\,R_0^2}{5a^2},$$

$$A_{0,3}^C = \frac{2c^2m^2}{a^3},$$

$$A_{0,4}^C = \frac{m\,\omega_0\,R_0^2}{70\,a^4}\left(56\,c\,C\,\sqrt{m\,p} + 9\,\omega_0 R_0^2 S^2 - 6\,\omega_0 R_0^2\right),$$

$$A_{1,2}^C = -\frac{2c^2m^2e^2}{a^2p},$$

$$A_{1,4}^C = -\frac{9m\,\omega_0^2\,R_0^4 S^2\mathrm{Cos}2\omega}{70a^4},$$

$$A_{1,4}^S = \frac{9m\,\omega_0^2\,R_0^4 S^2\mathrm{Sin}2\omega}{70a^4},$$

$$B_1^C = 0,$$

$$B_1^S = \frac{4\,c\,m\,e}{a^3}\left(cm - \frac{m^{1/2}\,\omega_0\,R_0^2\,C}{5\,p^{1/2}}\right),$$

$$B_2^S = \frac{12\, m\, \omega_0^2\, R_0^4\, S^2}{35a^4}\,\text{Cos}2\omega,$$

$$B_2^C = \frac{12\, m\, \omega_0^2\, R_0^4\, S^2}{35a^4}\,\text{Sin}2\omega,$$

$$C_0^C = \frac{2\, c\, p^{-1/2} m^{3/2}\, \omega_0\, R_0^2\, Se\,\text{Sin}\omega}{5a^3},$$

$$C_1^C = \frac{8c\, p^{-1/2} m^{3/2}\, \omega_0\, R_0^2\, S\,\text{Sin}\omega}{5a^3},$$

$$C_2^C = \frac{6c\, p^{-1/2} m^{3/2}\, \omega_0\, R_0^2\, Se\,\text{Sin}\omega}{5a^3},$$

$$C_3^C = -\frac{6m\, \omega_0^2\, R_0^4\, S\, C\,\text{Sin}\omega}{35a^4},$$

$$C_1^S = \frac{8c\, p^{-1/2} m^{3/2}\, \omega_0\, R_0^2\, S\,\text{Cos}\omega}{5a^3},$$

$$C_2^S = \frac{6c\, p^{-1/2} m^{3/2}\, \omega_0\, R_0^2\, Se\,\text{Cos}\omega}{5a^3},$$

$$C_3^S = -\frac{6m\, \omega_0^2\, R_0^4\, S\, C\,\text{Cos}\omega}{35a^4}. \tag{6}$$

2.2. Lorentz Acceleration. The motion of charged body, in the magnetic field with charge per unit mass q and magnetic dipole moment B, will be affected by a force called the Lorentz force. Lorentz acceleration, of a body in a rotating magnetic field, with rotational speed ω_0, can be written in radial, R_{LA}, transverse, T_{LA}, and normal, W_{LA}, components, respectively, as follows [8]:

$$R_{\text{LA}} = -\left(\frac{q\,B}{r^2}\right)\left(\dot{\phi} - \omega_0\right)\text{Sin}^2\theta,$$

$$T_{\text{LA}} = \left(\frac{q\,B}{\sqrt{\mu p}}\right)\left[\left(\frac{\dot{r}\dot{\phi}}{r}\right)\text{Sin}^2\theta - 2\omega_0\dot{\theta}\,\text{Sin}\theta\,\text{Cos}\theta\right],$$

$$W_{\text{LA}} = \left(\frac{q\,B}{\sqrt{\mu p}}\right)\left[-2\dot{\phi}\left(\dot{\phi} - \omega_0\right)\text{Sin}^2\theta\,\text{Cos}\,\theta + \left(\frac{\dot{r}\dot{\phi}}{r}\right)\text{Sin}\theta - 2\dot{\theta}^2\text{Cos}\theta\right], \tag{7}$$

with r, ϕ, θ, $\dot{r}$, $\dot{\phi}$, and $\dot{\theta}$ are the spherical coordinate variables and their rate of changes while μ is the gravitational parameter of the central planet.

The last equations, with the help of the relation between the orbital elements and the spherical coordinate variables, can be written as

$$R_{\text{LA}} = \sum_{j=0}^{1}\sum_{i=0}^{1} \Phi^{2(i+1)}\left(D_{j,i}^C\,\text{Cos}2j\nu + D_{j,i}^S\,\text{Sin}2j\nu\right), \tag{8}$$

$$T_{\text{LA}} = \sum_{i=0}^{1} \Phi^{(i+2)}\left(F_i^C\,\text{Cos}(i-2)\nu + F_i^S\,\text{Sin}(i-2)\nu\right), \tag{9}$$

$$W_{\text{LA}} = \sum_{i=1}^{2} \Phi^{2\,i}\left(G_i^C\,\text{Cos}\upsilon + G_i^S\,\text{Sin}\upsilon\right) + \sum_{i=0}^{1}\Phi^3\left(G_i'C\,\text{Cos}2\,i\upsilon + G_i'S\,\text{Sin}2\,i\upsilon\right), \tag{10}$$

with

$$D_{1,1}^C = D_{1,1}^S = 0,$$

$$D_{0,0}^C = \frac{Bq\omega_0}{a^2}\left(1 - \frac{1}{2}S^2\right),$$

$$D_{0,1}^C = -\frac{Bq\,C\,\sqrt{\mu\,p}}{a^4},$$

$$D_{1,0}^C = \frac{Bq\omega_0\,S^2}{2\,a^2}\,\text{Cos}2\,\omega,$$

$$D_{1,0}^S = -\frac{Bq\omega_0\,S^2}{2\,a^2}\,\text{Sin}2\,\omega,$$

$$F_1^C = 0,\; F_0^S = -\frac{BqS^2\omega_0}{a^2}\,\text{Cos}2\,\omega,$$

$$F_0^C = \frac{BqS^2\omega_0}{a^2}\,\text{Sin}2\,\omega,$$

$$F_1^S = -\frac{Bq\,e\,C}{a^3}\sqrt{\frac{\mu}{p}}, \tag{11}$$

$$G_0'C = \frac{B\,S\,q\,e}{2\,a^3}\sqrt{\frac{\mu}{p}}\text{Sin}\omega,$$

$$G_1'C = -G_0'C,$$

$$G_1^C = \frac{2\,B\,S\,q\,C\,\omega_0}{a^2}\,\text{Sin}\omega,$$

$$G_2^C = -\frac{B\,S\,q\,\sqrt{\mu\,p}}{a^4}\,\text{Sin}\omega,$$

$$G_1'S = -\frac{B\,S\,q\,e}{2\,a^3}\sqrt{\frac{\mu}{p}}\text{Cos}\omega,$$

$$G_1^S = \frac{2\,B\,S\,q\,C\,\omega_0}{a^2}\,\text{Cos}\omega,$$

$$G_2^S = -\frac{B\,S\,q\,\sqrt{\mu\,p}}{a^4}\,\text{Cos}\omega. \tag{12}$$

3. Orbital Perturbations

The perturbations on the orbital elements due to the considered perturbing forces with radial, transverse, and normal components R, T, and W, respectively, can be computed using LPEs. Many sources describe and formulate LPEs such as Gangestad et al. [8], Fitzpatrick [21], and Bate et al. [22], as follows:

$$\dot{a} = \frac{2a^2}{h}\left[T + e(T\,Cos\,\nu + R\,\text{Sin}\nu)\right], \tag{13}$$

$$\dot{e} = \frac{\sqrt{1-e^2}}{n\,a}\left[R\,\mathrm{Sin}\nu + T\,(\mathrm{Cos}\nu + \mathrm{Cos}E)\right], \tag{14}$$

$$\dot{I} = \frac{W}{h}\, r\,\mathrm{Cos}(\omega+\nu), \tag{15}$$

$$\dot{\Omega} = \frac{W}{h\,S}\, r\,\mathrm{Sin}(\omega+\nu), \tag{16}$$

$$\dot{\omega} = \frac{h}{\mu\, e}\left[-R\,\mathrm{Cos}\nu + T\left(\mathrm{Sin}\nu + \frac{1}{\sqrt{1-e^2}}\,\mathrm{Sin}E\right)\right] - C\,\dot{\Omega}, \tag{17}$$

where E, n, and h are the eccentric anomaly, the mean motion and specific angular momentum, respectively. The dot (.) over the elements means the rate of change of the element with respect to the time.

