Research Article

The Analysis of Using Spatial Smoothing for DOA Estimation of Coherent Signals in Sparse Arrays

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When there is coexistence of uncorrelated and coherent signals in sparse arrays, the conventional algorithms for direction-of-arrival (DOA) estimation using difference coarray fail. In order to solve the problems, this paper analyzes the feasibility of using spatial smoothing in sparse arrays. Firstly, we summarize the two types of sparse arrays, one consisting of identical sparse subarrays and the other consisting of several uniform linear subarrays. When, we give the feasibility analysis and the processes of applying spatial smoothing. At last, we discuss the performance of the number of detectable coherent signals in different sparse arrays. Numerical experiments prove the conclusions proposed by the paper.

1. Introduction

Direction of arrival (DOA) is the one of the key parameters of the wireless positioning technique. It has been widely used in the fifth generation communication systems, military early warning, radar monitoring, sonar targets positioning, and so on [1, 2]. Traditionally, the super-resolution DOA estimation methods, such as subspace methods, mainly use uniform nonsparse arrays like uniform linear arrays (ULAs). In recent years, many experts and scholars focus on non-uniform sparse arrays, which can provide larger array aperture with the same number of sensors. The typical sparse arrays are minimum redundancy arrays (MRAs) [3], coprime linear arrays (CLAs) [4], and nested linear arrays (NLAs) [5]. Through transforming the sparse array to a virtual ULA based on difference coarray, spatial smoothing multiple signal classification (SS-MUSIC) [6] and direct augmentation approach (DAA) [7] are proposed to solve DOAs. Moreover, compressed sensing can be directly applied to difference coarray to estimate DOAs [8–10].

Based on the DOA estimation methods mentioned above, many researchers proposed the improved array design methods. One design thought is based on the model of CLA. Coprime array with compressed interelement spacing (CACIS), coprime array with displaced subarrays (CADIS) [11, 12], coprime array with multiperiod subarrays (CAMpS) [13], shifted coprime array (SCA) [14], generalized nested array (GNA) [15], and novel sparse arrays with two uniform arrays (NSA-U2) [16] were proposed, where they all consist of two ULA-subarrays and have larger array aperture than that of CLA. CACIS and CADIS show that setting one subarray with a smaller interelement spacing can have larger aperture of virtual ULA. CAMpS demonstrates that in order to expand the aperture of virtual ULA, only one subarray can have a compressed interelement spacing. SCA reveals that the displacement between two subarrays is the main factor to the aperture of virtual ULA. Although GNA has the same degree of freedom as NLA, it owns a sparser array structure. NSA-U2 presents the solution to have the maximum degree of freedom for sparse arrays with two uniform arrays. In order to further improve the array aperture, the sparse arrays with multiple ULA-subarrays are proposed. Super-nested arrays (SNAs) [17, 18], the augmented nested array (ANA) [19], and the maximum interelement spacing constraint (MISC) [20] divide the dense subarray of CLA into several sparse ULA-subarrays. Another design thought is combining several identical sparse arrays, such as nested MRA (NMRA) [21, 22], generalized nested subarray (GNSA) [23], and displaced multistage cascade subarrays (MSC-DiSA) [24]. The subarrays can be any sparse array and the
2. Signal Model

Suppose that there are $K$ far-field narrow-band signals impinging on a sparse array with $M$ sensors. Define the unit interelement spacing as $\lambda/2$, where $\lambda$ is the wavelength of signals, and a integer set corresponding to the sensors location is given by $D = \{0, d_1, d_2, \ldots, d_{M-1}\}$ (generally assuming $d_1 < d_2 < \ldots < d_{M-1}$). Assume that there are $P$ coherent signal groups, where the $p$th group has $L_p$ signals. The coherent signal coming from $\theta_{p,t}$ is corresponding to the $t$th multipath propagation of $S_p(t)$ with power $\sigma_{p,t}^2$ ($p = 1, \ldots, P$). The signals within each group are coherent to each other and uncorrelated to those in different groups. The total number of coherent signals is $K_c = \sum_{p=1}^{P} L_p$. In addition, the remaining signals, $S_i(t)$ coming from $\theta_i$ with the power $\sigma_i^2$ ($k = K_c + 1, \ldots, K$), are uncorrelated to each other. The number of those signals is $K_u = K - K_c$. Thus, the received signals is

