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Research Article

Performance Analysis of TDOA and FDOA Estimation for Pulse Signals

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Time difference of arrival (TDOA) and frequency difference of arrival (FDOA) are effective measurements for localizing emitters' radiating pulse signals, such as radar. Temporal sparsity of pulse signals makes their TDOA and FDOA estimation precisions much different from that of continuous communication and acoustic signals. The way how the precisions are affected by various parameters, e.g., temporal duration of signals, may also deviate significantly in scenarios of pulse signals from that of continuous ones. In this paper, theoretical analyses are carried out to reveal the Cramer–Rao lower bounds (CRLBs) of TDOA and FDOA estimation precisions of pulse signals and also to obtain insights on how the CRLBs are affected by various parameters, including pulse number, signal-to-noise ratio (SNR), pulse width, and pulse repetition interval (PRI). Simulation results verify the correctness of the derived CRLBs and their variations with different parameters.

1. Introduction

Passive localization has been an important research topic in the field of signal processing for decades [1, 2], and source locations are always determined by measuring various intermediate parameters, such as time difference of arrival (TDOA), frequency difference of arrival (FDOA), and angle of arrival (AOA) [3–5]. When multiple receivers are spatially distributed and they have relative motions with respect to (w.r.t.) the emitter, TDOA and FDOA between signals collected by different receivers can be estimated for high-precision emitter localization [6–8].

Existing TDOA and FDOA estimation literature mostly focuses on continuous signals in areas of communications and acoustics [6–11]. Such signals usually span durations on the order of a few milliseconds to tens of milliseconds. During such a short period of time, only negligible position changes are introduced between emitters and receivers. Therefore, it is reasonable to treat the TDOA and FDOA as static parameters. Theoretical analyses have been carried out to give Cramer–Rao lower bounds (CRLBs) of the TDOA and FDOA estimation precisions for continuous signals, which reveal how the precisions are affected by various factors [12].

The theoretical performance of TDOA and FDOA estimation for pulse signals can hardly be obtained by directly extending existing conclusions for continuous signals [12]. At first sight, an extension can be realized by treating pulse trains, which consist of temporally sparse pulse signals, as amplitude-modulated continuous signals, i.e., the inter-pulse intervals are considered to be zero-amplitude signals. A huge challenge blocks this extension, i.e., the additive noises within inter-pulse intervals should not be taken into account for TDOA and FDOA estimation of pulse signals, as each pulse is sampled only after being detected and there are gaps (instead of noises) between adjacent pulses. Compared with the commonly used continuous signal model [6–11], the pulse signals are temporally sparse, and the signal energy distributes along the time axis in the form of discrete pulses. In such scenarios, various factors, such as the time span and the cumulative duration of the pulse signals, affect the TDOA and FDOA estimation precisions in much different ways for pulse signals than continuous ones. This paper provides in-depth analyses to make clear how these factors affect the TDOA and FDOA estimation precisions for pulse signals.

In this paper, we analyze the CRLBs of TDOA and FDOA estimation accuracies for pulse signals. The results degenerate to the existing result [12] for continuous signals when the pulse number decreases to 1. Based on the formulations of the CRLBs, the paper analyzes how the TDOA and FDOA estimation precisions are affected by various factors, including signal-to-noise ratio (SNR), pulse number, pulse width, and pulse repetition interval (PRI). As an extension of the results for coherent pulses, the TDOA and FDOA estimation CRLBs for independent pulses are also provided, together with influence analyses of the above-mentioned factors on the estimation precisions.

The rest of the paper is organized as follows. Section 2 formulates the problem of TDOA and FDOA estimation for pulse signals. Section 3 analyzes the CRLBs of the TDOA and FDOA estimation precisions for coherent pulse signals, establishes a relationship between the derived results and the existing results for continuous signals, and gives the corresponding CRLBs for independent pulse signals. Section 4 analyzes the influences of various factors, including signal-to-noise ratio (SNR), pulse number, pulse width, and pulse repetition interval (PRI), on the TDOA and FDOA estimation performance. In Section 5, simulations are carried out to verify the correctness of theoretical results of the CRLBs and how they are affected by various factors. Section 6 summarizes the whole paper.

2. Problem Formulation for TDOA and FDOA Estimation of Pulse Signals

Pulse-radiating emitters like radar are mainly used for energy-dependent applications such as target detection, and their signals usually do not contain complicated modulations [13]. The modulation parameters of each pulse are usually consistent, e.g., linear frequency modulated (LFM) signals with the same modulation parameters. Such pulse trains can be modeled with coherent signals [14, 15]. Furthermore, assume that the signal bandwidth is very small when compared with the carrier frequency; then, the signals can be approximated as narrowband ones, and the doppler frequency shifts of the signals at different stations keep unchanged during the observation time.

Suppose that K pulse signals are received by two sensors; the TDOA and FDOA of the pulses between the two receivers are t_d and f_d. The samples of the kth pulse at the two receivers can be written in a vector form as follows:

$$\begin{aligned}\mathbf{x}_{1,k} &= a_k e^{j\varphi_k}\mathbf{s} + \mathbf{u}_{1,k},\\ \mathbf{x}_{2,k} &= b_k e^{j\varphi_k} e^{j\phi_0} e^{j\phi_k(\nu)}\mathbf{D}_\nu \mathbf{s}(-\tau) + \mathbf{u}_{2,k},\ k = 1,\dots,K,\end{aligned} \tag{1}$$

where $\mathbf{x}_{i,k} = [x_{i,k}(1),\dots,x_{i,k}(N_0)]^T$ and $\mathbf{u}_{i,k} = [u_{i,k}(1),\dots,u_{i,k}(N_0)]^T$ for $i = 1,2$ denote the observation samples and noise samples with respect to a sampling interval T, respectively, and N_0 is the sample number within each pulse. $\mathbf{s} = [s(1),\dots,s(N_0)]^T$ stands for the signal samples, $\mathbf{s}(-\tau) = [s(1-\tau),\dots,s(N_0-\tau)]^T$, $\mathbf{D}_\nu = \mathrm{diag}\{\exp(j\nu\mathbf{l})\}$, with $\mathbf{l} = [0,1,\dots,N_0-1]^T$, and $\tau = t_d/T$ and $\nu = 2\pi f_d T$ are TDOA and FDOA-dependent parameters. $a_k, b_k \in \mathbb{R}$ denote the relative amplitudes of the kth pulse signal received by the two receivers w.r.t. that of the first pulse received by the first receiver, which satisfies $a_1 = 1$. The Gaussian white noises at the two receivers are denoted by $u_{1,k}(t)$ and $u_{2,k}(t)$, respectively, which satisfy $E\{|u_{1,k}(t)|^2\} = E\{|u_{2,k}(t)|^2\} = \sigma^2$ with the noise variance σ^2 assumed to be known. When the noise variance is unknown, similar conclusions can be obtained as these for known σ^2 following the guidelines of analyses in [12]. ϕ_0 stands for the initial phase offset between the two receivers caused by the asynchronization between different sampling clocks, and $\varphi_k \in [0, 2\pi)$ represents the additional phase shift of the kth pulse signal relative to the first one satisfying $\varphi_1 = 0$. $\phi_k(\nu) = 2\pi f_d T_k^{(1)} \triangleq \nu \cdot \overline{n}_k$ represents the additional phase shift caused by the doppler frequency shift at the beginning of the k th pulse, where $\overline{n}_k = T_k^{(1)}/T$ and $\overline{n}_1 = 0$.