3.1. RC Perturbation. Now, the perturbations of the RC on the orbital elements can be calculated using equations (3)–(5) in equations (13)–(17). After mathematical manipulations, we get

$$\therefore \dot{a}_{\mathrm{RC}} = \sum_{j=1}^{3}\sum_{i=2}^{4}\Phi^i\left(G^C_{j,i}\,\mathrm{Cos}j\,\nu + G^S_{j,i}\,\mathrm{Sin}j\nu\right) + G_0\Phi^3, \tag{18}$$

$$\therefore \dot{e}_{\mathrm{RC}} = \sum_{j=0}^{3}\sum_{i=2}^{4}\Phi^i\left(H^C_{j,i}\,\mathrm{Cos}j\,\nu + H^S_{j,i}\,\mathrm{Sin}j\,\nu\right), \tag{19}$$

$$\therefore \dot{I}_{\mathrm{RC}} = \sum_{j=0}^{3}\sum_{i=2}^{3}\Phi^i\left(K^C_{j,i}\,\mathrm{Cos}j\,\nu + K^S_{j,i}\,\mathrm{Sin}j\,\nu\right), \tag{20}$$

$$\therefore \dot{\Omega}_{\mathrm{RC}} = \sum_{j=0}^{3}\sum_{i=2}^{3}\Phi^i\left(Q^C_{j,i}\,\mathrm{Cos}j\,\nu + Q^S_{j,i}\,\mathrm{Sin}j\,\nu\right), \tag{21}$$

$$\therefore \dot{\omega}_{\mathrm{RC}} = \sum_{j=0}^{3}\sum_{i=2}^{4}\Phi^i\left(\chi^C_{j,i}\,\mathrm{Cos}j\,\nu + \chi^S_{j,i}\,\mathrm{Sin}j\,\nu\right), \tag{22}$$

where

$$G^C_{1,2} = \frac{ea^2}{h}A^S_{1,2},$$

$$G^C_{1,3} = \frac{a^2}{h}\left(2B^C_1 + eA^S_{1,3}\right),$$

$$G^C_{1,4} = \frac{ea^2}{h}\left(B^C_2 + A^S_{1,4}\right),$$

$$G^C_{2,2} = G^S_{2,2} = 0,$$

$$G^C_{2,3} = G_0 = \frac{a^2}{h}eB^C_1,$$

$$G^C_{2,4} = 2\frac{a^2}{h}B^C_2,$$

$$G^C_{3,2} = -\frac{ea^2}{h}A^S_{1,2},$$

$$G^C_{3,3} = -\frac{ea^2}{h}A^S_{1,3},$$

$$G^C_{3,4} = \frac{ea^2}{h}\left(B^C_2 - A^S_{1,4}\right),$$

$$G^S_{1,2} = \frac{ea^2}{h}\left(2A^C_{0,2} - A^C_{1,2}\right),$$

$$G^S_{1,3} = \frac{a^2}{h}\left(2B^S_1 + 2eA^C_{0,3} - eA^C_{1,3}\right),$$

$$G^S_{1,4} = \frac{ea^2}{h}\left(B^S_2 + 2A^C_{0,4} - A^C_{1,4}\right),$$

$$G^S_{2,3} = \frac{ea^2}{h}B^S_1,$$

$$G^S_{2,4} = 2\frac{a^2}{h}B^S_2,$$

$$G^S_{3,2} = \frac{ea^2}{h}A^C_{1,2},$$

$$G^S_{3,3} = \frac{ea^2}{h}A^C_{1,3},$$

$$G^S_{3,4} = \frac{ea^2}{h}\left(A^C_{1,4} + B^S_2\right),$$

$$H^C_{0,2} = H^C_{0,3} = H^C_{2,2} = H^C_{2,3} = \xi B^C_1,\ H^C_{0,4} = 0,\ H^C_{1,2} = \xi A^S_{1,2},$$

$$H^C_{1,3} = \xi\left(A^S_{1,3} + 2eB^C_1 + B^C_2\right),$$

$$H^C_{1,4} = \xi\left(B^C_2 + A^S_{1,4}\right),$$

$$H^C_{2,4} = 2e\,\xi B^C_2,$$

$$H^C_{3,2} = -\xi A^S_{1,2},$$

$$H^C_{3,3} = \xi\left(B^C_2 - A^S_{1,3}\right),$$

$$H^C_{3,4} = \xi\left(B^C_2 - A^S_{1,4}\right),$$

$$H^S_{1,2} = \xi\left(2A^C_{0,2} - A^C_{1,2}\right),$$

$$H^S_{1,3} = \xi\left(2A^C_{0,3} - A^C_{1,3} + 2eB^S_1 + B^S_2\right),$$

$$H^S_{1,4} = \xi\left(B^S_2 - A^C_{1,4} + 2A^C_{0,4}\right),$$

$$H^S_{2,2} = H^S_{2,3} = \xi B^S_1,$$

$$H^S_{2,4} = 2\,e\,\xi B^S_2,$$

$$H^S_{3,2} = \xi A^C_{1,2},$$

$$H^S_{3,3} = \xi\left(A^C_{1,3} + B^S_2\right),$$

$$H^S_{3,4} = \xi\left(B^S_2 + A^C_{1,4}\right),$$

$$K^C_{1,3} = K^S_{1,3} = K^C_{3,3} = K^S_{3,3} = 0,$$

$$K^C_{0,2} = \frac{a}{2h}\left(C^C_1 \mathrm{Cos}\omega - C^S_1 \mathrm{Sin}\omega\right),$$

$$K^C_{0,3} = \frac{a}{2h}\left(C^C_3 \mathrm{Cos}\omega - C^S_3 \mathrm{Sin}\omega\right),$$

$$K^C_{1,2} = \frac{a}{2h}\left[\left(2C^C_0 + C^C_2\right)\mathrm{Cos}\omega - C^S_2 \mathrm{Sin}\omega\right],$$

$$K^C_{2,2} = \frac{a}{2h}\left(C^C_1 \mathrm{Cos}\omega + C^S_1 \mathrm{Sin}\omega\right),$$

$$K^C_{2,3} = \frac{a}{2h}\left(C^C_3 \mathrm{Cos}\omega + C^S_3 \mathrm{Sin}\omega\right),$$

$$K^C_{3,2} = \frac{a}{2h}\left(C^C_2 \mathrm{Cos}\omega + C^S_2 \mathrm{Sin}\omega\right),$$

$$K^S_{1,2} = \frac{a}{2h}\left[-\left(2C^C_0 - C^C_2\right)\mathrm{Sin}\omega + C^S_2 \mathrm{Cos}\omega\right],$$

$$K^S_{2,2} = \frac{a}{2h}\left(-C^C_1 \mathrm{Sin}\omega + C^S_1 \mathrm{Cos}\omega\right),$$

$$K^S_{2,3} = \frac{a}{2h}\left(-C^C_3 \mathrm{Sin}\omega + C^S_3 \mathrm{Cos}\omega\right),$$

$$K^S_{3,2} = \frac{a}{2h}\left(-C^C_2 \mathrm{Sin}\omega + C^S_2 \mathrm{Cos}\omega\right),$$

$$Q^C_{1,3} = Q^S_{1,3} = Q^C_{3,3} = Q^S_{3,3} = 0,$$

$$Q^C_{0,2} = \frac{a}{2hS}\left(C^C_1 \mathrm{Sin}\omega + C^S_1 \mathrm{Cos}\omega\right),$$

$$Q^C_{0,3} = \frac{a}{2hS}\left(C^S_3 \mathrm{Cos}\omega + C^C_3 \mathrm{Sin}\omega\right),$$

$$Q^C_{1,2} = \frac{a}{2hS}\left[\left(2C^C_0 + C^C_2\right)\mathrm{Sin}\omega + C^S_2 \mathrm{Cos}\omega\right],$$

$$Q^C_{2,2} = \frac{a}{2hS}\left(C^C_1 \mathrm{Sin}\omega - C^S_1 \mathrm{Cos}\omega\right),$$

$$Q^C_{2,3} = \frac{a}{2hS}\left(C^C_3 \mathrm{Sin}\omega - C^S_3 \mathrm{Cos}\omega\right),$$

$$Q^C_{3,2} = \frac{a}{2hS}\left(C^C_2 \mathrm{Sin}\omega - C^S_2 \mathrm{Cos}\omega\right),$$

$$Q^S_{1,2} = \frac{a}{2hS}\left[\left(2C^C_0 - C^C_2\right)\mathrm{Cos}\omega + C^S_2 \mathrm{Sin}\omega\right],$$

$$Q^S_{2,2} = \frac{a}{2hS}\left(C^C_1 \mathrm{Cos}\omega + C^S_1 \mathrm{Sin}\omega\right),$$

$$Q^S_{2,3} = \frac{a}{2hS}\left(C^C_3 \mathrm{Cos}\omega + C^S_3 \mathrm{Sin}\omega\right),$$