\[
X(t) = \sum_{p=1}^{P} L_p a_D(\theta_{p,t})\beta_{p,t}S_p(t) + \sum_{k=K_c+1}^{K} a_D(\theta_k)S_k(t) + \mathbf{N}(t)
\]

\[
= A(\theta)\mathbf{S}(t) + \mathbf{N}(t),
\]

where the manifold matrix $A(\theta)$ is denoted as

\[
A(\theta) = [a_D(\theta_{1,1}), \ldots, a_D(\theta_{P,1}), a_D(\theta_{K,c+1}), \ldots, a_D(\theta_K)],
\]

and the steering vector $a_D(\theta_k)$ can be given by

\[
a_D(\theta_k) = [1, e^{-j d_1 \sin \theta_k}, \ldots, e^{-j d_{M-1} \sin \theta_k}]^T,
\]

and $\beta_{p,t}$ is the complex fading coefficient of the $t$th coherent signal in the $p$th group. The signal data vector is

\[
\mathbf{S}(t) = [S_{1,1}(t), \ldots, S_{P,1}(t), S_{K_c+1}(t), \ldots, S_K(t)]^T,
\]

where $t = 1, \ldots, J$, and $J$ is the number of snapshots. The noise vector is usually a Gaussian random variable with zero mean and variance $\sigma_n^2$.

From (1), the covariance matrix is denoted as

\[
R_X = \frac{1}{T} \mathbf{XX}^H = \mathbf{A} \Sigma \mathbf{A}^H + \sigma_n^2 \mathbf{I}_M,
\]

where $\Sigma$ can be written as a block-diagonal matrix given by

\[
\Sigma = \begin{bmatrix}
R_1 & \cdots & \cdots \\
\vdots & R_p & \vdots \\
\cdots & \cdots & \cdots \\
\sigma_{K_c+1}^2 & \cdots & \sigma_K^2
\end{bmatrix}
\]

\[
R_p = \begin{bmatrix}
\beta_{p,1}^2 \sigma_p^2 & \beta_{p,1}\beta_{p,2}^* \sigma_p^2 & \cdots & \beta_{p,1}\beta_{p,L_p}^* \sigma_p^2 \\
\beta_{p,2} \beta_{p,1}^* \sigma_p^2 & \beta_{p,2}^2 \sigma_p^2 & \cdots & \beta_{p,2}\beta_{p,L_p}^* \sigma_p^2 \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{p,L_p} \beta_{p,1}^* \sigma_p^2 & \beta_{p,L_p}\beta_{p,2}^* \sigma_p^2 & \cdots & \beta_{p,L_p}^2 \sigma_p^2
\end{bmatrix}
\]

Because $\text{rank}(R_p) = 1$, $\text{rank}(R_k) = P + K_u$, and $\text{rank}(R_X) = P + K_u < K$. Thus, the conventional methods for

<table>
<thead>
<tr>
<th>Notations</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\cdot)^T$</td>
<td>Transpose</td>
</tr>
<tr>
<td>$(\cdot)^*$</td>
<td>Conjugate</td>
</tr>
<tr>
<td>$(\cdot)^H$</td>
<td>Hermitian transpose</td>
</tr>
<tr>
<td>$\gcd(\cdot)$</td>
<td>Greatest common divisor operation</td>
</tr>
<tr>
<td>$\text{diag}[\cdot]$</td>
<td>Diagonal matrix</td>
</tr>
<tr>
<td>$\text{rank}(\cdot)$</td>
<td>Rank of a matrix</td>
</tr>
<tr>
<td>$\lfloor \cdot \rfloor$</td>
<td>Floor integer</td>
</tr>
</tbody>
</table>

Table 1: Key notations used in this paper.
DOA estimation fail. Spatial smoothing can let rank \( \mathbf{R}_X = K \) to satisfy the requirement of subspace methods, but it requires that the array can be divided into several same subarrays. Thus, in the next section, we will discuss the decomposition of sparse arrays.

3. Spatial Smoothing in Sparse Arrays

In this section, we try to divide any sparse array into several subarrays and summarize two situations. The first is that the sparse array is composed of several same sparse subarrays. The second is that the sparse array can be divided into several uniform sparse linear arrays, where although the structure of ULA is different, the process of spatial smoothing is applied to each ULA, and the ULAs with different interelement spacings are applied to remove the ambiguous values.

