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Review Article

Review of Single Bubble Motion Characteristics Rising in Viscoelastic Liquids

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The emphasis of this review is to discuss three peculiar phenomena of bubbles rising in viscoelastic fluids, namely, the formation of a cusp, negative wake, and velocity jump discontinuity, and to highlight the possible future directions of the subject. The mechanism and influencing factors of these three peculiar phenomena have been discussed in detail in this review. The evolution of the bubble shape is mainly related to the viscoelasticity of the fluid. However, the mechanisms of the two-dimensional cusp, tip-streaming, "blade-edge" tip, "fish-bone" tip, and the phenomenon of the tail breaking into two different threads, in some special viscoelastic fluids, are not understood clearly. The origin of the negative wake behind the bubbles rising in a viscoelastic fluid can be attributed to the synergistic effect of the liquid-phase viscoelasticity, and the bubbles are large enough; thus, leading to a very long relaxation time taken by the viscoelastic stresses. For the phenomenon of bubble velocity jump discontinuity, viscoelasticity is the most critical factor, and the cusp of the bubbles and the surface modifications play only ancillary roles. It has also been observed that a negative wake does not cause velocity jump discontinuity.

1. Introduction

The phenomenon of gas-liquid two-phase flow is frequently encountered in many industrial processes such as mineral processing, distillation, absorption, anaerobic fermentation, wastewater treatment, and petroleum extraction [1–6]. In these processes, the dispersed gas phase usually exists in the form of bubbles, and characteristics of the bubble motion such as velocity, shape, trajectory, and flow field are the key parameters in the design of the gas-liquid contact processes. While the abovementioned processes encounter bubble swarms, understanding the behavior of the motion of a single bubble can provide important insights for optimally designing the processes involving bubble swarms. Consequently, efforts have been made to explore the velocity, shape, trajectory, drag coefficients, and flow field of a single bubble rising in the liquid phase [7–11]. Nevertheless, most of these studies focus on Newtonian fluids, whereas a viscoelastic fluid is a very important type of non-Newtonian fluid that possesses viscosity as well as elasticity, and the behavior of bubbles in this fluid type is quite different from that in Newtonian fluids.

In this review, the behaviors of viscoelastic fluids have been briefly introduced. Furthermore, the mechanisms of three peculiar phenomena, namely, the formation of a cusp, negative wake, and velocity jump discontinuity, have been discussed, and future development directions have been proposed. This provides a reference for the process of bubble movement in viscoelastic fluids and the optimal design of the relevant equipment in the industry.

2. Viscoelastic Fluid Behavior

For Newtonian fluids, the shear stress is proportional to the shear rate, and the proportionality constant is the dynamic viscosity of the fluid, which is nothing but the classical Newton's law of internal friction. However, most fluids in industrial processes, such as polymer solutions, blood, various paints and coatings, butter, sludge, coal water slurry, and fermentation liquid, do not follow Newtonian internal friction laws. Fluids that do not follow Newton's law of internal friction are called non-Newtonian fluids. Non-Newtonian fluids are divided into pseudoplastic fluid, dilatant fluid, Bingham fluid, and viscoelastic fluid [12]. A viscoelastic fluid is one of the most common non-Newtonian fluids in the industry and shows peculiar characteristics, such as the die swell effect [13], rod-climbing effect (Weissenberg effect) [14], and tubeless siphon effect [15], as shown in Figure 1. These peculiar characteristics are mainly related to the viscoelasticity of the fluid. In the process of fluid flow, the molecular chain is not completely relaxed in terms of the unrecoverable plastic deformation and the recoverable elastic deformation. From the point of view of the molecular structure, this is the result of the molecular chain expansion and reorientation due to the action of a high shear field in the flow process.

The commonly used constitutive equations of viscoelastic fluids mainly include linear and nonlinear models, where nonlinear models can be subdivided into retarded-motion expansion models, differential models, and integral models. In the linear constitutive equation, the elastic behavior of the fluid is usually characterized by a linear elastomer (spring), and the viscous behavior of the fluid is described by Newton's law of viscosity or the constitutive equation of a pure viscous fluid (damper). Linear models are different combinations of springs and dampers, including the Maxwell model, the Kelvin–Voigt model, and the Jeffreys model. The retarded-motion expansion constitutive equation is an extension of the Newtonian fluid, in which the deviation from the Newtonian fluid is considered to be caused by the elastic effect, and it is a second-order fluid model. The differential constitutive model has good generality for all viscoelastic fluids and can truly describe the flow characteristics of viscoelastic fluids. Classical differential models include the Oldroyd-B model [16], the Giesekus model [17], the finitely extensible nonlinear elastic in Peterlin approximation (FENE-P) model [18], and the Phan–Thien and Tanner (PTT) model [19]. The integral viscoelastic fluid model also exhibits good generality, and its representative constitutive models include the Lodge model [20] and the Rivlin–Sawyers model [21].