Based on time-frequency transformation, the temporally delayed signal $\mathbf{s}(-\tau)$ can be approximated by a spectrally group-delayed form of the original signal [12], which is given by

$$\mathbf{s}(-\tau) = \mathbf{F}^H \mathbf{D}_\tau \mathbf{F}\mathbf{s}, \tag{2}$$

where $\mathbf{F} = 1/\sqrt{N_0}\exp(-j(2\pi/N_0)\mathbf{l}\mathbf{l}^T)$ and $\mathbf{D}_\tau = \mathrm{diag}\{\exp(-j2\pi\tau/N_0\mathbf{l})\}$. The superscripts $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$ denote transpose, conjugate, and conjugate transpose operators, respectively. In (2), the time-delayed signal $\mathbf{s}(-\tau)$ is interpreted as a variant of $\mathbf{s}$ after a series of transformations, where $\mathbf{s}$ is first transformed to the spectral domain by left multiplying the Fourier matrix $\mathbf{F}$; then, each spectral component is group-delayed by a TDOA τ and finally transformed back to the temporal domain by left multiplying the conjugate transposed Fourier matrix $\mathbf{F}^H$. One can demonstrate (2) by left multiplying both sides with $\mathbf{F}$, and then it becomes more apparent that each spectral component of $\mathbf{s}(-\tau)$ is a phase-shifted replica of the corresponding component of $\mathbf{s}$, and the shifted phases are time-delay and frequency-dependent and they form $\mathbf{D}_\tau$.

Unknown parameters contained in the above observation model include amplitudes $\mathbf{a} = [a_2,\dots,a_K]^T$ and $\mathbf{b} = [b_1, b_2,\dots,b_K]^T$ and phases $\varphi = [\varphi_2,\dots,\varphi_K]^T$ for different pulses, and $\eta = [\mathbf{s}_r^T, \mathbf{s}_i^T]^T$ and $\theta = [\phi_0, \tau, \nu]^T$ shared by all pulses, where subscripts $(\cdot)_r$ and $(\cdot)_i$ denote the real and imaginary parts of a variable, respectively. The whole parameter set in the observation model is

$$\boldsymbol{\xi} = \left[\boldsymbol{\eta}^T, \mathbf{a}^T, \mathbf{b}^T, \boldsymbol{\varphi}^T, \boldsymbol{\theta}^T\right]^T. \tag{3}$$

Denote $\mathbf{x}_k = [\mathbf{x}_{1,k}^T, \mathbf{x}_{2,k}^T]^T$, $\mathbf{x} = [\mathbf{x}_1^T,\dots,\mathbf{x}_K^T]^T$; then, $\mathbf{x}$ has a complex Gaussian distribution in the case of white Gaussian noise, which is given by

$$\mathbf{x} \sim \mathbb{CN}\left(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_{2KN_0}\right), \tag{4}$$

where

$$\boldsymbol{\mu} = \left[\boldsymbol{\mu}_1^T, \cdots, \boldsymbol{\mu}_K^T\right]^T,$$
$$\boldsymbol{\mu}_k = \begin{bmatrix} a_k e^{j\varphi_k}\mathbf{s} \\ b_k e^{j\varphi_k} e^{j\phi_0} e^{j\phi_k(\nu)} \mathbf{D}_\nu \mathbf{F}^H \mathbf{D}_\tau \mathbf{F}\mathbf{s} \end{bmatrix} \triangleq \begin{bmatrix} \boldsymbol{\mu}_{k,1} \\ \boldsymbol{\mu}_{k,2} \end{bmatrix}. \tag{5}$$

The task of TDOA and FDOA estimation is to estimate τ and ν based on the observations of the two sensors, i.e., $\mathbf{x}_{1,1}, \mathbf{x}_{1,2}, \ldots, \mathbf{x}_{1,K}$ and $\mathbf{x}_{2,1}, \mathbf{x}_{2,2}, \ldots, \mathbf{x}_{2,K}$.

3. Cramer–Rao Lower Bound of TDOA and FDOA Estimation

This section analyzes the theoretical lower bounds of the TDOA and FDOA estimation precisions for coherent pulse trains and establishes a relationship between the bounds and the existing results for continuous signals [12]. Finally, the results are extended to independent pulses.

3.1. Case of Coherent Pulses. The Fisher information matrix (FIM) of the parameter set ξ, denoted by $\mathbf{J}_\xi$, can be obtained according to (4) as follows [16]:

$$\mathbf{J}_{\boldsymbol{\xi}} = 2 \cdot \mathrm{Re}\left\{ \left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\xi}}\right)^H \cdot \sigma^{-2}\mathbf{I}_{2KN_0} \cdot \left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\xi}}\right)\right\}. \tag{6}$$

After straightforward derivations, the FIM of θ can be obtained from (6) as follows:

$$\mathbf{J}_{\boldsymbol{\theta}} = \frac{2}{\sigma^2}\left(\mathbf{J}_{2,2} - \mathbf{J}_{1,2}^T \mathbf{J}_{1,1}^{-1}\mathbf{J}_{1,2}\right), \tag{7}$$

where

$$\mathbf{J}_{1,1} = \|\mathbf{s}\|_2^2 \left\{ \mathrm{diag}(\boldsymbol{\alpha}) - \gamma^{-1}\boldsymbol{\alpha}\boldsymbol{\alpha}^T \right\}, \tag{8}$$

$$\mathbf{J}_{1,2} = \begin{bmatrix} \gamma^{-1}\left(\gamma_1 b_2^2 - \gamma_2 a_2^2\right)\|\mathbf{s}\|_2^2 & -\frac{2\pi}{N_0}\gamma^{-1}\left(\gamma_1 b_2^2 - \gamma_2 a_2^2\right)\beta^{(1)} & b_2^2\eta_2^{(1)} - \gamma^{-1}\left(a_2^2 + b_2^2\right)\left(\sum_{k=1}^K b_k^2 \eta_k^{(1)}\right) \\ \vdots & \vdots & \vdots \\ \gamma^{-1}\left(\gamma_1 b_K^2 - \gamma_2 a_K^2\right)\|\mathbf{s}\|_2^2 & -\frac{2\pi}{N_0}\gamma^{-1}\left(\gamma_1 b_K^2 - \gamma_2 a_K^2\right)\beta^{(1)} & b_K^2\eta_K^{(1)} - \gamma^{-1}\left(a_K^2 + b_K^2\right)\left(\sum_{k=1}^K b_k^2 \eta_k^{(1)}\right) \end{bmatrix}, \tag{9}$$

$$\mathbf{J}_{2,2} = \begin{bmatrix} \frac{\gamma_1\gamma_2}{\gamma}\|\mathbf{s}\|_2^2 & -\frac{2\pi}{N_0}\frac{\gamma_1\gamma_2}{\gamma}\beta^{(1)} & \frac{\gamma_1}{\gamma}\sum_{k=1}^K b_k^2\eta_k^{(1)} \\ -\frac{2\pi}{N_0}\frac{\gamma_1\gamma_2}{\gamma}\beta^{(1)} & \frac{\gamma_1\gamma_2}{\gamma}\left(\frac{2\pi}{N_0}\right)^2\beta^{(2)} & -\frac{2\pi}{N_0}\frac{\gamma_1}{\gamma}\left[\mathbf{s}^H\left(\sum_{k=1}^K b_k^2\mathbf{r}_k\right)\right]_r \\ \frac{\gamma_1}{\gamma}\sum_{k=1}^K b_k^2\eta_k^{(1)} & -\frac{2\pi}{N_0}\frac{\gamma_1}{\gamma}\left[\mathbf{s}^H\left(\sum_{k=1}^K b_k^2\mathbf{r}_k\right)\right]_r & \mathbf{s}^H(-\tau)\left[\left(\sum_{k=1}^K b_k^2\widetilde{\mathbf{L}}_k^2\right) - \gamma^{-1}\left(\sum_{k=1}^K b_k^2\widetilde{\mathbf{L}}_k\right)^2\right]\mathbf{s}(-\tau) \end{bmatrix}, \tag{10}$$