$$Q^S_{3,2} = \frac{a}{2hS}\left(C^C_2 \mathrm{Cos}\omega + C^S_2 \mathrm{Sin}\omega\right),$$

$$\chi^C_{0,4} = \chi^C_{2,4} = \chi^S_{2,4} = 0,$$

$$\chi^C_{0,2} = -\frac{a\,C}{2\,h\,S}\left(C^C_1 \mathrm{Sin}\omega + C^S_1 \mathrm{Cos}\omega\right) + \frac{h}{2e\mu}\frac{\eta^2 B^S_1}{(1-e^2)},$$

$$\chi^C_{0,3} = -\frac{a\,C}{2\,h\,S}\left(C^C_3 \mathrm{Sin}\omega + C^S_3 \mathrm{Cos}\omega\right) + \frac{h\,B^S_1}{2e\mu},$$

$$\chi^C_{1,2} = -\frac{a\,C}{2\,h\,S}\left[\left(2C^C_0 + C^C_2\right)\mathrm{Sin}\omega + C^S_2 \mathrm{Cos}\omega\right] - \frac{h}{2e\mu}\left(2A^C_{0,2} + A^C_{1,2}\right),$$

$$\chi^C_{1,3} = \frac{h}{2e\mu}\left(\eta^2 B^S_2 - 2A^C_{0,3} - A^C_{1,3}\right),$$

$$\chi^C_{1,4} = \frac{h}{2e\mu}\left(B^S_2 - 2A^C_{0,4} - A^C_{1,4}\right),$$

$$\chi^C_{2,2} = -\frac{a\,C}{2\,h\,S}\left(C^C_1 \mathrm{Sin}\omega - C^S_1 \mathrm{Cos}\omega\right) - \frac{h}{2e\mu}\eta^2 B^S_1,$$

$$\chi^C_{2,3} = -\frac{a\,C}{2\,h\,S}\left(C^C_3 \mathrm{Sin}\omega - C^S_3 \mathrm{Cos}\omega\right) - \frac{h\,B^S_1}{2e\mu},$$

$$\chi^C_{3,2} = -\frac{a\,C}{2\,h\,S}\left(C^C_2 \mathrm{Sin}\omega - C^S_2 \mathrm{Cos}\omega\right) - \frac{h\,A^C_{1,2}}{2e\mu},$$

$$\chi^C_{3,3} = -\frac{h}{2e\mu}\left(\eta^2 B^S_2 + A^C_{1,3}\right),$$

$$\chi^C_{3,4} = -\frac{h}{2e\mu}\left(B^S_2 + A^C_{1,4}\right),$$

$$\chi^S_{1,2} = -\frac{a\,C}{2\,h\,S}\left[\left(2C^C_0 - C^C_2\right)\mathrm{Cos}\omega + C^S_2 \mathrm{Sin}\omega\right] - \frac{h}{2e\mu}A^S_{1,2},$$

$$\chi^S_{1,3} = -\frac{h}{2e\mu}\left(\eta^2 B^C_2 + A^S_{1,3}\right),$$

$$\chi^S_{1,4} = -\frac{h}{2e\mu}\left(B^C_2 + A^S_{1,4}\right),$$

$$\chi^S_{2,2} = -\frac{a\,C}{2\,h\,S}\left(C^C_1 \mathrm{Cos}\omega + C^S_1 \mathrm{Sin}\omega\right) + \frac{h}{2e\mu}\eta^2 B^C_1,$$

$$\chi^S_{2,3} = -\frac{a\,C}{2\,h\,S}\left(C^C_3 \mathrm{Cos}\omega + C^S_3 \mathrm{Sin}\omega\right) + \frac{h\,B^C_1}{2e\mu},$$

$$\chi^S_{3,2} = -\frac{a\,C}{2\,h\,S}\left(C^C_2 \mathrm{Cos}\omega + C^S_2 \mathrm{Sin}\omega\right) - \frac{h\,A^S_{1,2}}{2e\mu},$$

$$\chi^S_{3,3} = \frac{h}{2e\mu}\left(\eta^2 B^C_2 - A^S_{1,3}\right),$$

$$\chi^S_{3,4} = \frac{h}{2e\mu}\left(B^C_2 - A^S_{1,4}\right),$$

$$\eta = \frac{1}{\sqrt{1-e^2}},$$

$$\xi = \frac{1}{2\eta n a}. \qquad (23)$$

3.2. LA Perturbation. Again, the perturbation of the LA is obtained for each orbital element by applying equations (8)–(10) into equations (13)–(17). After calculating the required derivatives and mathematical operations, we get

$$\therefore \dot{a}_{LA}=\sum_{j=0}^{3}\sum_{i=2}^{4}\Phi^{i}\left(\widetilde{G^{C}_{j,i}}\,\mathrm{Cos}j\,\nu+\widetilde{G^{S}_{j,i}}\,\mathrm{Sin}j\,\nu\right), \tag{24}$$

$$\therefore \dot{e}_{LA}=\sum_{j=0}^{3}\sum_{i=1}^{4}\Phi^{i}\left(\widetilde{H^{C}_{j,i}}\,\mathrm{Cos}j\,\nu+\widetilde{H^{S}_{j,i}}\,\mathrm{Sin}j\,\nu\right), \tag{25}$$

$$\therefore \dot{I}_{LA}=\sum_{j=0}^{3}\sum_{i=1}^{3}\Phi^{i}\left(\widetilde{K^{C}_{j,i}}\,\mathrm{Cos}j\,\nu+\widetilde{K^{S}_{j,i}}\,\mathrm{Sin}j\,\nu\right), \tag{26}$$

$$\therefore \dot{\Omega}_{LA}=\sum_{j=0}^{3}\sum_{i=1}^{3}\Phi^{i}\left(\widetilde{Q^{C}_{j,i}}\,\mathrm{Cos}j\,\nu+\widetilde{Q^{S}_{j,i}}\,\mathrm{Sin}j\,\nu\right), \tag{27}$$

$$\therefore \dot{\omega}_{LA}=\sum_{j=0}^{3}\sum_{i=1}^{4}\Phi^{i}\left(\widetilde{\chi^{C}_{j,i}}\,\mathrm{Cos}j\,\nu+\widetilde{\chi^{S}_{j,i}}\,\mathrm{Sin}j\,\nu\right), \tag{28}$$

where

$$\widetilde{G^{C}_{0,2}}=\widetilde{G^{C}_{0,4}}=\widetilde{G^{C}_{2,4}}=\widetilde{G^{C}_{3,3}}=\widetilde{G^{S}_{2,4}}=\widetilde{G^{S}_{3,3}}=0,$$

$$\widetilde{G^{C}_{0,3}}=\widetilde{G^{C}_{2,3}}=\frac{ea^{2}}{h}F^{C}_{1},$$

$$\widetilde{G^{C}_{1,2}}=\frac{ea^{2}}{h}\left(D^{S}_{1,0}+F^{C}_{0}\right),$$

$$\widetilde{G^{C}_{1,3}}=\frac{2a^{2}}{h}F^{C}_{1},$$

$$\widetilde{G^{C}_{1,4}}=\frac{ea^{2}}{h}D^{S}_{1,1},$$

$$\widetilde{G^{C}_{2,2}}=\frac{2a^{2}}{h}F^{C}_{0},$$

$$\widetilde{G^{C}_{3,2}}=\frac{ea^{2}}{h}\left(F^{C}_{0}-D^{S}_{1,0}\right),$$

$$\widetilde{G^{C}_{3,4}}=-\frac{e\,a^{2}}{h}D^{S}_{1,1},$$

$$\widetilde{G^{S}_{1,2}}=\frac{ea^{2}}{h}\left(-F^{S}_{0}+2D^{C}_{0,0}-D^{C}_{1,0}\right),$$

$$\widetilde{G^{S}_{1,3}}=-\frac{2a^{2}}{h}F^{S}_{1},$$

$$\widetilde{G^{S}_{1,4}}=\frac{ea^{2}}{h}\left(2D^{C}_{0,1}-D^{C}_{1,1}\right),$$

$$\widetilde{G^{S}_{2,2}}=-\frac{2a^{2}}{h}F^{S}_{0},$$

$$\widetilde{G^{S}_{2,3}}=-\frac{ea^{2}}{h}F^{S}_{1},$$

$$\widetilde{G^{S}_{3,2}}=\frac{ea^{2}}{h}\left(-F^{S}_{0}+D^{C}_{1,0}\right),$$

$$\widetilde{G^{S}_{3,4}}=\frac{ea^{2}}{h}D^{C}_{1,1},$$

$$\widetilde{H^{C}_{0,1}}=\widetilde{H^{C}_{0,4}}=\widetilde{H^{C}_{2,1}}=\widetilde{H^{C}_{2,4}}=\widetilde{H^{C}_{3,3}}=\widetilde{H^{S}_{2,1}}=\widetilde{H^{S}_{2,4}}=\widetilde{H^{S}_{3,3}}=0,$$