3.1. Sparse Arrays with Identical Sparse Subarrays. There exists a type of sparse arrays, which are made up of several same arrays. The sparse subarray can be CLAs, NLAs, MRAs, and so on. If the cascade number of array is \( M \) and \( M \geq 3 \). So, when \( \Delta d_{q} = d_{qM} - d_{0} \) is displacement between the qth subarray and first subarray and \( q = 0, \ldots, Q - 1 \). Thus, the spatial smoothing covariance matrix is defined as

\[
\mathbf{R}_{SS} = \frac{1}{JQ} \sum_{q=0}^{Q-1} \mathbf{X}_q \mathbf{X}_q^H,
\]

\[
= \frac{1}{JQ} \mathbf{A}_0 \left( \sum_{q=0}^{Q-1} \Phi_q \mathbf{SS}^H \Phi_q^H \right) \mathbf{A}_0^H + \sigma_n^2 \mathbf{I}_{M}
\]

where \( \Phi_q = \text{diag} \left[ e^{-j\Delta d_1 \sin \theta_1}, \ldots, e^{-j\Delta d_{Q-1} \sin \theta_{Q-1}} \right] \).

Theorem 1. If \( \gcd(\Delta d_1, \ldots, \Delta d_{Q-1}) = 1 \), \( Q \geq \max(L_p) \), and \( M \geq K \), \( \text{rank} \left( \mathbf{R}_{SS} \right) = K \).

Proof. See Appendix A.

Considering the requirement for setting the displacement between the subarrays [15], we have known that \( \gcd(\Delta d_1, \ldots, \Delta d_{Q-1}) = 1 \). So, when \( Q \geq \max(L_p) \) and \( M \geq K \), we can estimate all DOAs \( \theta_k \) by applying subspace methods [32] to \( \mathbf{R}_{SS} \).

3.2. Sparse Arrays with ULA-Subarrays. The sparse arrays, which consist of Q ULA-subarrays, are capable to use spatial smoothing algorithm to solve coherent signals. We first give the general model of sensors location denoted as

\[
\mathbf{D} = \bigcup_{q=1}^{Q} \left\{ \mathbf{G}_q + m \mathbf{g}_q | 0 \leq m \leq M_q - 1 \right\},
\]

where \( \mathbf{G}_q \) is the displacement between the qth ULA-subarray and the first ULA-subarray, \( M_q \) is the sensor number of qth ULA-subarray, and \( g_q \) is the interelement spacing of qth ULA-subarray.

3.2.1. Sparse Arrays with Two ULA-Subarrays. Figure 1(b) shows four arrays with 12 sensors using two ULAs. Thus, two subarrays, respectively, have \( M_1 \) and \( M_2 \) sensors. The location of subarrays can be denoted as \( \mathbf{D}_1 = \{ m_1 \mathbf{g}_1 | 0 \leq m_1 \leq M_1 - 1 \} \), \( \mathbf{D}_2 = \{ m_2 \mathbf{g}_2 | 0 \leq m_2 \leq M_2 - 1 \} \), where \( g_1, g_2 \) are coprime integers, and generally \( G_1 = 0 \). The parameters of arrays are defined in Table 2. Then, the received data in (1) can be rewritten as

\[
\mathbf{X} = \left[ \begin{array}{c} \mathbf{X}_1 \\ \mathbf{X}_2 \end{array} \right] = \left[ \begin{array}{c} \mathbf{A}_1 \\ \mathbf{A}_2 \end{array} \right] \mathbf{S} + \left[ \begin{array}{c} \mathbf{N}_1 \\ \mathbf{N}_2 \end{array} \right],
\]

where
Table 2: Parameters of sparse arrays.

<table>
<thead>
<tr>
<th>Type of sparse array</th>
<th>( M_1, M_2, g_1, g_2, G_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLA</td>
<td>7, 5, 4, 5, 0</td>
</tr>
<tr>
<td>CADis</td>
<td>5, 7, 7, 3, 10</td>
</tr>
<tr>
<td>SCA</td>
<td>5, 7, 5, 4, 8</td>
</tr>
<tr>
<td>NLA</td>
<td>6, 6, 1, 6, 0</td>
</tr>
</tbody>
</table>

\[
\mathbf{A}_i = \begin{bmatrix}
\mathbf{a}_{D_1}(\theta_{1,1}), & \cdots & \mathbf{a}_{D_1}(\theta_{P,1}), & \mathbf{a}_{D_2}(\theta_{K+1}), & \cdots & \mathbf{a}_{D_2}(\theta_K)
\end{bmatrix}.
\]

(14)