where $\alpha_k = a_k^2 + b_k^2$, $\alpha = [\alpha_2, \ldots, \alpha_K]^T$, $\gamma_1 = \sum_{k=1}^K a_k^2$, $\gamma_2 = \sum_{k=1}^K b_k^2$, $\gamma = \sum_{k=1}^K (a_k^2 + b_k^2)$, $\eta_k^{(1)} = \mathbf{s}^H(-\tau)\widetilde{\mathbf{L}}_k\mathbf{s}(-\tau)$, $\beta^{(1)} = \mathbf{s}^H\mathbf{F}^H\mathbf{L}\mathbf{F}\mathbf{s}$, $\beta^{(2)} = \mathbf{s}^H\mathbf{F}^H\mathbf{L}^2\mathbf{F}\mathbf{s}$, $\mathbf{r}_k = \mathbf{F}^H\mathbf{D}_\tau^H\mathbf{L}\mathbf{F}\widetilde{\mathbf{L}}_k\mathbf{F}^H\mathbf{D}_\tau\mathbf{F}\mathbf{s}$, $\widetilde{\mathbf{L}}_k = \overline{n}_k\mathbf{I}_{N_0} + \mathbf{L}$, and $\mathbf{L} = \mathrm{diag}\{\mathbf{l}\}$. Detailed derivations of (7) are listed in Appendix A.

The CRLB of θ can finally be obtained by computing the inversion of (7), i.e., $\mathrm{CRLB}_\theta = \{\mathbf{J}_\theta\}^{-1}$, whose 2nd and 3rd diagonal elements represent the theoretical lower bound for the mean square errors of the estimations of $\tau = t_d/T$ and $\nu = 2\pi f_d T$, respectively. Linear transformations can then be executed on the two diagonal elements to obtain the CRLB of t_d and f_d.

3.2. Particular Case of a Single Pulse. By setting the pulse number $K = 1$, the problem studied in this paper degenerates to the particular one containing a single pulse, which is equivalent to the widely studied problem for continuous signals [6–11]. In this particular case, $\mathbf{J}_{1,1}$ and $\mathbf{J}_{1,2}$ become empty matrices in (7), $\widetilde{\mathbf{L}}_1 = \mathbf{L}$, $\gamma_1 = 1$, $\gamma_2 = b_1^2$, and $\gamma = 1 + b_1^2$, and thus $\mathbf{J}_\theta$ simplifies to the following form:

$$\mathbf{J}_{\boldsymbol{\theta}} = \frac{2}{\sigma^2}\frac{b_1^2}{1+b_1^2}\begin{bmatrix} \|\mathbf{s}\|_2^2 & -\frac{2\pi}{N_0}\beta^{(1)} & \eta_1^{(1)} \\ -\frac{2\pi}{N_0}\beta^{(1)} & \left(\frac{2\pi}{N_0}\right)^2\beta^{(2)} & -\frac{2\pi}{N_0}\left[\mathbf{s}^H\mathbf{r}_1\right]_r \\ \eta_1^{(1)} & -\frac{2\pi}{N_0}\left[\mathbf{s}^H\mathbf{r}_1\right]_r & \mathbf{s}^H(-\tau)\mathbf{L}^2\mathbf{s}(-\tau) \end{bmatrix}. \tag{11}$$

By eliminating the differences in variable definitions between (11) and equation (20) in [12], it can be proved via straightforward mathematical derivations that the two FIMs are consistent, which indicates that the derived CRLB result is equivalent to the existing one in [12] in the particular case of a single pulse.

However, in addition to what have been revealed in the results in [12], the CRLB in (A2) also provides details about how the TDOA and FDOA estimation precisions are affected by the parameters of pulse trains. These details provide clues for making clear how the TDOA and FDOA estimation precisions will be improved when more and more pulses are being accumulated.

3.3. Case of Independent Pulses. When signals of multiple pulses are modulation-embedded and independent of each other, which is the case for emitters such as modern radars [17], the vector form of the signal sampling within each pulse is

$$\begin{aligned}\mathbf{x}_{1,k} &= \mathbf{s}_k + \mathbf{u}_{1,k}\\ \mathbf{x}_{2,k} &= b_k e^{j\phi_0} e^{j\phi_k(\nu)} \mathbf{D}_{\nu,k}\mathbf{s}_k(-\tau) + \mathbf{u}_{2,k}\ k = 1,\cdots,K.\end{aligned} \tag{12}$$

The definitions of the variables in (12) are similar to those in (3) with some slight differences, the waveforms of different pulses in (12) are distinguished by adding pulse index k as subscripts, the doppler shift matrix $\mathbf{D}_{\nu,k}$ corresponding to different pulses may differ due to unequal pulse lengths, all the amplitudes of the pulse signals at the first receiver are $a_k = 1$ as they are taken as references of the signals at the second receiver, and no random phase shift φ_k is introduced between different pulses at a certain receiver as it is contained in the reference signal.

In this observation model, the unknown parameters of the k th pulse are $\xi_k = [\mathbf{s}_{k,r}^T, \mathbf{s}_{k,i}^T, b_k]^T$, and the parameters shared by all pulses are $\theta = [\phi_0, \tau, \nu]^T$. The observation data in (12) obey a Gaussian distribution similar to that in (4), and the FIM of θ can be concluded following similar derivations as that in the first subsection of this part, which is given by

$$\mathbf{J}_{\boldsymbol{\theta}}' = \frac{2}{\sigma^2}\sum_{k=1}^{K}\frac{b_k^2}{1+b_k^2}\begin{bmatrix} \left\|\mathbf{s}_k\right\|_2^2 & -\frac{2\pi}{N_k}\mathbf{s}_k^H\mathbf{F}_k^H\mathbf{L}_k\mathbf{F}_k\mathbf{s}_k & \mathbf{s}_k^H(-\tau)\widetilde{\mathbf{L}}_k\mathbf{s}_k(-\tau) \\ -\frac{2\pi}{N_k}\mathbf{s}_k^H\mathbf{F}_k^H\mathbf{L}_k\mathbf{F}_k\mathbf{s}_k & \left\|\frac{2\pi}{N_k}\mathbf{L}_k\mathbf{F}_k\mathbf{s}_k\right\|_2^2 & -\frac{2\pi}{N_k}\left[\mathbf{s}_k^H\mathbf{F}_k^H\mathbf{D}_{\tau,k}^H\mathbf{L}_k\mathbf{F}_k\widetilde{\mathbf{L}}_k\mathbf{s}_k(-\tau)\right]_r \\ \mathbf{s}_k^H(-\tau)\widetilde{\mathbf{L}}_k\mathbf{s}_k(-\tau) & -\frac{2\pi}{N_k}\left[\mathbf{s}_k^H(-\tau)\widetilde{\mathbf{L}}_k\mathbf{F}_k^H\mathbf{L}_k\mathbf{D}_{\tau,k}\mathbf{F}_k\mathbf{s}_k\right]_r & \left\|\widetilde{\mathbf{L}}_k\mathbf{s}_k(-\tau)\right\|_2^2 \end{bmatrix}, \tag{13}$$

where $\mathbf{l}_k = [0, 1, \ldots, N_k - 1]^T$, N_k is the number of samplings within the kth pulse, $\mathbf{L}_k = \operatorname{diag}(\mathbf{l}_k)$, $\widetilde{\mathbf{L}}_k = \overline{n}_k\mathbf{I}_{N_k} + \mathbf{L}_k$, $\mathbf{D}_{\nu,k} = \operatorname{diag}\{\exp(j\nu\mathbf{l}_k)\}$, and $\mathbf{s}_k(-\tau) = \mathbf{F}_k^H\mathbf{D}_{\tau,k}\mathbf{F}_k\mathbf{s}_k$. The CRLB of θ can be obtained by computing the inverse of (13) as $CRLB_\theta' = \{\mathbf{J}_{\boldsymbol{\theta}}'\}^{-1}$, in which the 2nd and 3rd diagonal elements correspond to the lower bounds of the mean squared error of $\tau = t_d/T$ and $\nu = 2\pi f_d T$, and they give the precision bounds of t_d and f_d after linear transformations. By setting $K = 1$, it is not difficult to prove that the result in (13) in the case of a single pulse is also consistent with the results given in [12].