$$\widetilde{H^{C}_{1,1}}=\widetilde{H^{C}_{3,1}}=\xi F^{C}_{0},$$

$$\widetilde{H^{C}_{0,2}}=\widetilde{H^{C}_{0,3}}=\widetilde{H^{C}_{2,3}}=\xi F^{C}_{1},$$

$$\widetilde{H^{C}_{1,2}}=\xi\left(F^{C}_{0}+D^{S}_{1,0}\right),$$

$$\widetilde{H^{C}_{1,3}}=2\,e\,\xi F^{C}_{1},$$

$$\widetilde{H^{C}_{1,4}}=\xi D^{S}_{1,1}=-\widetilde{H^{C}_{3,4}},$$

$$\widetilde{H^{C}_{2,2}}=\xi\left(2eF^{C}_{0}+F^{C}_{1}\right),$$

$$\widetilde{H^{C}_{3,2}}=\xi\left(F^{C}_{0}-D^{S}_{1,0}\right),$$

$$\widetilde{H^{S}_{1,1}}=\widetilde{H^{S}_{3,1}}=-\xi F^{S}_{0},$$

$$\widetilde{H^{S}_{1,2}}=\xi\left(2D^{C}_{0,0}-F^{S}_{0}-D^{C}_{1,0}\right),$$

$$\widetilde{H^{S}_{1,3}}=-2\,e\,\xi F^{S}_{1},$$

$$\widetilde{H^{S}_{1,4}}=\xi\left(2D^{C}_{0,1}-D^{C}_{1,1}\right),$$

$$\widetilde{H^{S}_{2,2}}=-\xi\left(2eF^{S}_{0}+F^{S}_{1}\right),$$

$$\widetilde{H^{S}_{2,3}}=-\xi F^{S}_{1},$$

$$\widetilde{H^{S}_{3,2}}=\xi\left(D^{C}_{1,0}-F^{S}_{0}\right),$$

$$\widetilde{H^{S}_{3,4}}=\xi D^{C}_{1,1},$$

$$\widetilde{K^{C}_{0,2}}=\widetilde{K^{C}_{1,1}}=\widetilde{K^{C}_{1,3}}=\widetilde{K^{C}_{2,2}}=\widetilde{K^{C}_{3,1}}=\widetilde{K^{C}_{3,3}}=\widetilde{K^{S}_{1,1}}=\widetilde{K^{S}_{1,3}}$$
$$=\widetilde{K^{S}_{2,2}}=\widetilde{K^{S}_{3,1}}=\widetilde{K^{S}_{3,3}}=0,$$

$$\widetilde{K^{C}_{0,1}}=\frac{a}{2h}\left(G^{C}_{1}\mathrm{Cos}\omega-G^{S}_{1}\mathrm{Sin}\omega\right),$$

$$\widetilde{K^{C}_{0,3}}=\frac{a}{2h}\left(G^{C}_{2}\mathrm{Cos}\omega-G^{S}_{2}\mathrm{Sin}\omega\right),$$

$$\widetilde{K^{C}_{1,2}}=\frac{a}{2h}\left[\left(2{G'}^{C}_{0}+G'_{1}C\right)\mathrm{Cos}\omega-G'_{1}S\mathrm{Sin}\omega\right],$$

$$\widetilde{K^{C}_{2,1}}=\frac{a}{2h}\left(G^{C}_{1}\mathrm{Cos}\omega+G^{S}_{1}\mathrm{Sin}\omega\right),$$

$$\widetilde{K^{C}_{2,3}}=\frac{a}{2h}\left(G^{C}_{2}\mathrm{Cos}\omega+G^{S}_{2}\mathrm{Sin}\omega\right),$$

$$\widetilde{K_{3,2}^{C}} = \frac{a}{2h}\left(G_1'C\mathrm{Cos}\omega + G_1'S\mathrm{Sin}\omega\right),$$

$$\widetilde{K_{1,2}^{S}} = \frac{a}{2h}\left[-\left(2G_0'C - G_1'C\right)\mathrm{Sin}\omega + G_1'S\mathrm{Cos}\omega\right],$$

$$\widetilde{K_{2,1}^{S}} = \frac{a}{2h}\left(-G_1^C\mathrm{Sin}\omega + G_1^S\mathrm{Cos}\omega\right),$$

$$\widetilde{K_{2,3}^{S}} = \frac{a}{2h}\left(-G_2^C\mathrm{Sin}\omega + G_2^S\mathrm{Cos}\omega\right),$$

$$\widetilde{K_{3,2}^{S}} = \frac{a}{2h}\left(-G_1'C\mathrm{Sin}\omega + G_1'S\mathrm{Cos}\omega\right),$$

$$\widetilde{Q_{0,2}^{C}} = \widetilde{Q_{1,1}^{C}} = \widetilde{Q_{1,3}^{C}} = \widetilde{Q_{2,2}^{C}} = \widetilde{Q_{3,1}^{C}} = \widetilde{Q_{3,3}^{C}} = \widetilde{Q_{1,1}^{S}} = \widetilde{Q_{1,3}^{S}} = \widetilde{Q_{2,2}^{S}} = \widetilde{Q_{3,1}^{S}} = \widetilde{Q_{3,3}^{S}} = 0,$$

$$\widetilde{Q_{0,1}^{C}} = \frac{a}{2hS}\left(G_1^C\mathrm{Sin}\omega + G_1^S\mathrm{Cos}\omega\right),$$

$$\widetilde{Q_{0,3}^{C}} = \frac{a}{2hS}\left(G_2^C\mathrm{Sin}\omega + G_2^S\mathrm{Cos}\omega\right),$$

$$\widetilde{Q_{1,2}^{C}} = \frac{a}{2hS}\left[\left(2G'^{C}_{0} + G'^{C}_{1}\right)\mathrm{Sin}\omega + G'^{S}_{1}\mathrm{Cos}\omega\right],$$

$$\widetilde{Q_{2,1}^{C}} = \frac{a}{2hS}\left(G_1^C\mathrm{Sin}\omega - G_1^S\mathrm{Cos}\omega\right),$$

$$\widetilde{Q_{2,3}^{C}} = \frac{a}{2hS}\left(G_2^C\mathrm{Sin}\omega - G_2^S\mathrm{Cos}\omega\right),$$

$$\widetilde{Q_{3,2}^{C}} = \frac{a}{2hS}\left(G'^{C}_{1}\mathrm{Sin}\omega - G'^{S}_{1}\mathrm{Cos}\omega\right),$$

$$\widetilde{Q_{1,2}^{S}} = \frac{a}{2hS}\left[\left(2G'^{C}_{0} - G'^{C}_{1}\right)\mathrm{Cos}\omega + G'^{S}_{1}\mathrm{Sin}\omega\right],$$

$$\widetilde{Q_{2,1}^{S}} = \frac{a}{2hS}\left(G_1^C\mathrm{Cos}\omega + G_1^S\mathrm{Sin}\omega\right),$$

$$\widetilde{Q_{2,3}^{S}} = \frac{a}{2hS}\left(G_2^C\mathrm{Cos}\omega + G_2^S\mathrm{Sin}\omega\right),$$

$$\widetilde{Q_{3,2}^{S}} = \frac{a}{2hS}\left(G'^{C}_{1}\mathrm{Cos}\omega + G'^{S}_{1}\mathrm{Sin}\omega\right),$$

$$\widetilde{\chi_{0,4}^{C}} = \widetilde{\chi_{1,3}^{C}} = \widetilde{\chi_{2,4}^{C}} = \widetilde{\chi_{3,3}^{C}} = \widetilde{\chi_{1,3}^{S}} = \widetilde{\chi_{2,4}^{S}} = \widetilde{\chi_{3,3}^{S}} = 0,$$

$$\widetilde{\chi_{0,1}^{C}} = -\frac{aC}{2hS}\left(G_1^C\mathrm{Sin}\omega + G_1^S\mathrm{Cos}\omega\right),$$

$$\widetilde{\chi_{0,3}^{C}} = -\frac{aC}{2hS}\left(G_2^C\mathrm{Sin}\omega + G_2^S\mathrm{Cos}\omega\right) - \frac{h}{2e\mu}F_1^S,$$

$$\widetilde{\chi_{1,2}^{C}} = -\frac{aC}{2hS}\left[\left(2G'^{C}_{0} + G'^{C}_{1}\right)\mathrm{Sin}\omega + G'^{S}_{1}\mathrm{Cos}\omega\right] + \frac{h}{2e\mu}\left(-F_0^S - 2D_{0,0}^C - D_{1,0}^C\right),$$