Then, the spatial smoothing covariance matrix with size \( M_S \times M_S \) of \( i \)th subarray is given by

\[
\mathbf{R}_{SS_i} = \frac{1}{f(M_i - M_S + 1)} \sum_{m=0}^{M_i - M_S} (\Gamma_m^i \mathbf{X}_i) (\Gamma_m^i \mathbf{X}_i)^H, \tag{15}
\]

where

\[
\mathbf{R}_{SS_i} = \mathbf{A}_S^H \mathbf{R}_S \mathbf{A}_S + \sigma_n^2 \mathbf{I}_{M_S},
\]

(15)

where \( \Gamma_m^i = [0_{M_S \times (M_i - M_S - m)}, \mathbf{I}_{M_S}, 0_{M_S \times m}] \). \( \mathbf{A}_S \) is given by

\[
\mathbf{A}_S = \begin{bmatrix}
\mathbf{a}_{D_1}(\theta_{1,1}), & \cdots & \mathbf{a}_{D_1}(\theta_{P,1}), & \mathbf{a}_{D_2}(\theta_{K+1}), & \cdots & \mathbf{a}_{D_2}(\theta_K)
\end{bmatrix}, \tag{16}
\]

(16)

where \( D_S = \{m \mid m \geq 0 \} \). Also, we define \( A_S = A_S D \), where

\[
\mathbf{A}_S^2 = \begin{bmatrix}
\mathbf{a}_{D_2}(\theta_{1,1}), & \cdots & \mathbf{a}_{D_2}(\theta_{P,1}), & \mathbf{a}_{D_2}(\theta_{K+1}), & \cdots & \mathbf{a}_{D_2}(\theta_K)
\end{bmatrix},
\]

(17)

where \( D = \{m \mid m \geq 0 \} \).

\[
\mathbf{D} = \text{diag}\{e^{j\pi g_1 \sin \theta_{1,1}}, \ldots, e^{j\pi g_2 \sin \theta_{K,1}}, \ldots, e^{j\pi g_2 \sin \theta_{K,K}}\}. \tag{18}
\]

Moreover,

\[
\mathbf{R}_S = \frac{1}{f(M_i - M_S + 1)} \sum_{m=0}^{M_i - M_S} \mathbf{W}_i^m \mathbf{S}_i^H \mathbf{W}_i^m \mathbf{S}_i^H, \tag{19}
\]

where

\[
\mathbf{W}_i = \text{diag}\{e^{j\pi g_1 \sin \theta_{1,1}}, \ldots, e^{j\pi g_2 \sin \theta_{K,1}}, \ldots, e^{j\pi g_2 \sin \theta_{K,K}}\}
\]

and

\[
\mathbf{W}_2 = \text{diag}\{e^{j\pi g_1 \sin \theta_{1,1}}, \ldots, e^{j\pi g_2 \sin \theta_{K,1}}, \ldots, e^{j\pi g_2 \sin \theta_{K,K}}\}.
\]

Based on the model above, the following statement gives the applicability of using spatial smoothing in this type of sparse array. Firstly, we introduce Theorem 2 in [30] and propose Theorem 3.

**Theorem 2.** Let \( g_1 \) and \( g_2 \) denote the interelement spacing of two ULAs, respectively. Let \( \Theta_k^{(1)} = \{\theta_k^{(1)}, \ldots, \theta_k^{(1)}\} \) and \( \Theta_k^{(2)} = \{\theta_k^{(2)}, \ldots, \theta_k^{(2)}\} \) denote the estimated DOA values of the \( k \)th signal of two subarrays, respectively, where each set has multiple ambiguous values and one real value. If \( \gcd(g_1, g_2) = 1 \), then \( \Theta_k^{(1)} \cap \Theta_k^{(2)} = \emptyset \).
Theorem 3. Let the directions of sources have a random distribution. When \( M_1 - M_S + 1 \geq \max(L_p) \) \((p = 1, \ldots, P)\) and \( M_S \geq K \), rank \((R_{SS}) = K\).

Proof. See Appendix B. \(\square\)

When \( M_1 - M_S + 1 \geq \max(L_p) \) \((p = 1, \ldots, P)\) and \( M_S > K \), rank \((\overrightarrow{R}_S) = K\) and the requirement of using subspace methods is satisfied. Also, rank \((D\overrightarrow{R}_S D^H) = \text{rank}(\overrightarrow{R}_s)\). Then, we know that \( g_1, g_2 \) are coprime integers, so we use the subspace method to \(\overrightarrow{R}_S\) and obtain the estimated values of the \(k\)th signals, defined as \( \Theta_k^{(1)} = \{\theta_{k,1}, \ldots, \theta_{k,2\beta_k - 1}\} \) and \( \Theta_k^{(2)} = \{\theta_{k,1}, \ldots, \theta_{k,2\beta_k - 1}\}\). From Theorem 2, common peak finding [30] tells that the real value \( \hat{\theta}_k = \Theta_k^{(1)} \cap \Theta_k^{(2)} \).