4. Influences of Various Factors on TDOA and FDOA Estimation Precisions

The FIM of the TDOA and FDOA in (7) indicate that the TDOA and FDOA estimation precisions for pulse trains are affected by various factors, such as signal-to-noise ratio (SNR), pulse width (PW), pulse number, and pulse repetition interval (PRI). When the number of pulses is large, the influence of a certain factor on the parameter estimation performances is affected combinatorially by the amplitudes of all pulses, making it much too complicated or even impossible to analyze theoretically the influence of each pulse's parameters on the overall TDOA and FDOA estimation precisions. In order to establish a relationship between the various factors and the parameter estimation performances, we introduce an additional assumption that the SNR of all pulses is equal, so that in-depth analyses will be available.

4.1. Simplification of FIM. When the SNR of different pulses is equal, the FIMs of coherent and independent pulses can be simplified to the formulations in (14) and (15), respectively:

$$\mathbf{J}_{\boldsymbol{\theta}} \approx \frac{2}{\sigma^2}\frac{b_1^2}{1+b_1^2}\begin{bmatrix} \operatorname{tr}(\mathbf{R}_1) & -\frac{2\pi}{N_0}\operatorname{tr}\left(\mathbf{F}^H\mathbf{LFR}_1\right) & \operatorname{tr}(\mathbf{R}_3) \\ -\frac{2\pi}{N_0}\operatorname{tr}\left(\mathbf{F}^H\mathbf{LFR}_1\right) & \left(\frac{2\pi}{N_0}\right)^2\operatorname{tr}\left(\mathbf{F}^H\mathbf{L}^2\mathbf{FR}_1\right) & -\frac{2\pi}{N_0}\left[\operatorname{tr}\left(\mathbf{F}^H\mathbf{LD}_\tau\mathbf{FR}_2\right)\right]_r \\ \operatorname{tr}(\mathbf{R}_3) & -\frac{2\pi}{N_0}\left[\operatorname{tr}\left(\mathbf{F}^H\mathbf{LD}_\tau\mathbf{FR}_2\right)\right]_r & \operatorname{tr}(\mathbf{R}_4)\end{bmatrix}, \quad (14)$$

$$\mathbf{J}'_{\boldsymbol{\theta}} \approx \frac{2}{\sigma^2}\frac{b_1^2}{1+b_1^2}\begin{bmatrix} \operatorname{tr}(\mathbf{R}'_1) & -\frac{2\pi}{N_1}\operatorname{tr}\left(\mathbf{F}_1^H\mathbf{L}_1\mathbf{F}_1\mathbf{R}'_1\right) & \operatorname{tr}(\mathbf{R}'_3) \\ -\frac{2\pi}{N_1}\operatorname{tr}\left(\mathbf{F}_1^H\mathbf{L}_1\mathbf{F}_1\mathbf{R}'_1\right) & \left(\frac{2\pi}{N_1}\right)^2\operatorname{tr}\left(\mathbf{F}_1^H\mathbf{L}_1^2\mathbf{F}_1\mathbf{R}'_1\right) & -\frac{2\pi}{N_1}\left[\operatorname{tr}\left(\mathbf{F}_1^H\mathbf{D}_{\tau,1}^H\mathbf{L}_1\mathbf{F}_1\mathbf{R}'_2\right)\right]_r \\ \operatorname{tr}(\mathbf{R}'_3) & -\frac{2\pi}{N_1}\left[\operatorname{tr}\left(\mathbf{F}_1^H\mathbf{D}_{\tau,1}^H\mathbf{L}_1\mathbf{F}_1\mathbf{R}'_2\right)\right]_r & \operatorname{tr}(\mathbf{R}'_4)\end{bmatrix}, \quad (15)$$

where $\operatorname{tr}(\cdot)$ represents the trace operator, $\mathbf{R}_1 = K\mathbf{s}\mathbf{s}^H$, $\mathbf{R}_2 = \overline{n}_0 K\mathbf{s}(-\tau)\mathbf{s}^H$, $\mathbf{R}_3 = \overline{n}_0 K\mathbf{s}(-\tau)\mathbf{s}^H(-\tau)$, $\mathbf{R}_4 = \sum_{k=1}^K \overline{n}_k^2\mathbf{s}(-\tau)\mathbf{s}^H(-\tau)$, $\mathbf{R}'_1 = \sum_{k=1}^K \mathbf{s}_k\mathbf{s}_k^H$, $\mathbf{R}'_2 = \sum_{k=1}^K \overline{n}_k\mathbf{s}_k(-\tau)\mathbf{s}_k^H$, $\mathbf{R}'_3 = \sum_{k=1}^K \overline{n}_k\mathbf{s}_k(-\tau)\mathbf{s}_k^H(-\tau)$, and $\mathbf{R}'_4 = \sum_{k=1}^K \overline{n}_k^2\mathbf{s}_k(-\tau)\mathbf{s}_k^H(-\tau)$. Detailed derivations for the simplifications are listed in Appendix B.

The FIMs in (14) and (15) corresponding to cases of coherent and independent pulse trains have very similar formulations. The only difference lies in the matrix sets of $\mathbf{R} = \{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4\}$ and $\mathbf{R}' = \{\mathbf{R}'_1, \mathbf{R}'_2, \mathbf{R}'_3, \mathbf{R}'_4\}$, which is caused by the distinction in waveforms of independent pulses. When the lengths and SNR of independent pulses are equal and the number of pulses is very large, the difference is very weak, which becomes negligible according to the law of large numbers. In both cases, the CRLB obtained by FIM inversion will also be approximately equivalent. This result indicates that although coherent pulse trains have much fewer unknown parameters than independent pulse trains due to repeated pulse waveforms, the reduction of the unknown parameter dimension does not lead to significant improvements in TDOA and FDOA estimation precisions. This conclusion is similar to the one in [14], which points out that prior information about the signal waveform contributes only slightly to the frequency estimation precision.

4.2. Influence Analyses of Different Factors. Based on the simplified FIM, this section analyzes how the TDOA and FDOA estimation precisions are affected by various factors, including SNR, pulse width, pulse number, and PRI.

When the SNR of the pulses varies, the values of the matrices in $\mathbf{R}$ change with it, which results in different FIMs in (14). However, the relative pulse amplitudes at the two receivers, i.e., b_1^2, keep fixed, and the value of each element in $\mathbf{J}_\theta$ is proportional to the SNR. Therefore, the values of the elements in $CRLB_\theta$ obtained by FIM inversion are inversely proportional to the SNR, and the root mean squared errors (RMSEs) of the TDOA and the FDOA estimates, i.e., $CRLB_{\tau_d}$ and $CRLB_{f_d}$, are inversely proportional to the square root of the SNR, i.e., $SNR^{-1/2}$.