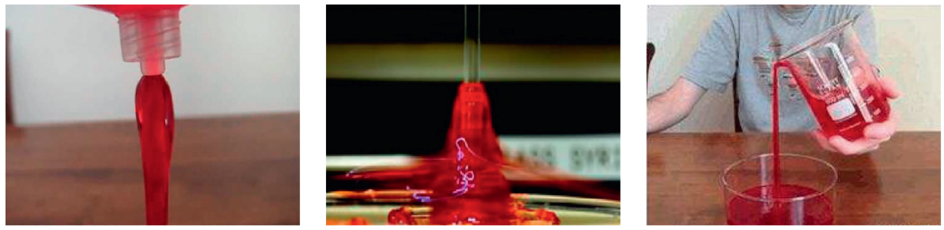
3. Bubble Shapes in Viscoelastic Fluids

The shape of a bubble affects not only the bubble velocity but also the heat and mass transfer rate of the gas-liquid phase. Due to the variability of the interface, its shape often depends on the balance of forces acting on the bubble. Clift et al. [22] divided the bubble shapes in the Newtonian fluids into spherical, ellipsoid, and spherical cap shapes and drew a bubble shape diagram based on the Re number, the Eötvös number (Eo), and the Morton number (Mo), as shown in Figure 2. Compared with the Newtonian fluids, the bubbles in the viscoelastic fluids show more peculiar shapes, such as prolate and teardrop shapes [23]. Figure 3 shows a typical prolate and teardrop shape with a cusp in a viscoelastic fluid, namely, the polyacrylamide (PAAm) solution [24] (the cusp shape was first discovered by Philippoff [25]). As shown in Figure 3 (the critical volume is approximately 20 mm^3), for bubble volumes below the critical volume, the shape of the bubble is spherical and transforms into a prolate shape with increasing volume, which is entirely convex. The bubble volume above the critical volume shows a teardrop shape with a concave cusp at the rear of the bubble. Thus, for bubbles below and above the critical volume, two differences exist: a sharp cusp rear and a concave boundary around the cusp. In the early stage, some researchers made only qualitative observations on the occurrence of bubble cusps in viscoelastic fluids [26, 27], and only a few researchers have conducted detailed experimental observations [28, 29]. The bubble shape is greatly influenced by the rheological properties, and qualitative observations are greatly influenced by subjectivity and image clarity.

Recently, some scientists have observed the peculiar shapes of bubbles more clearly and found several exotic secondary wake structures by using high-speed camera technology. For example, Liu et al. [30] observed two-dimensional (2D) cusps formed by rising bubbles in polyox solutions, as shown in Figure 4, which is not common in most viscoelastic fluids, as polyox solutions are worm-like micellar fluids. Soto et al. [31, 32] found that bubbles in aqueous solutions of associating polymers above a critical volume have long and strange tails, such as a "horseshoe crab tail," "knife-edge," and "fish backbone," as shown in Figure 5. To date, no incisive mechanism explanation has been put forth for the strange tails observed in the associating polymer solutions; the associative polymer solutions have viscoelasticity and intermolecular association [33]. Therefore, this phenomenon could be related to the specific associative use of associative polymers formed by a hydrophilic principal backbone and some pendant hydrophobic groups in a comb-like arrangement. However, these 2D cusps or strange bubble tails occur only in a few special types of viscoelastic fluids, not in the most common viscoelastic fluids.

A few numerical investigations involving the bubble shape in viscoelastic fluids have been performed. Pillapakkam et al. [34] used the level set method, and Wagner et al. [35] and Frank and Li [36] used the 2D lattice Boltzmann method to study the bubbles rising in a Maxwell-type fluid for observing the cusp. Papaioannou et al. [37] studied bubbles in the axisymmetric extensional flow of a viscoelastic liquid filament, which is described by the exponential PTT viscoelastic model, and observed the onset of cusps at the two bubble poles. For the peculiar shapes of the bubble tails, You et al. [38] performed a linear stability analysis to simulate the trailing-edge three-dimensional (3D) cusp formation, which was also experimentally observed by Liu et al. [30] using a camera. You et al. also considered that the development of a 3D cusp is not a Hopf-type bifurcation but an exchange of stabilities. However, the tip-streaming, "blade-edge" tip, "fish-bone" tip, and the phenomenon of the tail breaking into two different threads have not been simulated by researchers thus far, and there are no experimental studies either.

The above studies are direct image observations of the bubble shape. In terms of quantifying the bubble shape, Tadaki and Maeda [39] proposed the parameter of bubble eccentricity ($e = D_X/D_Y$), where D_X is the horizontal diameter and D_Y is the vertical diameter of the bubble. Calderbank [40] compared the bubble shapes in the water and viscoelastic fluids based on their eccentricities (e).



(a) (b) (c)

Figure 1: The extraordinary phenomena of viscoelastic fluids. (a) Die swell effect. (b) Rod-climbing effect. (c) Tubeless siphon effect.