3.2.2. Sparse Arrays with \( Q \) ULA-Subarrays. When it comes to the sparse arrays with \( Q > 2 \) ULAs-subarrays, we need to point out that the sensor location of existing sparse arrays satisfies (13). We just give the examples about ANAI-1, MISC, ANAI-1, and MRA with \( M = 12 \) in Table 3.

Based on the analysis in sparse arrays with two subarrays, we just need to select two subarrays, whose interelement spacings are coprime integers, to solve the DOA estimation. Besides the requirement for setting the inter-element spacing, another two criteria for subarray parameters selection are the number of sensors that are as large as possible and the subarray apertures that are as large as possible. The former is to estimate as many sources as possible, and the latter is to ensure the accuracy of estimation. Thus, we can obtain the receiving data \( \mathbf{X}_k \) and \( \mathbf{X}_f \) of two chosen subarrays from \( \mathbf{X} \). Then, we can use (16) to calculate the spatial smoothing covariance matrix and apply common peak finding [30] to find the DOAs of coherent signals.

4. Performance Analysis and Simulation Experiments

4.1. Performance Analysis. We discuss \( \max(L_p) \) of each sparse array, that is, the maximum number of detectable coherent signals in one group. For GNSA, \( \max(L_p) \leq Q = M/M_1 \), so when \( Q \) is the maximum integer no more than \( \lfloor \sqrt{M} \rfloor \), \( \max(L_p) \) achieving maximum is equal to \( Q \). For any sparse array consisting of \( Q \) ULAs, we arrange the number of subarray sensors \( \{M_1, \ldots, M_Q\} \) in descending order and have the set \( \{\overline{M}_1, \ldots, \overline{M}_Q\} \), where \( \overline{M}_1 > \cdots > \overline{M}_Q \). Based on the theorem that the paper proposes, we can have \( \max(L_p) \leq \lfloor \overline{M}_2/2 \rfloor \) and \( K_u \leq \lfloor \overline{M}_2/2 \rfloor \). Thus, when \( P = 1 \) and \( K_u = 0 \), \( \max(L_p) = \lfloor (M - Q + 2)/2 \rfloor \) achieves maximum in theory. So, the more the number of ULAs-subarrays, the smaller the value of \( \max(L_p) \).

Next, we compare the value of \( \max(L_p) \) of GNSA with that of CLA, NLA, ANAI-1, MISC, ANAI-1, and MRA. We vary \( M \) from 8 to 20 with 2 intervals, and the results are shown in Figure 2. \( \max(L_p) \) becomes bigger with the increase of \( M \). The CLA and NLA with two ULA-subarrays have the biggest value of \( \max(L_p) \). But the ANAI-1 and MRA with 5 or more ULAs-subarrays have the smallest value, and only when \( M > 12 \), they can use spatial smoothing to estimate coherent signals. Hence, with a fixed number of sensors, the conclusion that the less subarrays can have the bigger \( \max(L_p) \) is corrected.

4.2. Simulation Experiments. We use root mean square error (RMSE) to quantify the accuracy of DOA estimation, given by

\[
\text{RMSE} = \sqrt{\frac{1}{FK} \sum_{f=1}^{F} \sum_{k=1}^{K} (\hat{\theta}_{k,f} - \theta_{k,f})^2},
\]

where \( F \) is the Monte Carlo number, \( K \) is the number of target signals, and \( \hat{\theta}_{k,f} \) is the DOA of the \( k \)th estimated source by the \( f \)th Monte Carlo experiment. The simulation conditions are shown in Table 4.

Simulation 1. feasibility of estimate coherent signals in different sparse arrays.

In first simulation, we show the feasibility of estimate coherent signals with maximum number in one group. Thus, assume that \( P = 1 \), \( K_u = 0 \), and \( L_1 \) of each sparse array can be seen in Figure 2. Set SNR = 0dB and \( J = 5000 \). The estimation values of 100 times experiments are shown in Figure 4. The figure demonstrates that the sparse arrays can use spatial smoothing to estimate DOAs of coherent signals. Moreover, in the condition of \( \max(L_p) \) and low SNR, all estimated values are still close to the real values, which means a favorable performance.