The variation of the pulse width changes the number of samplings in each pulse, i.e., N_0, and also changes the dimensions of the matrices in the trace operator $\operatorname{tr}(\cdot)$ and the diagonal weighting matrix $\mathbf{L}$ in (14). As the SNR of the pulses is assumed to be constant, the signal energy increases linearly with increasing pulse width. In addition, the carrier frequency f_0 of approximately narrowband pulse signals can be denoted coarsely as a proportion to the sampling frequency f_s, i.e., $f_0 = \rho f_s$ with $0 < \rho < 1$. Thus, in the time-frequency conversed vector $\mathbf{Fs}$ in (14), only the element on the (ρN_0)th row has a significant non-zero value, and only the weight coefficients of ρN_0 and $(\rho N_0)^2$ in $\mathbf{L}$ and $\mathbf{L}^2$ dominate the weighted sums of $\operatorname{tr}(\mathbf{F}^H\mathbf{LFR}_1)$ and $\operatorname{tr}(\mathbf{F}^H\mathbf{L}^2\mathbf{FR}_1)$. By jointly considering the linear increase of signal energy with pulse width, it can be concluded that the values of matrices in $\mathbf{R} = \{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4\}$ increase squarely or cubically with PW, which depends on whether the matrix is weighted linearly or squarely by $\mathbf{L}$. Therefore, the values of each element in $\mathbf{J}_\theta$ are linearly proportional to PW according to (14). By combining it with the computation process of matrix inversion, we can infer the relationship of $CRLB_{\tau_d}$ and $CRLB_{f_d}$ w.r.t. PW as $CRLB_{\tau_d} \propto PW^{-1/2}$ and $CRLB_{f_d} \propto PW^{-1/2}$.

In order to analyze the influences of pulse number and PRI on the TDOA and FDOA estimation precisions, we further assume that the pulses are equally spaced along the time axis, and the PRI is equal to $\overline{n}$ times the sampling interval, and then the proportionalities of the matrices in the set of $\mathbf{R}$ with the pulse number K and the PRI are $\mathbf{R}_1 = K\mathbf{s}\mathbf{s}^H \propto \{K^1, PRI^0\}$, $\mathbf{R}_2 = ((K-1)(K-2)/2)\overline{n}\mathbf{s}(-\tau)\mathbf{s}^H \approx \propto \{K^2, PRI^1\}$, $\mathbf{R}_3 = ((K-1)(K-2)/2)\overline{n}\mathbf{s}(-\tau)\mathbf{s}^H(-\tau) \approx \propto \{K^2, PRI^1\}$, and $\mathbf{R}_4 = ((K-1)(K-2)(2K-3)$

$/6)\overline{n}^2\mathbf{s}\ (-\tau)\mathbf{s}^H(-\tau) \approx \propto \{K^3, PRI^2\}$, where $\approx\propto$ stands for "approximately proportional to." The relative approximation error is roughly $1.5/K$, which is negligible in practical applications when K is as large as several tens or even thousands. Then, we have

$$\mathbf{J}_{\boldsymbol{\theta}} \approx \propto \begin{bmatrix} K^1 & K^1 & K^2 \\ K^1 & K^1 & K^2 \\ K^2 & K^2 & K^3 \end{bmatrix}, \mathbf{J}_{\boldsymbol{\theta}} \approx \propto \begin{bmatrix} PRI^0 & PRI^0 & PRI^1 \\ PRI^0 & PRI^0 & PRI^1 \\ PRI^1 & PRI^1 & PRI^2 \end{bmatrix}. \quad (16)$$

Furthermore, by combining it with the process of CRLB computation via FIM inversion, the relationship between the TDOA and FDOA estimation CRLBs with the pulse number and the PRI can be concluded to be $CRLB_{\tau_d} \propto \{K^{-1/2}, PRI^0\}$ and $CRLB_{f_d} \propto \{K^{-3/2}, PRI^{-1}\}$.

In summary, the TDOA and FDOA estimation precisions of coherent pulses are affected by various factors in a way as that shown in Table 1.

It is indicated by Table 1 that the PRI has no influence on the TDOA estimation accuracy, and the pulse number and the PRI (which jointly determine the time span of the observed pulse signals) affect the FDOA estimation accuracy very significantly. In practical TDOA and FDOA estimation applications, the signal parameters of SNR, PW, and PRI are determined by the emitter and the signal propagation path and are not revisable at the non-cooperative receiver. However, one can improve the TDOA and FDOA estimation precisions by increasing the pulse number K. An efficient way to enlarge K is gathering pulses of the same emitter, which may be scattered along the time axis when some pulses between them are not received due to low SNR. Gathering such pulses also increases PRI as a by-product and helps to further improve FDOA estimation precision according to Table 1.

In the case of independent pulses, the difference between the FIM in (15) and its counterpart for coherent pulses in (15) is caused by the difference between the matrix sets $\mathbf{R}$ and $\mathbf{R}'$. Each matrix in $\mathbf{R}'$ contains an invariant component that depends on pulse energy and some other stationary factors, together with a variable component that depends on specific modulations of different pulses. When the widths and amplitudes of different pulses are equal, the variable components of different pulses cancel each other, and the invariant components accumulate linearly or super-linearly according to the weighting matrices. As the number of pulses increases, the variable components cancel each other and tend to zero according to the law of large numbers. Therefore, the matrices in R' are affected by different factors in a similar way as the matrices in $\mathbf{R}$. Based on this approximated model, it can be proved by similar analyses as these in this subsection that the TDOA and FDOA estimation accuracies of independent pulses are affected roughly in the same way as that shown in Table 1.

5. Simulations and Analyses

In this part, we carry out simulations to demonstrate the correctness of the CRLB results and the conclusions on the influences of different factors on TDOA and FDOA estimation precisions. Assume that the pulse signals are linear frequency modulated (LFM) with a bandwidth of 1 MHz, and the signal carrier frequency is 1 GHz. Therefore, the signals are approximately narrowband. The received signals are down-converted to a low intermediate frequency and then sampled with a frequency of 10 MHz. The time delay of the signals at the two receivers is 0 s, and the frequency shift is 1 kHz.

In the first group of simulations, we fix the pulse width at 30 us, the SNR at 5 dB, and the PRI at 100 us and increase the pulse number from 5 to 80. The TDOA and FDOA estimation CRLBs obtained from (7) are shown in Table 2. The results show that when the number of pulses increases from 5 to 80, the theoretical accuracy of the TDOA estimate improves from 0.49 ns to 0.12 ns, and that of the FDOA estimate improves from 13.2 Hz to 0.24 Hz. The amplitudes of improvements are approximately 4 and 55 times, respectively, which are roughly inversely proportional to the 1/2 and 3/2 powers of the pulse number. This result is basically consistent with the results in Table 1, and the slight difference between them is mainly caused by the effect of finite pulses, which leads to a deviation between the accumulation operations in simulations and the expectation operations in theoretical analyses.

Based on the above simulations, we then fix the number of coherent pulses at 40 and vary the SNR of the pulse signals from −5 dB to 35 dB. The TDOA and FDOA estimation CRLBs are obtained and shown in Table 3. When the SNR increases by 40 dB in the simulations, the TDOA and FDOA estimation CRLB increases from 0.54 ns to $5.4e-3$ ns and from 2.16 Hz to 0.0216 Hz, respectively, which both improve by 100 times. The results indicate that the TDOA and FDOA estimation precisions are proportional to the inverse of the square root of the SNR, which is consistent with the results in Table 1.

Then, we fix the number of coherent pulses at 40 and the SNR on both receivers at 5 dB and then vary the pulse width from 5 us to 30 us. The TDOA and FDOA estimation CRLBs are shown in Table 4. When the pulse width increases by 6 times, the theoretical lower bounds of the TDOA and FDOA estimation precisions are reduced from 0.44 ns to 0.17 ns and from 1.7 Hz to 0.68 Hz, respectively, which are improved by 2.6 times and 2.5 times. The amplitudes of improvement are approximately equal to $\sqrt{6}$, which are consistent with the results in Table 1.

Finally, we fix the number of coherent pulses at 40, the SNR on both receivers at 5 dB, and the pulse width at 30 us and increase the PRI from 0.1 ms to 10 ms. The TDOA and FDOA estimation CRLBs are shown in Table 5. When the PRI increases by 100 times, the theoretical lower bound of the TDOA estimation accuracy keeps unchanged at about 0.17 ns, and that of the FDOA estimation accuracy improves from 0.68 Hz to 0.0068 Hz. The improvement amplitudes are 0 times and 100 times, respectively, which are immune to and linearly proportional to the PRI, respectively. The results are consistent with those in Table 1.