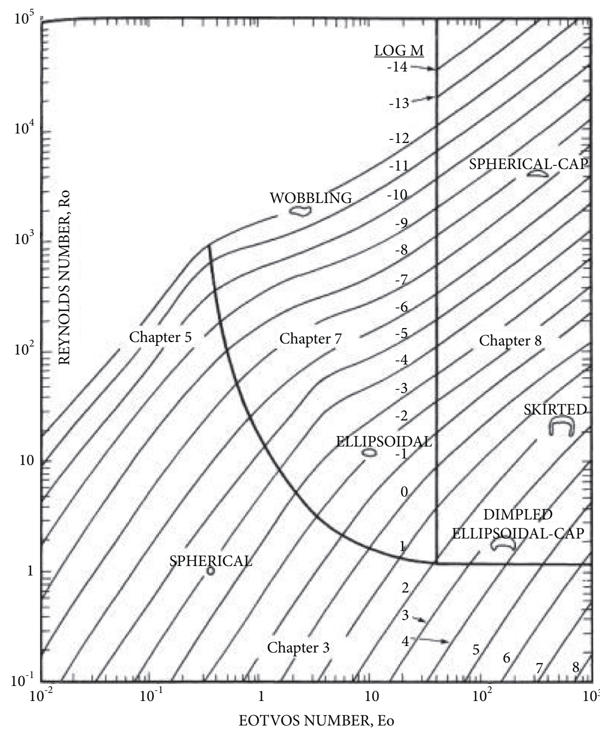
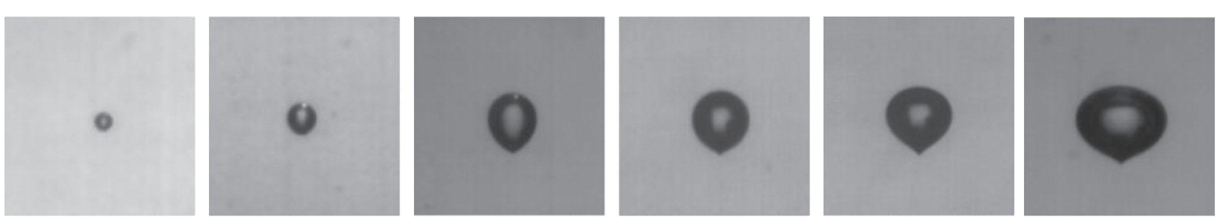


Figure 2: The diagram of the bubble shapes drawn by Clift et al. [22].



$V = 0.25\ \mathrm{mm}^3$ $V = 6.08\ \mathrm{mm}^3$ $V = 19.08\ \mathrm{mm}^3$ $V = 25.50\ \mathrm{mm}^3$ $V = 66.14\ \mathrm{mm}^3$ $V = 138.24\ \mathrm{mm}^3$

Figure 3: The bubble shape in 1.0 wt% PAAm solution (critical volume, $V_c = 22.29\ \mathrm{mm}^3$) [24].

Furthermore, some scholars used the aspect ratio (E), defined as the ratio of the vertical diameter to the horizontal diameter, i.e., $E = D_Y/D_X$, to describe the bubble shapes [41–44]. Unfortunately, most mathematical models of eccentricity (e) and aspect ratio (E) are for Newtonian fluids and viscous non-Newtonian fluids, and limited studies have been conducted on viscoelastic fluids. However, the bubble shape plays a very important role in the rate of heat and mass transfer in the industrial gas-liquid two-phase processes [45–47]. Therefore, it is necessary to establish mathematical models of the aspect ratio and eccentricity of bubbles in viscoelastic fluids. Acharya et al. [48] found that the bubbles become prolate in shape with a pointed tail ($E < 1$) at small Reynolds numbers and gradually become spherical to ellipsoid in shape with increasing Reynolds numbers before finally becoming spherical caps ($E \geq 1$). A correlation formula of the eccentricity (E) for the two deformation cases of the bubble is proposed as follows:

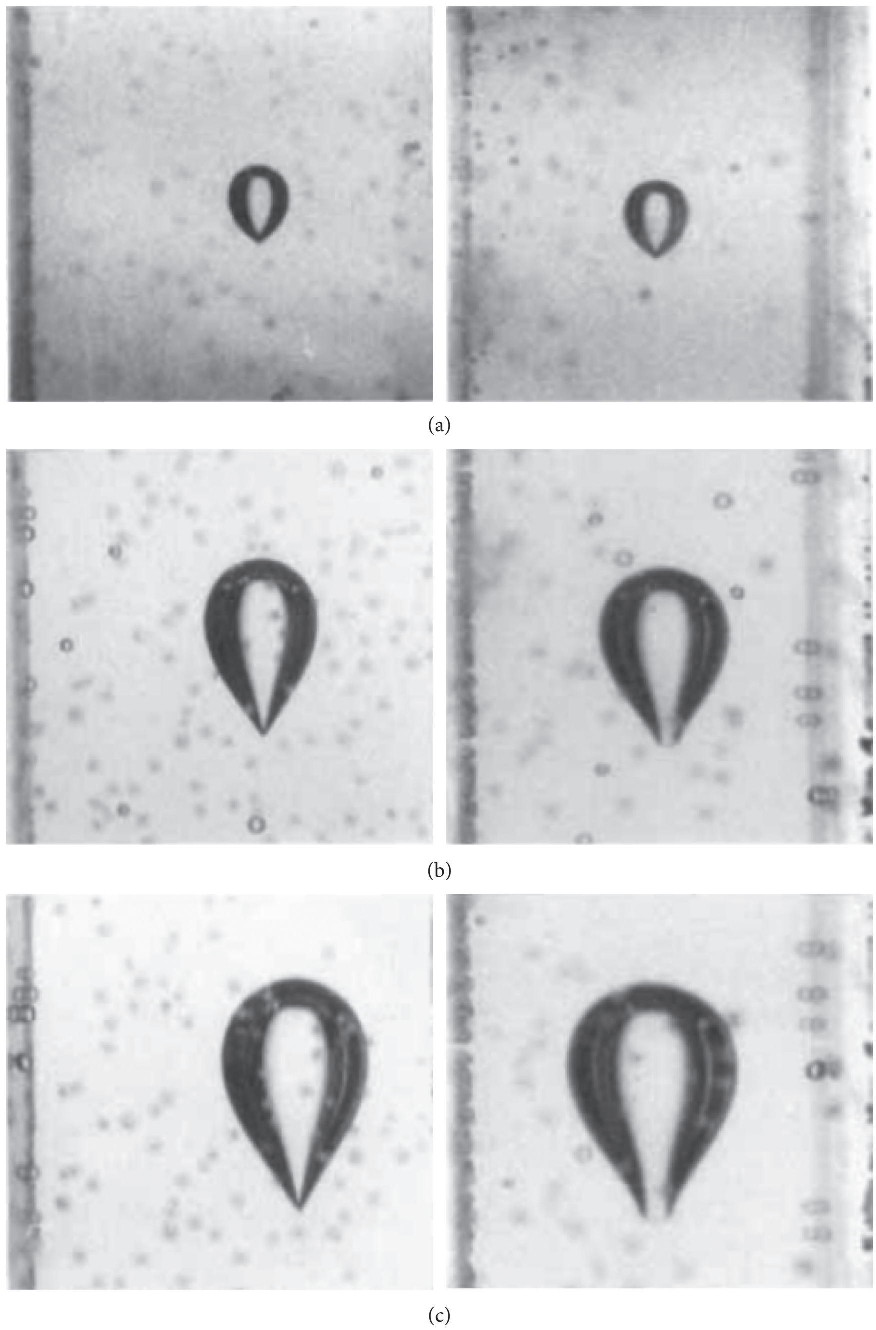


Figure 4: Cusped bubbles rising in a 1.5% polyox solution (a) below the critical volume and (b), (c) above the critical volume [30].

$$E = 0.616\left(\frac{\mathrm{Wi}}{\mathrm{ReWe}}\right)^{-0.17}, \quad 0.68 < E < 1,$$
$$E = 1 + 0.00083\left(\frac{\mathrm{Wi}}{\mathrm{ReWe}}\right)^{-0.87}, \quad 1 \le E < 1.5, \tag{1}$$

where Wi is the Weissenberg number, Re is the Reynolds number, and We is the Weber number. The related dimensionless numbers are defined as follows:

Weissenberg number:

$$\mathrm{Wi} = \frac{\lambda U}{d}. \tag{2}$$

Reynolds number:

$$\mathrm{Re} = \frac{\rho U d}{\mu}. \tag{3}$$

Weber number:

$$\mathrm{We} = \frac{\rho d U^2}{\sigma}, \tag{4}$$

Figure 5: Shapes of bubble tails in a viscoelastic fluid, namely, associative polymer aqueous solution. From the left, the horseshoe crab tail, knife-edge, axe, and fish backbone tail can be seen [32].

where λ is the rheological characteristic time of continuous fluids, σ is the surface tension of the continuous fluids, U is the bubble velocity, and d is the bubble diameter.

The main reason for the scarcity of bubble shape correlation in viscoelastic fluids is their peculiar cusp shape. Liu et al. [30] concluded from experimental results that the 2D cusps of bubbles follow a universal asymptotic relation of $z = a|r|^{2/3}$, as shown in Figure 6, where z and r are the ordinate and abscissa distances from the tip of the cusp, respectively. Lind and Phillips [49] simulated the trademark cusp at the trailing end of the bubbles and found that the curve $z = 1.01|r|^{0.381}$ agrees with the simulated cusp shape for $\mathrm{De} = 0.934$ and $\mathrm{Re} = 1.180$. This suggests that a large number of general cusps can probably be described via analytical expressions of the form $z = a|r|^n$. Nevertheless, nothing is known about the other peculiar shapes of bubble rears, such as the "long-thin tail," "knife-edge," "fish backbone," and "tail split" phenomena.