$$\widetilde{\chi_{1,4}^{C}} = \frac{h}{2e\mu}\left(-2D_{0,1}^C - D_{1,1}^C\right),$$

$$\widetilde{\chi_{2,1}^{C}} = -\frac{aC}{2hS}\left(G_1^C\mathrm{Sin}\omega - G_1^S\mathrm{Cos}\omega\right),$$

$$\widetilde{\chi_{2,2}^{C}} = \frac{\eta^2 h}{2e\mu}F_1^S = -\widetilde{\chi_{0,2}^{C}},$$

$$\widetilde{\chi_{2,3}^{C}} = -\frac{aC}{2hS}\left(G_2^C\mathrm{Sin}\omega - G_2^S\mathrm{Cos}\omega\right) + \frac{h}{2e\mu}F_1^S,$$

$$\widetilde{\chi_{3,1}^{C}} = \frac{\eta^2 h}{2e\mu}F_0^S = -\widetilde{\chi_{1,1}^{C}},$$

$$\widetilde{\chi_{3,2}^{C}} = -\frac{aC}{2hS}\left(G'^{C}_{1}\mathrm{Sin}\omega - G'^{S}_{1}\mathrm{Cos}\omega\right) + \frac{h}{2e\mu}\left(F_0^S - D_{1,0}^C\right),$$

$$\widetilde{\chi_{3,4}^{C}} = -\frac{h}{2e\mu}D_{1,1}^C,$$

$$\widetilde{\chi_{1,1}^{S}} = -\frac{\eta^2 h}{2e\mu}F_0^C,$$

$$\widetilde{\chi_{1,2}^{S}} = -\frac{aC}{2hS}\left(2G'^{C}_{0}\mathrm{Cos}\omega - G'^{C}_{1}\mathrm{Cos}\omega + G'^{S}_{1}\mathrm{Sin}\omega\right) - \frac{h}{2e\mu}\left(F_0^C + D_{1,0}^S\right),$$

$$\widetilde{\chi_{1,4}^{S}} = -\frac{h}{2e\mu}D_{1,1}^S,$$

$$\widetilde{\chi_{2,1}^{S}} = -\frac{aC}{2hS}\left(G_1^C\mathrm{Cos}\omega + G_1^S\mathrm{Sin}\omega\right),$$

$$\widetilde{\chi_{2,2}^{S}} = \frac{\eta^2 h}{2e\mu}F_1^C,$$

$$\widetilde{\chi_{2,3}^{S}} = -\frac{aC}{2hS}\left(G_2^C\mathrm{Cos}\omega + G_2^S\mathrm{Sin}\omega\right) + \frac{h}{2e\mu}F_1^C,$$

$$\widetilde{\chi_{3,1}^{S}} = \frac{\eta^2 h}{2e\mu}F_0^C,$$

$$\widetilde{\chi_{3,2}^{S}} = -\frac{aC}{2hS}\left(G'^{C}_{1}\mathrm{Cos}\omega + G'^{S}_{1}\mathrm{Sin}\omega\right) + \frac{h}{2e\mu}\left(F_0^C - D_{1,0}^S\right),$$

$$\widetilde{\chi_{3,4}^{S}} = -\frac{h}{2e\mu}D_{1,1}^S. \tag{29}$$

4. Short-Period Orbital Perturbations

The short-period perturbation is the perturbation during one revolution of the body in its orbit. The true anomaly, ν, is used as short-period variable. Since this kind of perturbations is fast periodic and has no cumulative effects on the orbit, we average over the fast periodic variable ν to get rid of this kind of perturbations and to keep only the long-period and secular terms.

The average of the function, $f(x)$, over a complete period for the variable x is defined as

$$\langle f(x)\rangle_x = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx. \tag{30}$$

Table 1: The parameters for the Earth.

$B\left(\frac{\text{kg m}^3}{s\,C}\right)$	$\mu\left(\frac{\text{m}^3}{s^2}\right)$	$\omega_0\left(\frac{\text{Rad}}{s}\right)$	$G\left(\frac{\text{m}^3}{s^2\text{kg}}\right)$
8×10^{15}	3.98601×10^{14}	7.272205217×10^{-5}	6.6732×10^{-11}

The relation between the average over the true anomaly v and the mean anomaly M is

$$dv = \eta^{-1}\,\Phi^2 dM. \tag{31}$$

Therefore, we can apply the last transformation for our short-period averaging process, so

$$\langle f(v)\rangle_v = \frac{1}{2\pi}\int_0^{2\pi} f(v)dv = \frac{1}{2\pi}\,\eta^{-1}\int_0^{2\pi}\Phi^2 f(v)dM = \eta^{-1}\langle \Phi^2 f(v)\rangle_M. \tag{32}$$

4.1. Averaged Perturbation of RC. To calculate the average variations in the orbital elements due to the RC, we apply the definition above in equations (18)–(22). The results are

$$\therefore\langle \dot a_{\text{RC}}\rangle_v = \frac{3e^2 m\,S^2 R_0^4\omega_0^2}{140\,a^2(1-e^2)^4\,h}\left[5e^2\text{Cos}2\omega + \left(92+41e^2\right)\text{Sin}2\omega\right], \tag{33}$$

$$\therefore\langle \dot e_{\text{RC}}\rangle_v = \frac{3e\left(34+35e^2\right)m\,S^2\,R_0^4\,\omega_0^2}{140\,a^5\,(1-e^2)^{9/2}\,n}\,\text{Sin}2\omega, \tag{34}$$

$$\therefore\left\langle \dot I_{\text{RC}}\right\rangle_v = -\frac{e^2 m\,S\,R_0^2\,\omega_0\left[-56\,c\sqrt{mp}+9CR_0^2\omega_0\right]}{140\,p^3 h}\,\text{Sin}2\omega, \tag{35}$$

$$\begin{aligned}\therefore\left\langle \dot\Omega_{\text{RC}}\right\rangle_v = \frac{mR_0^2\omega_0}{140\,p^3 h}&\left[\left(56\,c\sqrt{m\,p}-6\,C\,R_0^2\omega_0\right)\left(2+3e^2\right)\right.\\&\left.+\left(56\,c\sqrt{m\,p}+9\,C\,R_0^2\omega_0\right)e^2\text{Cos}\,2\omega\right],\end{aligned} \tag{36}$$

$$\therefore\langle \dot\omega_{\text{RC}}\rangle_v = \sum_{i=0}^{1}\Psi_i\text{Cos}2\,i\,\omega, \tag{37}$$

where

$$\begin{aligned}\Psi_0 = \frac{m}{280\,p^4\sqrt{\mu}}&\left[70\,c^2 m\sqrt{p}\left(8a+4ae^2-8p+e^2p\right)\right.\\&+R_0^2\omega_0\left(-336acC\sqrt{m}+56acCe^2\sqrt{m}+112acCe^4\sqrt{m}\right.\\&-112cC\sqrt{mp}-56cCe^2\sqrt{mp}-448cCp\sqrt{m}\\&\left.-560cCe^2p\sqrt{m}+112\,p^2\sqrt{p}\right)\\&\left.+6\sqrt{p}R_0^4\omega_0^2\left(-4-3e^2+16C^2+15C^2e^{2\cdot}\right)\right],\end{aligned} \tag{38}$$

$$\Psi_1 = \frac{mR_0^2\omega_0}{280\,p^3 h}\left[112cCe^2\sqrt{mp}+\left(204+96e^2-204C^2-114e^2C^2\right)R_0^2\omega_0\right]. \tag{39}$$

4.2. Averaged Perturbation of LA. Again, using the average definition in equations (24)–(28), it yields

$$\therefore\langle \dot a_{LA}\rangle_v = \frac{Be^2 q\,S^2\omega_0}{(1-e^2)^2\,h}\,\text{Sin}2\omega, \tag{40}$$

$$\therefore\langle \dot e_{LA}\rangle_v = \frac{BeqS^2\omega_0}{2\,a\,(1-e^2)\,h}\,\text{Sin}2\omega, \tag{41}$$

$$\therefore\langle \dot I_{LA}\rangle_v = -\frac{Be^2 qS}{8\,a^3(1-e^2)^3}\,\text{Sin}2\omega, \tag{42}$$

$$\begin{aligned}\therefore\left\langle \dot\Omega_{LA}\right\rangle_v = \frac{B\,q\,\eta^2}{8a^3 h}&\left[8a^2\text{C}\,\omega_0 - 2\left(2+3e^2\right)\eta^4 h\right.\\&\left.+e^2\eta^2 h\left(-\frac{2a}{\sqrt{p}}+3\eta^2\right)\text{Cos}2\omega\right],\end{aligned} \tag{43}$$

$$\begin{aligned}\therefore\langle \dot\omega_{LA}\rangle_v = \frac{Bq}{8\,p^3 h}&\left\{2\left[-4p^2\omega_0-2p^2\left(1+C^2\right)\omega_0\right.\right.\\&\left.\left.+5C\left(2+e^2\right)h\right]+\left(5\,C\,h\,e^2+4\,p^2S^2\omega_0\right)\text{Cos}2\omega\right\},\end{aligned} \tag{44}$$

Equations (33)–(37) and equations (40)–(44) are the desired equations to calculate the rate of change of orbital elements needs to be controlled.