Table 1: Influences of pulse parameters on TDOA and FDOA estimation precisions.

	SNR	PW	Pulse number (K)	PRI
$CRLB_{\tau_d}$	$SNR^{-1/2}$	$PW^{-1/2}$	$K^{-1/2}$	PRI^0
$CRLB_{f_d}$	$SNR^{-1/2}$	$PW^{-1/2}$	$K^{-3/2}$	PRI^{-1}

Table 2: TDOA and FDOA estimation precision for different pulse numbers.

Pulse number	5	10	20	40	80
TDOA (ns)	0.49	0.34	0.24	0.17	0.12
FDOA (Hz)	13.2	5.04	1.92	0.68	0.24

Table 3: TDOA and FDOA estimation precision for different SNRs.

SNR (dB)	−5	5	15	25	35
TDOA (ns)	0.54	0.171	0.054	0.017	$5.4e-3$
FDOA (Hz)	2.16	0.683	0.216	0.0683	0.0216

Table 4: TDOA and FDOA estimation precision for different pulse widths.

Pulse width (us)	5	10	20	30
TDOA (ns)	0.44	0.31	0.22	0.17
FDOA (Hz)	1.7	1.18	0.85	0.68

Table 5: TDOA and FDOA estimation precision for different PRIs.

PRI (ms)	0.1	1	10
TDOA (ns)	0.17	0.17	0.17
FDOA (Hz)	0.68	0.068	0.0068

6. Conclusions

This paper analyzes the TDOA and FDOA estimation CRLBs of pulse signals and provides deep insights to reveal how the CRLBs are affected by various factors, including signal-to-noise ratio (SNR), pulse width (PW), pulse number, and pulse repetition interval (PRI). Theoretical results indicate that the TDOA and FDOA estimation precisions of coherent pulse signals vary roughly in the same way with respect to different environmental factors as those of independent pulse signals. Simulation results in scenarios of varying pulse numbers, SNRs, pulse widths, and PRIs demonstrate the correctness of the theoretical results on how TDOA and FDOA estimation precisions are affected by these factors.

Appendix

A. Derivation of the Fisher Information Matrix in the Case of Coherent Pulses

The blocks in the partial derivative $\partial\mu/\partial\xi$ in (7) are given as follows:

$$
\begin{aligned}
\frac{\partial \boldsymbol{\mu}_k}{\partial \boldsymbol{\eta}} &= \begin{bmatrix} a_k e^{j\varphi_k}\mathbf{I}_{N_0} & j a_k e^{j\varphi_k}\mathbf{I}_{N_0} \\ b_k \mathbf{P}_k & j b_k \mathbf{P}_k \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{Q}_{k,1} \\ \mathbf{Q}_{k,2} \end{bmatrix} = \mathbf{Q}_k, \\
\frac{\partial \boldsymbol{\mu}_k}{\partial \mathbf{a}} \triangleq \mathbf{Y}_{1,k} &= \begin{cases} \mathbf{0}, & k = 1, \\ \begin{array}{c} \begin{array}{ccccccc} & & & \scriptstyle (k-1)-th & & & \end{array} \\ \begin{bmatrix} \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} & e^{j\varphi_k}\mathbf{s} & \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} \\ \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} & \mathbf{0}_{N_0\times 1} & \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} \end{bmatrix} \end{array}, & k > 1, \end{cases} \\
\frac{\partial \boldsymbol{\mu}_k}{\partial \mathbf{b}} \triangleq \mathbf{Y}_{2,k} &= \begin{array}{c} \begin{array}{ccccccc} & & & \scriptstyle k-th & & & \end{array} \\ \begin{bmatrix} \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} & \mathbf{0}_{N_0\times 1} & \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} \\ \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} & \mathbf{P}_k\mathbf{s} & \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} \end{bmatrix} \end{array} \\
\frac{\partial \boldsymbol{\mu}_k}{\partial \boldsymbol{\varphi}} \triangleq \mathbf{Y}_{3,k} &= \begin{cases} \mathbf{0}, & k = 1, \\ \begin{array}{c} \begin{array}{ccccccc} & & & \scriptstyle (k-1)-th & & & \end{array} \\ \begin{bmatrix} \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} & j a_k e^{j\varphi_k}\mathbf{s} & \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} \\ \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} & j b_k \mathbf{P}_k\mathbf{s} & \mathbf{0}_{N_0\times 1} & \cdots & \mathbf{0}_{N_0\times 1} \end{bmatrix} \end{array}, & k > 1, \end{cases} \\
\frac{\partial \boldsymbol{\mu}_k}{\partial \boldsymbol{\theta}} &= \begin{bmatrix} \mathbf{0}_{N_0\times 1} & \mathbf{0}_{N_0\times 1} & \mathbf{0}_{N_0\times 1} \\ \mathbf{g}_{k,1} & \mathbf{g}_{k,2} & \mathbf{g}_{k,3} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{0}_{N_0\times 3} \\ \mathbf{G}_k \end{bmatrix}, \\
&= \widetilde{\mathbf{G}}_k,
\end{aligned}
\tag{A.1}
$$

where $\mathbf{P}_k = e^{j\varphi_k} e^{j\phi_0} e^{j\phi_k(\nu)} \mathbf{D}_\nu \mathbf{F}^H \mathbf{D}_\tau \mathbf{F}$, $\mathbf{g}_{k,1} = j b_k \mathbf{P}_k \mathbf{s}$, $\mathbf{g}_{k,2} = -j2\pi/N_0 b_k e^{j\varphi_k} e^{j\phi_0} e^{j\phi_k(\nu)} \mathbf{D}_\nu \mathbf{F}^H \mathbf{L} \mathbf{D}_\tau \mathbf{F}\mathbf{s}$, $\mathbf{g}_{k,3} = j b_k (\overline{n}_k \mathbf{I}_{N_0} + \mathbf{L}) \mathbf{P}_k \mathbf{s}$, $\mathbf{L} = \mathrm{diag}\{\mathbf{l}\}$, and $\widetilde{\mathbf{L}}_k = \overline{n}_k \mathbf{I}_{N_0} + \mathbf{L}$. Thus,

$$
\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Y}_{1,1} & \mathbf{Y}_{2,1} & \mathbf{Y}_{3,1} & \widetilde{\mathbf{G}}_1 \\ \mathbf{Q}_2 & \mathbf{Y}_{1,2} & \mathbf{Y}_{2,2} & \mathbf{Y}_{3,2} & \widetilde{\mathbf{G}}_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{Q}_K & \mathbf{Y}_{1,K} & \mathbf{Y}_{2,K} & \mathbf{Y}_{3,K} & \widetilde{\mathbf{G}}_K \end{bmatrix}. \tag{A.2}
$$