4. The Negative Wake Phenomenon

A so-called "negative wake" occurs when bubbles flow in a viscoelastic fluid (this strange phenomenon was first observed by Hassager [50]). A negative wake has also been observed for solid spheres settling in viscoelastic fluids [51–53]. A large amount of research has been done on the "negative wake" of solid balls and discs compared with the experimental research done for bubbles, owing to the instability of the bubble surface [51, 54, 55]. These studies show that the so-called "negative wake" could be observed in the wake of solid spheres over a wide range of Deborah numbers. Here, the Deborah numbers are defined as the ratio of the first normal stress difference, N_1, and the tangential stress, shown in equation (9).

Some studies have reported observing a negative wake around bubbles rising in viscoelastic fluids. Funfschilling and Li [56] observed the detailed flow field around bubbles rising in viscoelastic liquids using particle image velocimetry (PIV). Three distinct zones of the flow field were proposed

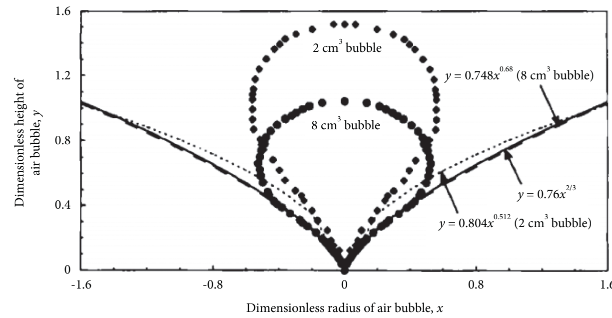


Figure 6: Analytical approximations of the cusp shape for a 2 cm^3 and 8 cm^3 bubble [30].

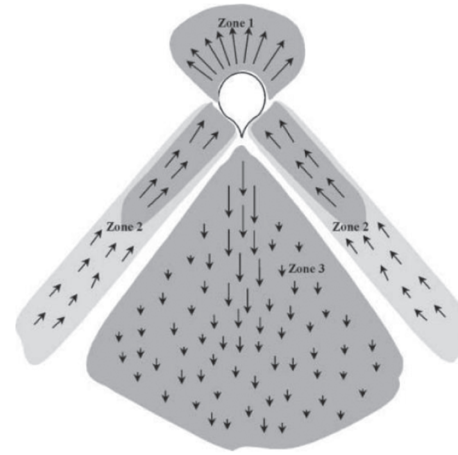


Figure 7: Three distinct flow field zones around a bubble in a viscoelastic fluid [56].

for the first time: the negative wake behind the bubble, a conical upward flow surrounding the negative wake zone, and an upward flow zone in front of the bubble, as shown in Figure 7. Herrera-Velarde et al. [57] and Soto et al. [32] observed the flow field around a bubble rising in a viscoelastic fluid by using the PIV technology and found that as the bubble volume is smaller than the critical volume, the flow field is similar to that in a Newtonian fluid, whereas a negative wake appears only for bubble sizes above the critical volume. The flow fields around bubbles in Newtonian fluids (glycerine solution) and viscoelastic fluids are shown in Figure 8 [56]. Frank and Li [58] concluded that the relaxation time of viscoelastic stresses is faster than that for small bubbles or spheres. However, the competition between these two antagonistic mechanisms favors stress relaxation for large bubbles or spheres as viscoelastic stresses require a very long time to relax. Sousa et al. [59] adopted PIV and shadowgraphy in order to study the negative wake behind the Taylor bubbles observed in carboxymethyl cellulose (CMC) solution and concluded that bubble deformation is responsible for the negative wake. Furthermore, some researchers simulated the peculiar motion behavior using different numerical methods, such as the coupled level set and volume of fluid (CLSVOF) method [60], the boundary element method [49], OpenFOAM software [61], and the lattice Boltzmann (LB) method [62]. For the negative wake generation