Simulation 2. RMSE performance comparison of different SNRs.

In this simulation, we compare the RMSEs of different sparse arrays and ULAs, when there are both uncorrelated and coherent signals. We set \( K = 3 \), where \( \theta_{1,1} = 25°, \theta_{1,2} = 0°, \theta_{1,3} = 15° \), and \( J = 5000 \). Because ANAI-1 and MRA cannot estimate 3 signals after spatial smoothing, we only compare the other arrays, and the results are shown in Figure 5. Obviously, the RMSEs decrease with the increase of SNR. GNSA has the highest RMSE due to the smallest \( \Pi \). ULA has the second smallest \( \Pi \). So, the RMSE of it is just lower than that of GNSA. The other three sparse arrays have the close RMSE due to their close values of \( \Pi \).
Table 3: Parameters of sparse arrays.

<table>
<thead>
<tr>
<th>Arrays</th>
<th>$D$</th>
<th>$Q$</th>
<th>$[M_q, g_q, G_q]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANAI-1</td>
<td>$[0, 1], [6, 13, 20, 27, 34, 41]$</td>
<td>3</td>
<td>$[2, 1, 0], [7, 6, 6], [5, 1, 41]$</td>
</tr>
<tr>
<td></td>
<td>$[41, 42, 43, 44, 45]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MISC</td>
<td>$[0, 1], [6, 14, 22, 30, 38]$</td>
<td>4</td>
<td>$[2, 1, 0], [5, 8, 6]$</td>
</tr>
<tr>
<td></td>
<td>$[38, 40, 42], [45, 47, 49]$</td>
<td></td>
<td>$[3, 2, 38], [3, 2, 45]$</td>
</tr>
<tr>
<td>ANAI-2</td>
<td>$[0, 1], [1, 5, 9], [9, 16, 23, 30, 37, 44]$</td>
<td>5</td>
<td>$[2, 1, 0], [3, 4, 1], [6, 7, 9], [2, 2, 47], [2, 1, 49]$</td>
</tr>
<tr>
<td></td>
<td>$[47, 49], [49, 50]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MRA</td>
<td>$[0, 1], [3, 6], [6, 13, 20, 27, 34, 41]$</td>
<td>5</td>
<td>$[2, 1, 0], [2, 3, 3], [6, 7, 6], [3, 4, 41], [2, 1, 49]$</td>
</tr>
<tr>
<td></td>
<td>$[41, 45, 49], [49, 50]$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 2: $\max(L_p)$ in different sparse arrays with different $M$.

Figure 3: The comparison of $\Pi$ and $\gamma$ in different sparse arrays.
Table 4: Simulation conditions for the experiments.

<table>
<thead>
<tr>
<th>Simulation parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Antenna number</td>
<td>20</td>
</tr>
<tr>
<td>The subarrays for smoothing in ULA</td>
<td>{0, 1, 2, 3, \ldots, 18, 19}</td>
</tr>
<tr>
<td>The subarrays for smoothing in GNSA</td>
<td>{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}</td>
</tr>
<tr>
<td>The subarrays for smoothing in CLA</td>
<td>{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}</td>
</tr>
<tr>
<td>The subarrays for smoothing in NLA</td>
<td>{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}</td>
</tr>
<tr>
<td>The subarrays for smoothing in ANAI-I</td>
<td>{35, 50, 65, 80, 95, 110}</td>
</tr>
<tr>
<td>The subarrays for smoothing in ANAI-II</td>
<td>{3, 11, 19, 27, 35}</td>
</tr>
<tr>
<td>The subarrays for smoothing in MRA</td>
<td>{28, 43, 58, 73, 88, 103}</td>
</tr>
<tr>
<td>Monte Carlo number</td>
<td>$F = 100$</td>
</tr>
<tr>
<td>SNR</td>
<td>$[-5, 0, 5, 10, 15]$</td>
</tr>
<tr>
<td>Snapshot</td>
<td>{50, 100, 200, 500, 1000, 2000, 5000}</td>
</tr>
</tbody>
</table>

Figure 4: The estimation values of coherent signals in different sparse arrays.
Simulation 3. RMSE performance comparison of different number of snapshots.