5. Controlling Conditions

To find the value and class of the charge per unit mass, the LPEs for both RC and LA must be equal in magnitude and different in sign. Suppose the desired charges per unit mass (q) for orbital elements are q_a, q_e, q_I, q_Ω, and q_ω, respectively. These charges make control over variation in semimajor axis (a), eccentricity (e), inclination (I), argument of ascending node (Ω), and argument of preside (ω), respectively.

Let $\dot x_{\text{RC}}$ and $\dot x_{\text{LA}}$ represent the rate of change of any orbital elements due to RC and LA, respectively, then

$$\langle \dot x_{\text{RC}}\rangle_v + \langle \dot x_{\text{LA}}\rangle_v = 0. \tag{45}$$

5.1. Controlling Orbital Elements. The conditions for the charge per unit mass to control the average variations in the orbital elements due to the considered force (forces)

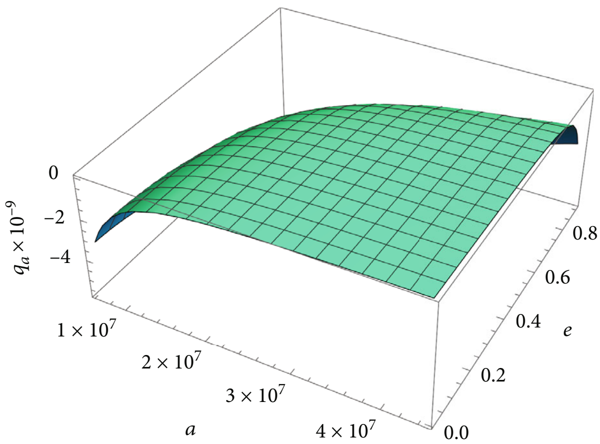

FIGURE 1: q_a values for $\omega = -(\pi/4)$.

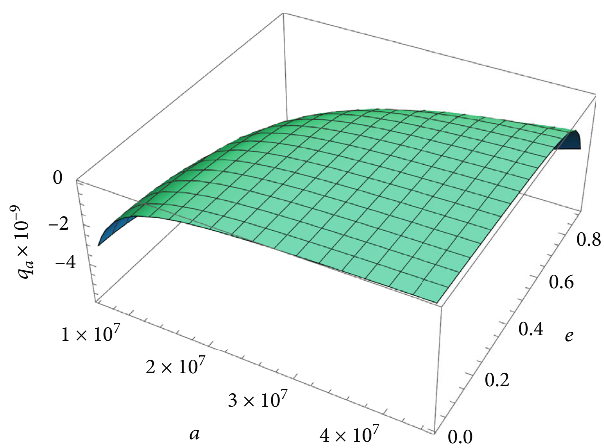

FIGURE 2: q_a values for $\omega = -(\pi/3)$.

can be obtained using equation (45). After solving for q, for each orbital element, we get

$$q_a = -\frac{3\,m\,R_0^4\,\omega_0}{140\,B\,a^2\,(1-e^2)^2}\left(92 + 41e^2 + 5e^2\mathrm{Cot}2\omega\right), \tag{46}$$

$$q_e = -\frac{3m\,R_0^4\omega_0}{70\,B\,a^2(1-e^2)^3}\left(34 + 35e^2\right), \tag{47}$$

$$q_I = \frac{2\,m\,R_0^2\,\omega_0}{35\,B\,h}\left(56\,c\sqrt{pm} - 9\,C\,R_0^2\,\omega_0\right), \tag{48}$$

$$q_\Omega = \frac{2\,m\,R_0^2\,\omega_0}{35\,B\,(1-e^2)^2}\left[\frac{\Psi_0 + \Psi_1\,\mathrm{Cos}2\omega}{\Psi_2 + \Psi_3\,\mathrm{Cos}2\omega}\right], \tag{49}$$

$$q_\omega = -\frac{2\,m}{35\,B}\,\frac{(\Gamma_0 + \Gamma_1\mathrm{Cos}2\omega)}{(\Gamma_2 + \Gamma_3\mathrm{Cos}2\omega)}, \tag{50}$$

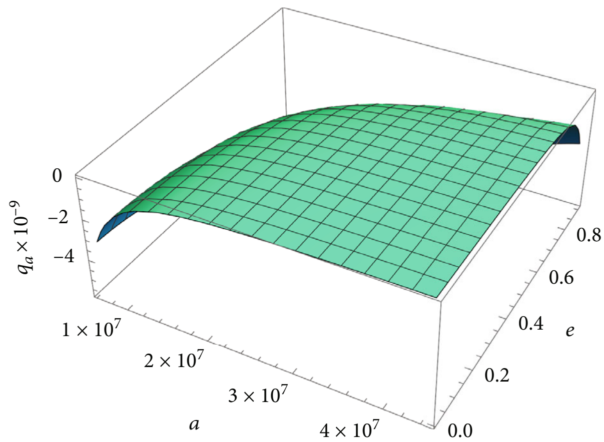

FIGURE 3: q_a values for $\omega = \pi/4$.

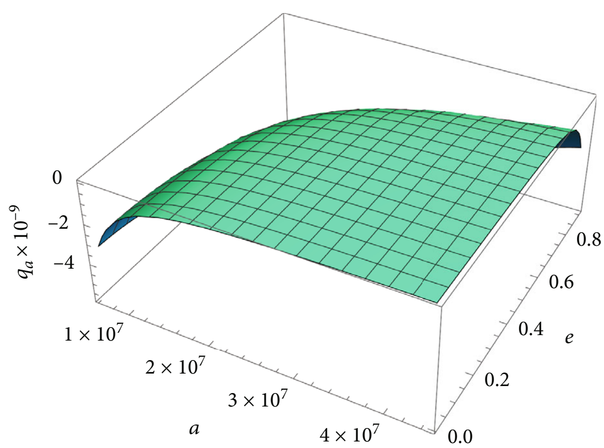

FIGURE 4: q_a values for $\omega = \pi/3$.

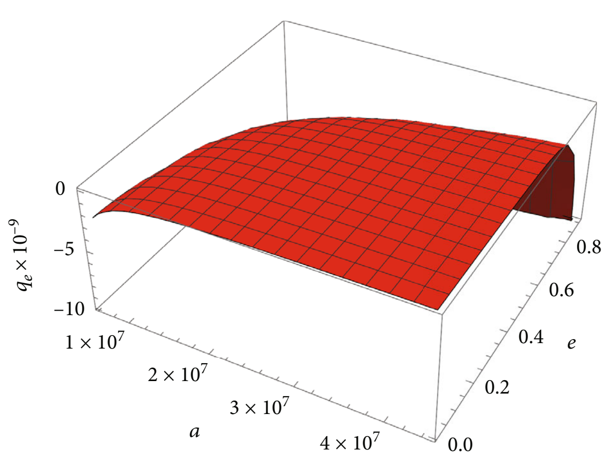

FIGURE 5: q_e values.

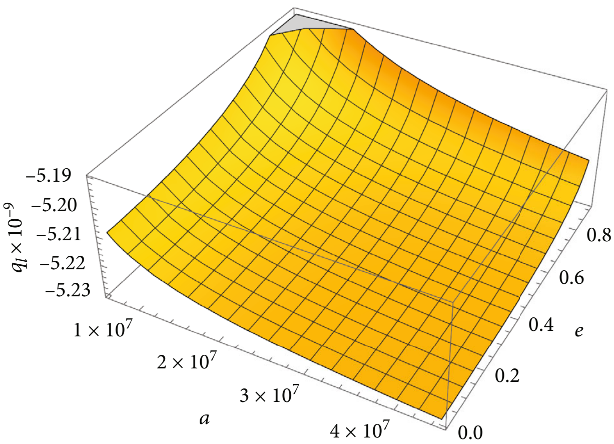

Figure 6: q_I values for $I = -(\pi/4)$.