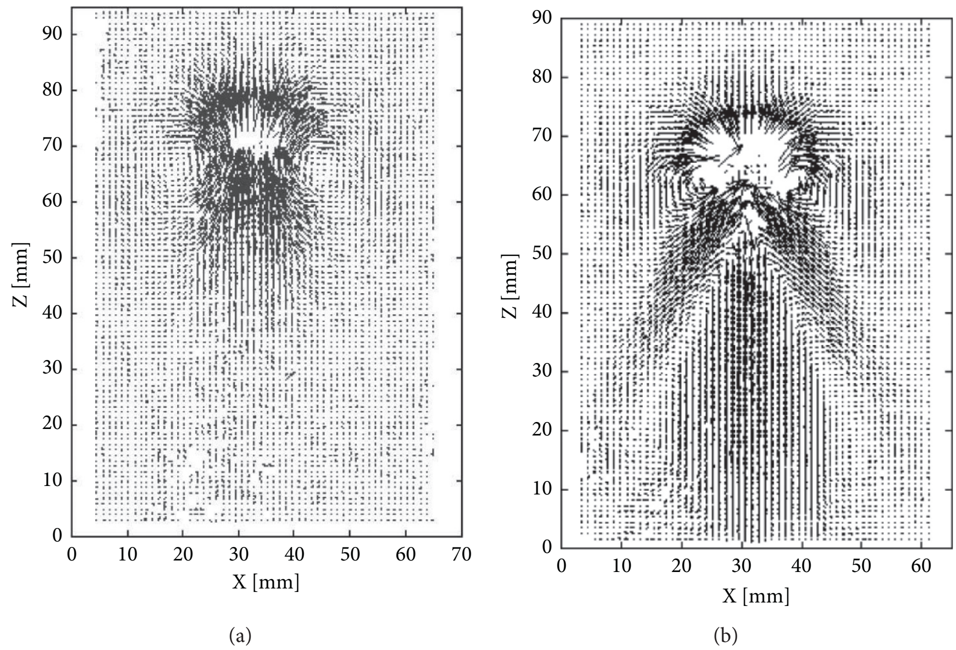


FIGURE 8: Flow field around a bubble in (a) a Newtonian fluid (glycerine solution) and (b) a viscoelastic fluid (0.75% PAAm solution) [56].

mechanism, Kemiha et al. [63] observed the flow field around a bubble and a solid sphere in viscoelastic fluids, as shown in Figure 9. It can be seen from the figure that negative wake cannot arise from a simple deformation of the interface, which should be attributed to the viscoelastic properties of liquids. Furthermore, it has been conjectured that thixotropy could be the cause of the negative wake. However, Sadeghy and Vahabi [64] simulated the effect of thixotropy on a rising bubble and did not find a negative wake behind the bubble. Further, Arigo et al. [52] and Vélez-Cordero et al. [62] did not observe a negative wake experimentally for the Boger-type fluids, which have a constant viscosity. Similarly, the phenomenon of a negative wake was not found in the Boger-type fluids from numerical simulation results either [65]. These studies suggest that only pure elasticity cannot be responsible for the negative wake observed behind a bubble or a solid sphere. The negative wake formation must be due to the coexistence of viscoelasticity and shear thinning properties. Common experimental fluids, such as sodium carboxymethyl cellulose, PAAm, xanthan gum, and guar gum, exhibit viscoelasticity and shear thinning properties.

Therefore, the emergence of a negative wake behind the rising bubbles in a viscoelastic fluid might require the synergistic effect of three mechanisms: (1) the continuous phase has viscoelasticity, (2) the continuous phase has a shear thinning property, and (3) the bubbles are large enough; thus, leading to a very long time required by the viscoelastic stresses to relax. However, there is no literature available on a comparative study that simultaneously considers these three aspects, and there is a lack of mathematical models to describe the phenomenon mathematically. Thus, further experimental and theoretical investigations need to be conducted to reveal the internal mechanism of the negative wake phenomenon.

5. The Phenomenon of Velocity Jump

5.1. The Mechanism of Velocity Jump. It is well-known that in Newtonian fluids or other types of inelastic non-Newtonian fluids, the velocity of bubbles is a continuous function of the bubble volume. However, it has been observed that a velocity discontinuity occurs in viscoelastic fluids as the bubble volume exceeds the critical bubble volume. A typical example of this phenomenon is shown in Figure 10.

The first studies on the velocity jump phenomenon were performed by Astarita and Apuzzo [26]. Henceforth, many researchers have investigated this phenomenon and explored various factors influencing the jump velocity and the critical volume [66]. In most studies, despite the variety of materials used, the critical diameter range of the bubble during the bubble velocity transition is very narrow, approximately 3–6 mm, and the critical volume is approximately 15–65 mm^3. The degree of the bubble velocity jump increases with viscoelasticity, whereas the critical volume decreases with viscoelasticity. In addition, the container wall has an important effect on the degree of bubble velocity jump but little effect on the critical volume of the bubble [58].