The item in the brace on the right-hand side of (7) can be expressed as

$$\left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\xi}}\right)^{H} \cdot \sigma^{-2}\mathbf{I}_{2KN_0} \cdot \left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\xi}}\right) = \sigma^{-2}\begin{bmatrix} \sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Q}_k & \sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Y}_{1,k} & \sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Y}_{2,k} & \sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Y}_{3,k} & \sum_{k=1}^{K}\mathbf{Q}_k^H\widetilde{\mathbf{G}}_k \\ \sum_{k=1}^{K}\mathbf{Y}_{1,k}^H\mathbf{Q}_k & \sum_{k=1}^{K}\mathbf{Y}_{1,k}^H\mathbf{Y}_{1,k} & \sum_{k=1}^{K}\mathbf{Y}_{1,k}^H\mathbf{Y}_{2,k} & \sum_{k=1}^{K}\mathbf{Y}_{1,k}^H\mathbf{Y}_{3,k} & \sum_{k=1}^{K}\mathbf{Y}_{1,k}^H\widetilde{\mathbf{G}}_k \\ \sum_{k=1}^{K}\mathbf{Y}_{2,k}^H\mathbf{Q}_k & \sum_{k=1}^{K}\mathbf{Y}_{2,k}^H\mathbf{Y}_{1,k} & \sum_{k=1}^{K}\mathbf{Y}_{2,k}^H\mathbf{Y}_{2,k} & \sum_{k=1}^{K}\mathbf{Y}_{2,k}^H\mathbf{Y}_{3,k} & \sum_{k=1}^{K}\mathbf{Y}_{2,k}^H\widetilde{\mathbf{G}}_k \\ \sum_{k=1}^{K}\mathbf{Y}_{3,k}^H\mathbf{Q}_k & \sum_{k=1}^{K}\mathbf{Y}_{3,k}^H\mathbf{Y}_{1,k} & \sum_{k=1}^{K}\mathbf{Y}_{3,k}^H\mathbf{Y}_{2,k} & \sum_{k=1}^{K}\mathbf{Y}_{3,k}^H\mathbf{Y}_{3,k} & \sum_{k=1}^{K}\mathbf{Y}_{3,k}^H\widetilde{\mathbf{G}}_k \\ \sum_{k=1}^{K}\widetilde{\mathbf{G}}_k^H\mathbf{Q}_k & \sum_{k=1}^{K}\widetilde{\mathbf{G}}_k^H\mathbf{Y}_{1,k} & \sum_{k=1}^{K}\widetilde{\mathbf{G}}_k^H\mathbf{Y}_{2,k} & \sum_{k=1}^{K}\widetilde{\mathbf{G}}_k^H\mathbf{Y}_{3,k} & \sum_{k=1}^{K}\widetilde{\mathbf{G}}_k^H\widetilde{\mathbf{G}}_k \end{bmatrix}. \tag{A.3}$$

Denote $\widetilde{\mathbf{B}} = [\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4]$, where

$$\mathbf{B}_1 = \mathrm{Re}\left\{\sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Y}_{1,k}\right\} = \begin{bmatrix} a_2\mathbf{s}_r & \cdots & a_K\mathbf{s}_r \\ a_2\mathbf{s}_i & \cdots & a_K\mathbf{s}_i \end{bmatrix}, \tag{A.4}$$

$$\mathbf{B}_2 = \mathrm{Re}\left\{\sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Y}_{2,k}\right\} = \begin{bmatrix} b_1\mathbf{s}_r & b_2\mathbf{s}_r & \cdots & b_K\mathbf{s}_r \\ b_1\mathbf{s}_i & b_2\mathbf{s}_i & \cdots & b_K\mathbf{s}_i \end{bmatrix}, \tag{A.5}$$

$$\mathbf{B}_3 = \mathrm{Re}\left\{\sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Y}_{3,k}\right\} = \begin{bmatrix} -\left(a_2^2+b_2^2\right)\mathbf{s}_i & \cdots & -\left(a_K^2+b_K^2\right)\mathbf{s}_i \\ \left(a_2^2+b_2^2\right)\mathbf{s}_r & \cdots & \left(a_K^2+b_K^2\right)\mathbf{s}_r \end{bmatrix}, \tag{A.6}$$

$$\mathbf{B}_4 = \mathrm{Re}\left\{\sum_{k=1}^{K}\mathbf{Q}_k^H\widetilde{\mathbf{G}}_k\right\} = \begin{bmatrix} -\gamma_2\mathbf{s}_i & \frac{2\pi}{N_0}\gamma_2(\widetilde{\mathbf{r}})_i & -\left(\mathbf{F}^H\mathbf{D}_\tau^H\mathbf{F}\left(\sum_{k=1}^{K}b_k^2\widetilde{\mathbf{L}}_k\right)\mathbf{s}(-\tau)\right)_i \\ \gamma_2\mathbf{s}_r & -\frac{2\pi}{N_0}\gamma_2(\widetilde{\mathbf{r}})_r & \left(\mathbf{F}^H\mathbf{D}_\tau^H\mathbf{F}\left(\sum_{k=1}^{K}b_k^2\widetilde{\mathbf{L}}_k\right)\mathbf{s}(-\tau)\right)_r \end{bmatrix}, \tag{A.7}$$

where $\gamma_1 = \sum_{k=1}^{K} a_k^2$, $\gamma_2 = \sum_{k=1}^{K} b_k^2$, $\gamma = \sum_{k=1}^{K}(a_k^2 + b_k^2)$, and $\widetilde{\mathbf{r}} = \mathbf{F}^H\mathbf{LFs}$. It can be concluded from (A.4)–(A.7) that $\begin{bmatrix}\mathbf{B}_1^T \\ \mathbf{B}_2^T\end{bmatrix}[\,\mathbf{B}_3 \;\; \mathbf{B}_4\,] = \mathbf{0}$, which means that the last two block elements in the second and third rows in (A.3) are all 0s, together with the submatrices in the symmetrical positions. Denote the real part of the lower-right submatrix consisting of 4×4 blocks in (A.3) by $\widetilde{\mathbf{Z}}$, and it can be concluded via straightforward calculations that

$$\widetilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Z}_3 & \mathbf{C} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}^T & \mathbf{Z}_4 \end{bmatrix}, \tag{A.8}$$

where

$$\mathbf{Z}_1 = \mathrm{Re}\left\{\sum_{k=1}^{K} \mathbf{Y}_{1,k}^H \mathbf{Y}_{1,k}\right\} = \|\mathbf{s}\|_2^2 \times \mathbf{I}_{K-1},$$

$$\mathbf{Z}_2 = \mathrm{Re}\left\{\sum_{k=1}^{K} \mathbf{Y}_{2,k}^H \mathbf{Y}_{2,k}\right\} = \|\mathbf{s}\|_2^2 \times \mathbf{I}_{K},$$

$$\mathbf{Z}_3 = \mathrm{Re}\left\{\sum_{k=1}^{K} \mathbf{Y}_{3,k}^H \mathbf{Y}_{3,k}\right\} = \|\mathbf{s}\|_2^2 \times \mathrm{diag}(\boldsymbol{\alpha}),$$

$$\mathbf{Z}_4 = \mathrm{Re}\left\{\sum_{k=1}^{K} \widetilde{\mathbf{G}}_k^H \widetilde{\mathbf{G}}_k\right\} = \begin{bmatrix} \gamma_2\|\mathbf{s}\|_2^2 & -\frac{2\pi}{N_0}\gamma_2\beta^{(1)} & \sum_{k=1}^{K} b_k^2\eta_k^{(1)} \\ -\frac{2\pi}{N_0}\gamma_2\beta^{(1)} & \gamma_2\left(\frac{2\pi}{N_0}\right)^2\beta^{(2)} & -\frac{2\pi}{N_0}\left[\mathbf{s}^H\left(\sum_{k=1}^{K} b_k^2\mathbf{r}_k\right)\right]_r \\ \sum_{k=1}^{K} b_k^2\eta_k^{(1)} & -\frac{2\pi}{N_0}\left[\mathbf{s}^H\left(\sum_{k=1}^{K} b_k^2\mathbf{r}_k\right)\right]_r & \sum_{k=1}^{K} b_k^2\eta_k^{(2)} \end{bmatrix}, \tag{A.9}$$

$$\mathbf{C} = \mathrm{Re}\left\{\sum_{k=1}^{K} \mathbf{Y}_{3,k}^H \widetilde{\mathbf{G}}_k\right\} = \begin{bmatrix} b_2^2\|\mathbf{s}\|_2^2 & -\frac{2\pi}{N_0}b_2^2\beta^{(1)} & b_2^2\eta_2^{(1)} \\ \vdots & \vdots & \vdots \\ b_K^2\|\mathbf{s}\|_2^2 & -\frac{2\pi}{N_0}b_K^2\beta^{(1)} & b_K^2\eta_K^{(1)} \end{bmatrix},$$

where $\alpha_k = a_k^2 + b_k^2$, $\alpha = [\alpha_2, \ldots, \alpha_K]^T$, $\beta^{(1)} = \mathbf{s}^H\mathbf{F}^H\mathbf{L}\mathbf{F}\mathbf{s}$, $\beta^{(2)} = \mathbf{s}^H\mathbf{F}^H\mathbf{L}^2\mathbf{F}\mathbf{s}$, $\eta_k^{(1)} = \mathbf{s}^H(-\tau)\widetilde{\mathbf{L}}_k\mathbf{s}(-\tau)$, $\eta_k^{(2)} = \mathbf{s}^H(-\tau)\widetilde{\mathbf{L}}_k^2\mathbf{s}(-\tau)$, and $\mathbf{r}_k = \mathbf{F}^H\mathbf{D}_\tau^H\mathbf{L}\mathbf{F}\widetilde{\mathbf{L}}_k\mathbf{F}^H\mathbf{D}_\tau\mathbf{F}\mathbf{s}$.