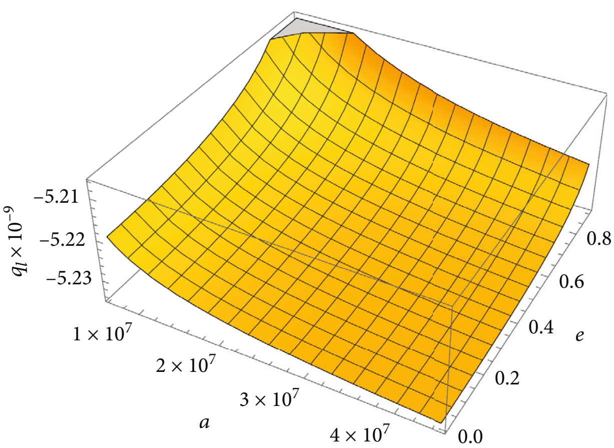

Figure 7: q_I values for $I = -(\pi/3)$.

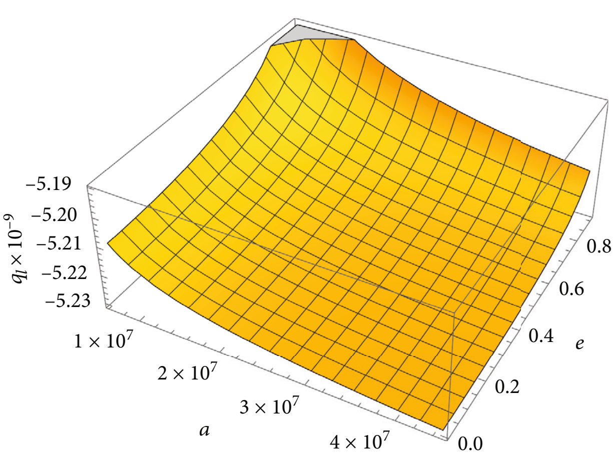

Figure 8: q_I values for $I = \pi/4$.

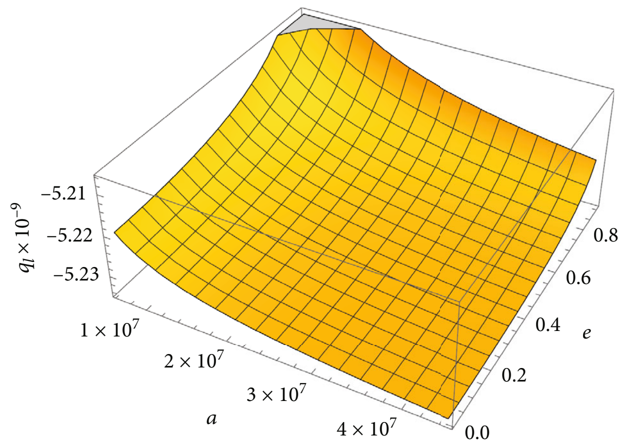

Figure 9: q_I values for $I = \pi/3$.

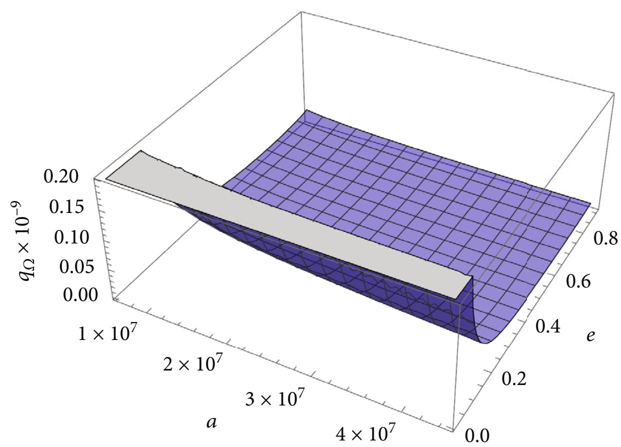

Figure 10: q_Ω values for $\omega = \pi/6, I = \pi/3$.

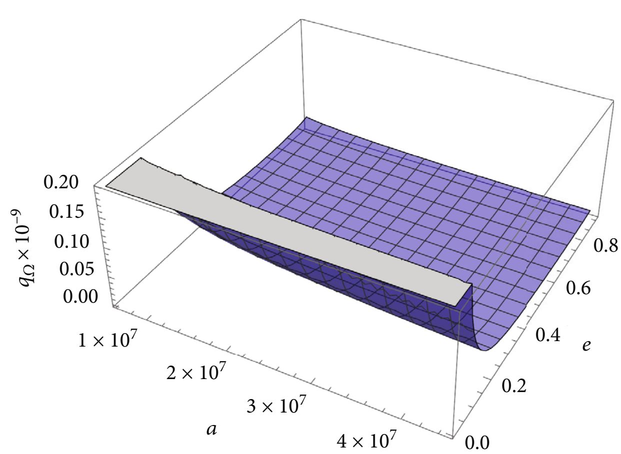

Figure 11: q_Ω values for $\omega = \pi/6, I = 2\pi/5$.

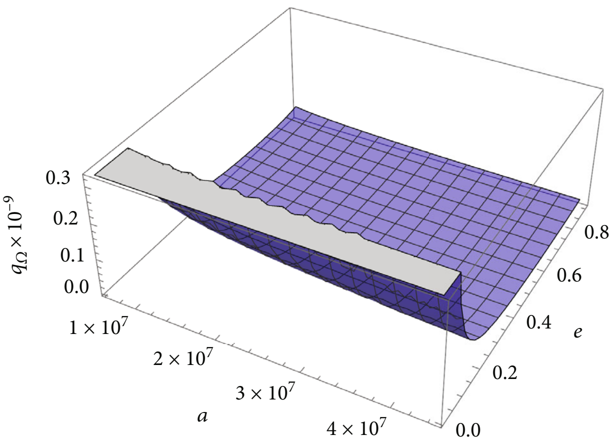

Figure 12: q_Ω values for $\omega = \pi/5, I = \pi/3$.

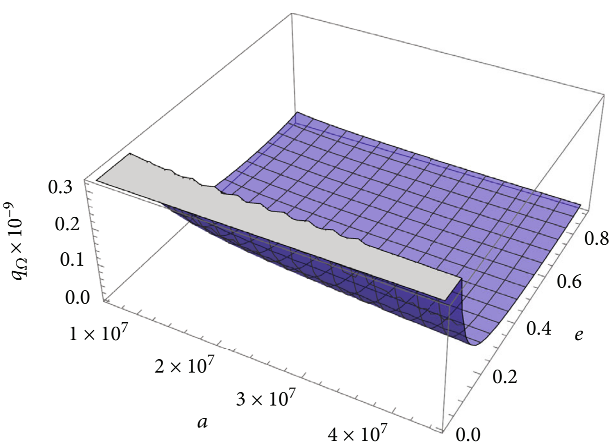

Figure 13: q_Ω values for $\omega = \pi/5, I = 2\pi/5$.

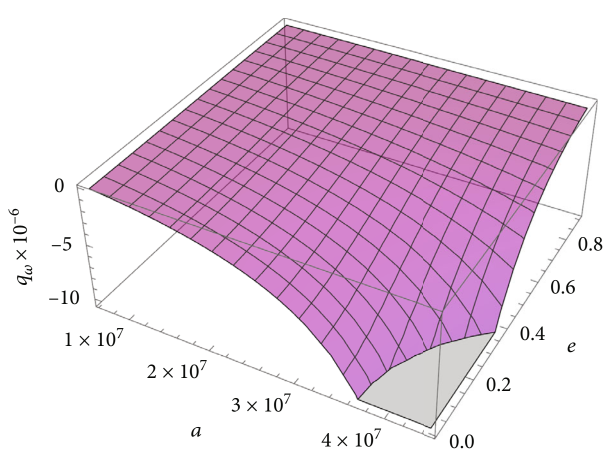

Figure 14: q_ω values for $\omega = \pi/6, I = \pi/6$.

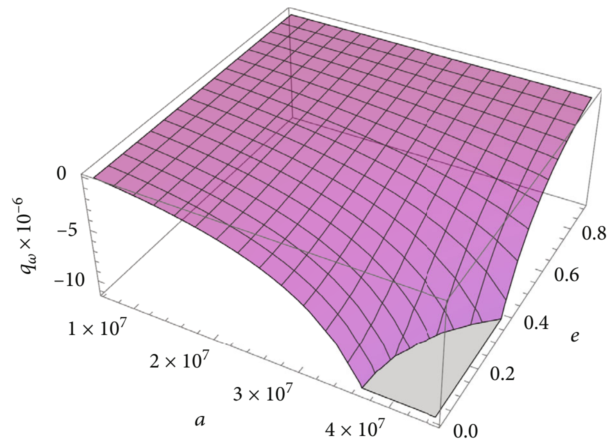

Figure 15: q_ω values for $\omega = \pi/6, I = \pi/4$.