Several mechanisms have been proposed for explaining the observation of the bubble velocity jump discontinuity, some of which are given below:

(1) The transition from no-slip to shear-free interface conditions: It is well known that small bubbles are equivalent to a solid rigid sphere, whereas large bubbles have a shear-free mobile interface. Liang et al. [2] attributed the phenomenon of the velocity jump of bubbles to the transition from no-slip to shear-free interface conditions. Leal et al. [67] investigated glass spheres and air bubbles rising in

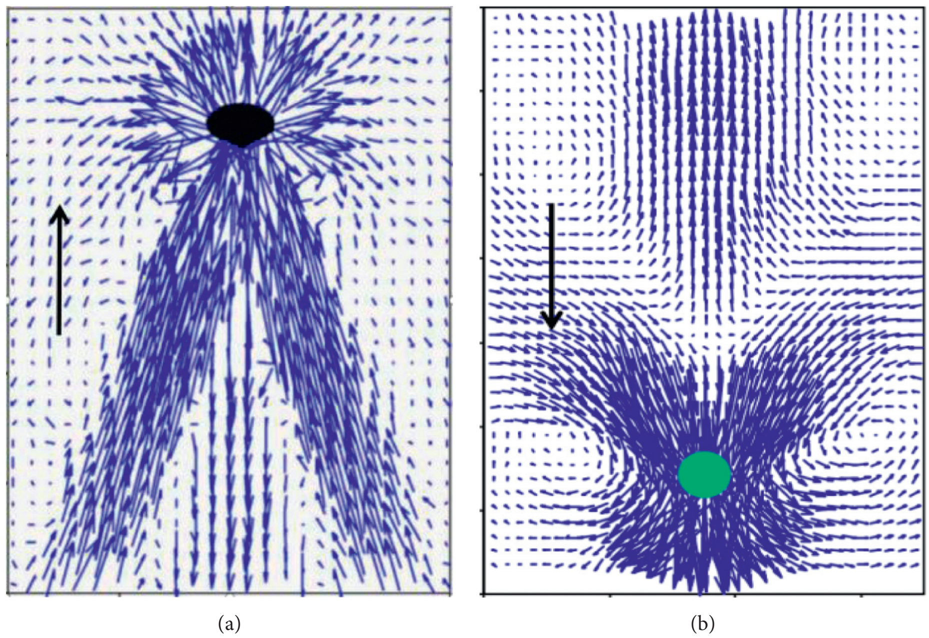


Figure 9: The negative wake around (a) a bubble and (b) a solid sphere in viscoelastic fluids [63].

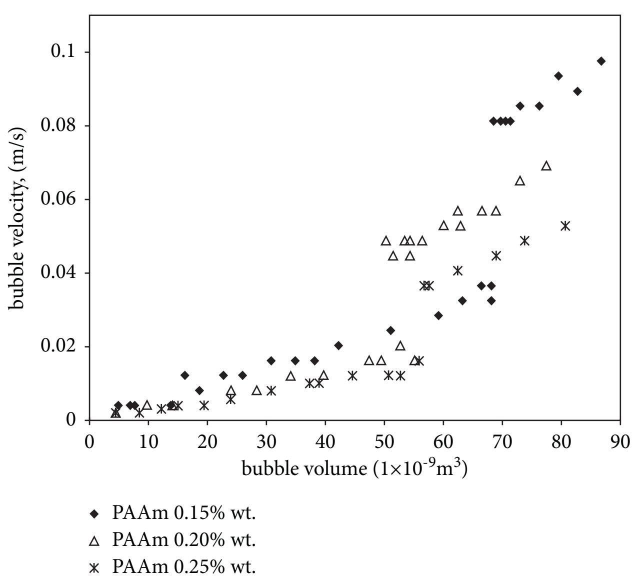


Figure 10: The rising bubble velocity as a function of the bubble volume for three different PAAm concentrations [57].

PAAm solutions and supported the hypothesis of Astarita et al. Nevertheless, this transition from the Stokes to Hadamard regime also occurs in bubbles rising in Newtonian and inelastic non-Newtonian fluids. Thus, the interface transition is not the main reason for the occurrence of a velocity jump.

(2) The reduction of drag: This view was put forward by Liu et al. [30], who argued that the bubble cusp and the associated reduction in drag should be responsible for the occurrence of a velocity jump. Belmonte [68] observed that this phenomenon coincides with the cusp of bubble formation. Recently, Vélez-Cordero et al. [62] also found that the phenomenon of velocity discontinuity was accompanied by the formation of bubble cusps in the Boger-type fluids.

(3) Surface modification due to the balance between elastic and Marangoni instability: Rodrigue et al. [69, 70] concluded that the surfactant and viscoelasticity must work together to produce velocity jump. The existence of a bubble cusp is a necessary but insufficient condition for the occurrence of velocity jump [71]. Soto et al. [31] investigated the motion of bubbles in hydrophobically modified associative polymer solutions and supported the arguments of Rodrigue et al. In their recent study, Enders et al. [61] did not observe the phenomenon of velocity jump in numerical calculations involving a volume of fluid solver in OpenFOAM. This supported the argument that the surfactants and viscoelastic effect must be simultaneously responsible for the occurrence of velocity jump. However, polymer molecules can act as surfactants, even if the surface-active agents are absent. This is probably the reason why some simulations cannot obtain this phenomenon.