Similar to the simulation 2, we present the RMSEs of different sparse arrays under different snapshots with SNR = 10dB. From Figure 6, the RMSEs decrease with the increase of snapshot numbers, but when $J > 500$, the downtrend of RMSEs become slow. Other conclusions are the same as those in simulation 2.

Simulation 4. RMSE performance comparison using FBSS.

In this simulation, we use FBSS to replace spatial smoothing and do simulations 2 and 3 again. FBSS can be seen as an improvement method of spatial smoothing, where II can be 1.5 times that of spatial smoothing [26]. Thus, we do not need to present the analysis in Part 1 of this section about using FBSS. But we should note that FBSS is not applicable to GNSA because it needs that the subarrays have uniform structure, where we still use spatial smoothing in GNSA. The results are shown in Figures 7 and 8. Compared with the results in simulations 2 and 3, FBSS has improved
the accuracy. Thus, we can use FBSS in sparse arrays to find more coherent signals and obtain higher accuracy.

5. Conclusions

In this paper, the DOA estimation methods using spatial smoothing for coherent signals in sparse arrays are proposed. We divide the sparse arrays into two parts. The first type consists of several identical sparse arrays. The second type can be decomposed of several ULA-subarrays. In view of subarrays, spatial smoothing can be applied in sparse arrays. Based on the analysis of the maximum number of detectable coherent signals in one group, the sparse arrays with less subarrays are capable to estimate more signals and own bigger smoothing array aperture. Also, the simulation experiments prove that CLA and NLA have better performance than other arrays.

Appendix

A

Proof of Theorem 1. The matrix \( \mathbf{R}_s \) can be rewritten as

\[
\mathbf{R}_s = \begin{bmatrix}
\mathbf{R}_1 \\
\mathbf{R}_2 \\
\vdots \\
\sigma_K^2
\end{bmatrix},
\]

where \( \mathbf{R}_s \) is a block-diagonal matrix and

\[
\mathbf{R}_p = \frac{1}{Q} \sum_{q=0}^{Q-1} \Phi_q \mathbf{R}_p(\Phi_q)^H.
\]

Define \( \Phi = \text{diag}[e^{-j\pi \sin \theta_1}, \ldots, e^{-j\pi \sin \theta_K}] \), and (20) can be denoted as

\[
\mathbf{R}_p = \frac{1}{Q} \sum_{q=0}^{Q-1} \Phi^{\Delta d} \mathbf{R}_p(\Phi^{\Delta d})^H.
\]

If rank(\( \mathbf{R}_p \)) = \( L_p \), rank(\( \mathbf{R}_s \)) = \( K \). Thus, we rewrite \( \mathbf{R}_p \) as

\[
\mathbf{R}_p = \begin{bmatrix}
\mathbf{I}_K & \Phi^{\Delta d_1} & \ldots & \Phi^{\Delta d_{K-1}}
\end{bmatrix}
\begin{bmatrix}
\mathbf{R}_p \\
\Phi^{\Delta d_1} \\
\vdots \\
\Phi^{\Delta d_{K-1}}
\end{bmatrix}^H.
\]

Based on (25), it can also be simplified to give

\[
\mathbf{R}_p = \mathbf{V}_p \mathbf{V}_p^H,
\]

with \( \mathbf{C}_p \) denoting the Hermitian square root of \( \mathbf{R}_p \).

Because rank(\( \mathbf{R}_p \)) = rank(\( \mathbf{V}_p \)), we need to prove that rank(\( \mathbf{V}_p \)) = \( L_p \). We take column permutations to \( \mathbf{V}_p \), which cannot change the rank of a matrix, and have

\[
\text{rank}(\mathbf{V}_p) = \text{rank}
\begin{bmatrix}
\mathbf{c}_{i1} \mathbf{v}_1 & \mathbf{c}_{i2} \mathbf{v}_1 & \ldots & \mathbf{c}_{ip} \mathbf{v}_1 \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{c}_{L_p,1} \mathbf{v}_{L_p} & \mathbf{c}_{L_p,2} \mathbf{v}_{L_p} & \ldots & \mathbf{c}_{L_p,L_p} \mathbf{v}_{L_p}
\end{bmatrix}.
\]

(A.6)

where \( \mathbf{c}_{i,j} \) represents element in the \( i \)th row and \( j \)th column of \( \mathbf{C}_p \) and \( \mathbf{v} \) is expressed as

\[
\mathbf{v}_k = [1, e^{j\pi \Delta d_1 \sin \theta_1}, \ldots, e^{j\pi \Delta d_{L_p} \sin \theta_1}]^T.
\]

(A.7)

When each row of \( \mathbf{C}_p \) has at least one nonzero element and vectors \( \{\mathbf{v}_1, \ldots, \mathbf{v}_{L_p}\} \) are linearly independent, rank(\( \mathbf{V}_p \)) = \( L_p \). Because every signal has the positive energy,
every row of $C$ cannot be all zeros. The matrix $[y_1, \ldots, y_p]$ can be seen as the manifold matrix of a sparse array and $\{0, \Delta d_1, \ldots, \Delta d_{Q-1}\}$ are denoted as the location of sensors.