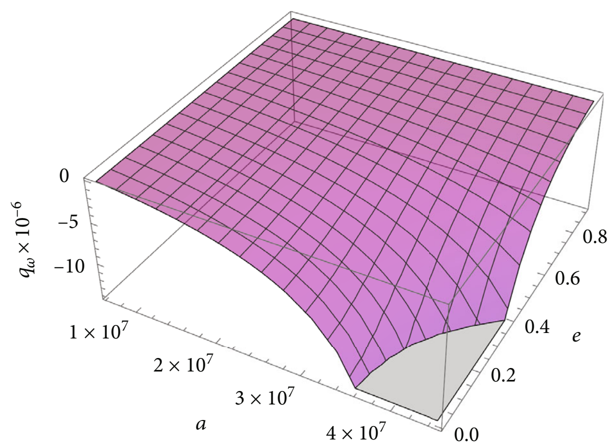

Figure 16: q_ω values for $\omega = \pi/4, I = \pi/6$.

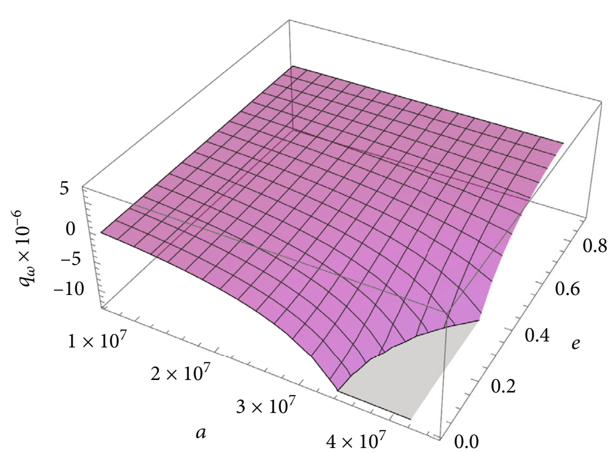

Figure 17: q_ω values for $\omega = \pi/4, I = \pi/4$.

where

$$\Psi_0 = 2\left(28\, c\sqrt{mp} - 3\, CR_0^2\omega_0\right)\left(2 + 3e^2\right),$$

$$\Psi_1 = e^2\left(9\, C R_0^2\, \omega_0 + 56\, c\sqrt{mp}\right),$$

$$\Psi_2 = \frac{2\, h\left(2 + 3e^2\right)}{\left(1 - e^2\right)^2} - 8a^2 C\omega_0,$$

$$\Psi_3 = \frac{e^2\sqrt{\mu}}{\left(1 - e^2\right)^2}\left[2a\left(1 - e^2\right) - 3\sqrt{p}\right],$$

$$\Gamma_0 = 35\, a\, c^2 e^2 m\left(13 - e^2\right) - 28\, R_0^2\, \omega_0\left[c\, C\sqrt{\frac{m}{p}}\left(3a + 13p + 13e^2 p\right) - 2p^2\right] + + 3R_0^4\omega_0^2\left(-4 - 3e^2 + 16C^2 + 15C^2 e^2\right),$$

$$\Gamma_1 = R_0^2\omega_0\left[56cCe^2\sqrt{mp} + 3\, R_0^2\omega_0\left(34 + 16e^2 - 34C^2 - 19e^2 C^2\right)\right],$$

$$\Gamma_2 = 2\left[-2p^2\omega_0\left(3 + C^2\right) + 5C\left(2 + e^2\right) h\right], \quad (51)$$

$$\Gamma_3 = 5\, C\, h\, e^2 + 4\, p^2 S^2 \omega_0. \quad (52)$$

Equations (46)–(50) are the required relations needed to calculate the desired charge per unit mass to balance the effects of RC on the body.

5.2. Controlling the Shape, Size, and Orientation of the Orbit. To control the shape and size of the orbit using the same charge per unit mass, q, we must equate equations (46) and (47) to get the condition (or conditions). So,

$$\omega = \frac{1}{2}\mathrm{Tan}^{-1}\left[\frac{5e^2\left(1 - e^2\right)}{41e^4 + 121e^2 - 24}\right]. \quad (53)$$

The last condition is valid for any kind of orbits except at $e = \{0.0, 0.43191954980459046\}$ at which $\omega = \{0.0, 45^o\}$, respectively. In addition, this condition does not depend on the physical parameters of the problem.

To control the orientation of the orbit in space, we need to use the same charge for balancing the inclination and argument of ascending node. So, $q_I = q_\Omega$, which leads to the conditions:

$$\omega = \pm\frac{1}{2}\mathrm{Cos}^{-1}\left[\frac{2\left[14\beta^2\mu\left(2 + 3e^2\right) - 112\sqrt{\mu}\omega_0 a^2 C\beta\left(1 - 2e^2 + e^4\right) - 3C\beta\sqrt{\mu}R_0\omega_0\left(2 + 3e^2\right) + 18a^2C^2R_0\omega_0^2\left(1 - 2e^2 + e^4\right)\right]}{56e^2\beta^2\mu(2 - \beta) - 9\, Ce^2\beta\sqrt{\mu}R_0^2\omega_0(1 - \beta)}\right], \quad (54)$$

with $\beta = \sqrt{a(1 - e^2)}$, which is valid for all kinds of orbits except at

$$I = \mathrm{Cos}^{-1}\left\{\frac{56\sqrt{a\left(1 - e^2\right)}\left[2 - \sqrt{a\left(1 - e^2\right)}\right]\sqrt{\mu}}{9\left[1 - \sqrt{a\left(1 - e^2\right)}\right]R_0^2\omega_0}\right\}. \quad (55)$$

Conditions (54) can be simplified for the case of polar orbits with $C = 0$ to be

$$\omega = \pm\frac{1}{2}\mathrm{Cos}^{-1}\left[\frac{\left(2 + 3e^2\right)}{2e^2\left(2 - \sqrt{a\left(1 - e^2\right)}\right)}\right], \quad (56)$$

and for equatorial orbits with $C = 1$, to be

$$\omega = \pm\frac{1}{2}\mathrm{Cos}^{-1}\left\{\frac{2\left[\beta\left(2 + 3e^2\right)\left(14\beta\mu - 3\sqrt{\mu}R_0\omega_0\right) + 2\omega_0 a^2\left(1 - 2e^2 + e^4\right)\left(9R_0\omega_0 - 56\sqrt{\mu}\beta\right)\right]}{e^2\beta\left[56\beta\,\mu(2 - \beta) - 9\,\sqrt{\mu}R_0^2\omega_0(1 - \beta)\right]}\right\}. \quad (57)$$

6. Numerical Application

In this section, we examine the charge per unit mass for different orbits of satellites. We use 3D graphs to cover as much possible orbits as we can.

The parameters for the Earth are given in Table 1 [23].

The following figures illustrate the q values (Coulomb/kg) against a (meter) and e (dimensionless) required to balance the relativistic effects on the orbital elements for Earth's artificial satellites for the range of the considered orbits. The range of a is from 1.01 Earth radius to 7 Earth radii which covers all the used altitudes of earth's satellites so far, and the range of e is taken from 0.01 to 0.9.

We address the following notes about the graphs:

(i) All the graphs are independent of Ω as clear from equations (46) to equation (50)

(ii) $q_a$is also independent of I, as clear from equation (46); therefore, Figures 1–4 illustrate $q_a$with a and e which present the principal part of its variation, while four different values of ω are chosen as examples. Since the dependence on ω is periodic with small amplitude, very small differences can be noticed among the four graphs of $q_a$with different values of ω

(iii) q_e is dependent on a and e only as clear from equation (47); therefore, the required values of it are presented only by Figure 5 which illustrates the variation of q_e with a and e

(iv) q_I depends on a, e, and I, (equation (48)); therefore. its values are presented by Figures 6–9, which illustrate the relation between q_I and a and e, for four different values of I

(v) Both q_Ω and q_ω depend on i and ω (equations (49) and (50)). Figures 10–13 show the variations of q_Ω against a and e for different values of I and ω, while Figures 14–17 show the variations of q_ω against a and e for different values of I and ω. Again, the principal variations come from a and e

Abbreviations

LA: Lorentz acceleration
RC: Relativistic correction
LPE: Lagrange planetary equation
PT: Perturbation technique.

Data Availability

The datasets generated and analyzed during the current study are available in Peng and Gao [23] (https://www.sciencedirect.com/science/article/abs/pii/S0094576512000732) and Milani et al. (1987) (https://ui.adsabs.harvard.edu/abs/1987npsg.book.....M).

Conflicts of Interest

The authors declare that they have no conflict of interest.

Authors' Contributions

All authors contributed to the study conception and design, material preparation, data collection, and analysis. The first draft of the manuscript was written by M. I. El-Saftawy, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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