In addition, denote $\mathbf{Z}_0 = \mathrm{Re}\{\sum_{k=1}^{K}\mathbf{Q}_k^H\mathbf{Q}_k\} = (\sum_{k=1}^{K}(a_k^2 + b_k^2)) \times \mathbf{I}_{2N_0}$; then, the FIM of the parameter set $\xi_1 = [\mathbf{a}^T, \mathbf{b}^T, \varphi^T, \theta^T]^T$ can be derived according to (16) using the matrix inversion lemma [18] as follows:

$$\mathbf{J}_{\xi_1} = \frac{2}{\sigma^2}\left(\widetilde{\mathbf{Z}} - \widetilde{\mathbf{B}}^T\mathbf{Z}_0^{-1}\widetilde{\mathbf{B}}\right). \tag{A.10}$$

Based on the formulations of the matrices in (A.10), the upper-right and lower-left $(2K-1) \times (K+2)$ matrices of $\mathbf{J}_{\xi_1}$ can be concluded to be 0, which implies that the estimation performances of the parameter sets $[\mathbf{a}^T, \mathbf{b}^T]^T$ and $[\varphi^T, \theta^T]^T$ are independent of each other. Denote $\xi_2 = [\varphi^T, \theta^T]^T$; then, its FIM can be obtained from (A.10) as follows [18]:

$$\mathbf{J}_{\xi_2} = \frac{2}{\sigma^2}\left(\begin{bmatrix} \mathbf{Z}_3 & \mathbf{C} \\ \mathbf{C}^T & \mathbf{Z}_4 \end{bmatrix} - \gamma^{-1}\begin{bmatrix} \mathbf{B}_3^T\mathbf{B}_3 & \mathbf{B}_3^T\mathbf{B}_4 \\ \mathbf{B}_4^T\mathbf{B}_3 & \mathbf{B}_4^T\mathbf{B}_4 \end{bmatrix}\right) \triangleq \frac{2}{\sigma^2}\begin{bmatrix} \mathbf{J}_{1,1} & \mathbf{J}_{1,2} \\ \mathbf{J}_{1,2}^T & \mathbf{J}_{2,2} \end{bmatrix}, \tag{A.11}$$

where $\mathbf{J}_{1,1}$, $\mathbf{J}_{1,2}$, and $\mathbf{J}_{2,2}$ are defined in (8)–(10).

The FIM of θ can be finally obtained from (A.11) by using the matrix inversion lemma [18] to conclude in the form given in (7).

B. Derivations for FIM Simplification

When the SNR of different pulses is equal, some simplifications hold, i.e., $\gamma_1 = Ka_1^2 = K$, $\gamma_2 = Kb_1^2$, $\gamma = K(1 + b_1^2)$, $\alpha_k = 1 + b_1^2$ for $k = 1, \ldots, K$, $\alpha = (1 + b_1^2)\mathbf{1}_{K-1}$, and thus (8)–(10) can be rewritten as

$$\mathbf{J}_{1,1} = \left(1 + b_1^2\right)\|\mathbf{s}\|_2^2 \left\{\mathbf{I}_{K-1} - \frac{1}{K}\mathbf{1}_{K-1}\mathbf{1}_{K-1}^T\right\},$$

$$\mathbf{J}_{1,2} = \left[\, \mathbf{0}_{K-1} \;\; \mathbf{0}_{K-1} \;\; b_1^2 \|\mathbf{s}(-\tau)\|_2^2 \left(\overline{\mathbf{n}} - \overline{n}_0 \mathbf{1}_{K-1}\right) \right],$$

$$\mathbf{J}_{2,2} = \frac{b_1^2}{1+b_1^2}\begin{bmatrix} K\|\mathbf{s}\|_2^2 & -\dfrac{2\pi}{N_0}K\beta^{(1)} & \displaystyle\sum_{k=1}^{K}\eta_k^{(1)} \\ -\dfrac{2\pi}{N_0}K\beta^{(1)} & \left(\dfrac{2\pi}{N_0}\right)^2 K\beta^{(2)} & -\dfrac{2\pi}{N_0}\left[\mathbf{s}^H\left(\displaystyle\sum_{k=1}^{K}\mathbf{r}_k\right)\right]_r \\ \displaystyle\sum_{k=1}^{K}\eta_k^{(1)} & -\dfrac{2\pi}{N_0}\left[\mathbf{s}^H\left(\displaystyle\sum_{k=1}^{K}\mathbf{r}_k\right)\right]_r & \mathbf{s}^H(-\tau)\left[\displaystyle\sum_{k=1}^{K}\widetilde{\mathbf{L}}_k^2 + b_1^2\left(\sum_{k=1}^{K}\overline{n}_k^2 - K\overline{n}_0^2\right)\mathbf{I}_{N_0}\right]\mathbf{s}(-\tau) \end{bmatrix}, \tag{A.12}$$

where $\overline{\mathbf{n}} = [\overline{n}_2, \ldots, \overline{n}_K]^T$ and $\overline{n}_0 = (1/K)\sum_{k=1}^{K}\overline{n}_k$. Moreover,

$$\mathbf{J}_{\boldsymbol{\theta}} = \frac{2}{\sigma^2}\frac{b_1^2}{1+b_1^2}\begin{bmatrix} K\|\mathbf{s}\|_2^2 & -\dfrac{2\pi}{N_0}K\beta^{(1)} & \displaystyle\sum_{k=1}^{K}\eta_k^{(1)} \\ -\dfrac{2\pi}{N_0}K\beta^{(1)} & \left(\dfrac{2\pi}{N_0}\right)^2 K\beta^{(2)} & -\dfrac{2\pi}{N_0}\left[\mathbf{s}^H\left(\displaystyle\sum_{k=1}^{K}\mathbf{r}_k\right)\right]_r \\ \displaystyle\sum_{k=1}^{K}\eta_k^{(1)} & -\dfrac{2\pi}{N_0}\left[\mathbf{s}^H\left(\displaystyle\sum_{k=1}^{K}\mathbf{r}_k\right)\right]_r & \mathbf{s}^H(-\tau)\left(\dfrac{1}{K}\displaystyle\sum_{k=1}^{K}\left(\mathbf{L}+\overline{n}_k\mathbf{I}_{N_0}\right)^2\right)\mathbf{s}(-\tau) \end{bmatrix}. \tag{A.13}$$

As the PRI is often much larger than the pulse width in most pulse-radiating systems such as radar [13, 17], $\overline{n}_k \gg N_0$ for $k \geq 2$, and the FIM in (A.13) can be approximated to the formulation in (13) by neglecting high-order minima. Similarly, the FIM in (13) corresponding to independent pulse trains can also be simplified to the formulation in (15).

Data Availability

The data used in this study were generated via simulations. Readers can get the data and repeat the simulations following the illustrations in our study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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