Hence, we introduce the theorem in [32, 33], which tells us that if $\gcd(\Delta d_1, \Delta d_2, \ldots, \Delta d_{Q-1}) = 1$, the array manifold $\nu_k$ is invertible. Invertibility means that if $\theta_i \neq \theta_j$, then $v_1 \neq v_2$. Then, we can obtain the conclusion that if $Q \geq L_p$, the rank of $[y_1, \ldots, y_p]$ is $L_p$. So, we prove that if $\gcd(\Delta d_1, \Delta d_2, \ldots, \Delta d_{Q-1}) = 1$ and $Q \geq \max(L_p)$, rank $(\nu_k) = L_p$, and rank $(\nu_k) = L_p$ for any $p$, then rank $(\nu_k) = K$. Hence, when $M \geq K$, rank $(\nu_k) = K$.

**B**

**Proof of Theorem 3.** In this situation, $\nu_k$ is also a block-diagonal matrix, given by

$$\nu_k = \begin{bmatrix} R_{1,j} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & R_{j,j} \end{bmatrix}$$

and $R_{p,j}$ is changed to

$$R_{p,j} = \frac{1}{M_j - M_S - 1} \sum_{i=0}^{M_j - M_S} \Psi_i^{\theta_i} R_p (\Psi_i^{\theta_i})^H.$$  \hspace{1cm} (B.2)

Similar to the Proof of Theorem 1, $\nu_k = V_{p,j} V_{p,j}^H$, where

$$V_{p,j} = \begin{bmatrix} C_{p,j} \Psi_i^{\theta_i} C_{p,j} \Psi_i^{\theta_i} \cdots \Psi_i^{\theta_i} (M_j - M_S) C_{p,j} \end{bmatrix},$$

with $C_{p,j}$ denoting the Hermitian square root of $R_{p,j}$. We just need to prove that rank $(\nu_k) = L_p$ because rank $(\nu_k) = L_p$. Considering the analysis in Appendix A, we should prove that each row of $C_{p,j}$ has at least one nonzero element and vectors $[v_{1,j}, \ldots, v_{L,j}]$ are linearly independent, where

$$v_{k,j} = [1, e^{j \pi \sin \theta_1}, \ldots, e^{j \pi \sin \theta_L}]^T.$$  \hspace{1cm} (B.4)

It is easy to obtain that each row of $C_{p,j}$ has at least one nonzero element. $[v_{1,j}, \ldots, v_{L,j}]$ is a Vandermonde matrix and $\{0, g_1, \ldots, g_L(M_j - M_S)\}$ are denoted as the location of sensors. Because $\gcd(g_1, \ldots, g_L(M_j - M_S)) = g_i \geq 1$, $v_k$ is not invertible. For example, if $g_1 = 2$, $\theta_1, 1 = 30^\circ$, and $\theta_1, 2 = -30^\circ$, then $v_1, 1 = v_2, 2$, and rank $(\nu_k) = 1$, while if $\theta_1, 2 = 30^\circ$, then $\nu_2, 1 = v_2, 2$, and rank $(\nu_k) = 2$. In general, assuming that $g_i = a$, if

$$\sin \theta_k - \sin \theta_l \neq \frac{2\pi n}{\alpha},$$  \hspace{1cm} (B.5)

for any integer $n$, $v_k \neq v_l$ and rank $(\nu_k) = L_p$. Considering that the directions of sources generally have a random distribution in real environment, the parameters $(\sin \theta_k, \sin \theta_l)$ satisfy (33) with probability one. In other words, if $g_1 > 1$ and $M_j - M_S + 1 \geq \max(L_p)$, rank $(\nu_k) = L_p$ and rank $(\nu_k) = K$ in this situation. Hence, when $M \geq K$, rank $(\nu_k) = K$.

**Data Availability**

The data used in this article are provided by our simulations and the data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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