(4) The negative wake: Herrera-Velarde et al. [57] concluded that the velocity jump phenomenon is due to the appearance of a negative wake for bubbles larger than the critical volume. They indicated that a negative wake could result in an additional thrust, which is similar to a jet engine. This is supported by the numerical study of Pillapakkam et al. [34], who concluded that there is an additional vortex ring in the surrounding flow, corresponding to the existence of a negative wake, which is the key factor for the occurrence of the velocity jump phenomenon. In

addition, they did not think that this phenomenon should be attributed to the surfactant effect. In the work by Lind and Phillips [49], the absence of the jump discontinuity and the negative wake corroborates the experimental findings of Herrera-Velarde et al. [57], who concluded that a negative wake was the main reason for the jump discontinuity. Furthermore, Yuan et al. [72, 73] verified that a negative wake is mainly responsible for the velocity jump via a direct numerical technique. However, Frank and Li [58] found the bubble velocity jump phenomenon in the absence of a negative wake.

(5) The coupled effects of viscoelastic and shear-thinning properties: Fraggedakis et al. [74] discovered that the bubble velocity jump phenomenon is related to two velocity hysteresis loops. One hysteresis loop corresponds to a smaller velocity jump linked with the bubble shape deformation and a negative wake. The other hysteresis loop is accompanied by a change in the shear rates in front of the bubble, which leads to a larger velocity jump. Their simulated results agree with the experimental results of Herrera-Velarde et al. [57]. The two hysteresis loops merge into one when the shear-thinning property of the fluid is important, resulting in a larger velocity jump. This is the reason for the bubbles in a Boger fluid to have a smaller velocity jump than those in a shear thinning-viscoelastic fluid. Thus, elasticity is the main factor for the occurrence of a velocity jump, and shear thinning plays a supplementary role.

The abovementioned mechanisms seem to explain the phenomenon of bubble velocity jump to a certain extent. However, no mechanism is satisfactory because of the contradictions by other studies. Recently, Zenit and Feng [75] integrated the previous literature and proposed a comprehensive explanation for the phenomena, as shown in Figure 11. An increase in the bubble volume leads to an increase in the buoyancy force, which further leads to an increase in the bubble velocity (step 1 in Figure 11). The increase in the bubble velocity has two directions, namely, an increase in the extension rate and the shear rate (step 2 in Figure 11). An increase in the expansion rate causes an elastic response of the liquid phase and produces normal stress, which brings about three changes compared to the inelastic fluids. First, the bubble is stretched in the direction of the flow and squeezed inward at the equator, and this shape reduces the bubble drag. Second, when the bubble volume crosses a critical value, the formation of a cusp can cause a drag reduction of approximately 50%, according to the calculations of Soto et al. [32]. Finally, the occurrence of a negative wake is also important for bubble velocity jumps. Thus, liquid elasticity reduces the bubble drag and increases the bubble velocity in the abovementioned three ways (step 3a in Figure 11). With regard to the increase in the shear rate, the viscosity of the fluid will decrease due to the shear-thinning property [76], which will reduce the drag and increase the velocity of the bubbles (step 3b in Figure 11), and the surfactant at the bubble interface will be cleaned by shear force, resulting in the transition of the boundary condition from a no-slip condition to a slip-free condition (step 3c in Figure 11).

All the abovementioned mechanisms might contribute to the formation of the bubble jump phenomenon. However, from our point of view, the decisive factor is viscoelasticity. For the viscous effect, the velocity jump also occurs in the Boger-type fluids, which have a constant viscosity [62]. For the surface effect, the bubble surface transition from the no-slip to the slip condition occurs not only in viscoelastic fluids but also in Newtonian fluids. For the cusp effect, Lind and Phillips [49] obtained the formation of the cusp but did not observe any velocity jump. A negative wake should also be a consequence of viscoelasticity, similar to the velocity jump discontinuity. The mechanism for this could be as follows: The polymer molecule chains twist with one another in a stagnant viscoelastic liquid phase, but when the fluid begins to flow, the polymer molecule chains elongate along the direction of flow and induce stress, which tends to relax as the flow stops [77]. When a bubble rises in a stagnant liquid, elastic stress originates at the head of the bubble and tends to relax at the rear of the bubble. This elastic stress is the normal stress that acts on the surface of the bubble. For small bubbles, the polymer chains show small stretched conformations with less normal stress, of which the relaxation time scale is less than that of the liquid flow characteristic time scale caused by bubbles rising, and the normal stress rapidly dissipates. However, the bubble velocity and the degree of stretching of the polymer molecules increase with increasing bubble size. On the one hand, the large degree of stretching of the polymer molecules induces large elastic stress, whereas on the other hand, increasing the bubble velocity causes a decrease in the bubble motion time scale. When the relaxation time scale is larger than the bubble motion time scale, the residual elastic stresses accumulate in the wake of the bubble. The relaxation of the residual stresses, which is closely related to the decrease in drag, results in a bubble velocity discontinuity. In addition, the residual stress compresses the bubble wake downward, resulting in a negative wake. Thus, the cusp of the bubbles is also due to viscoelasticity. As mentioned by Zenit and Feng [75], the cusp is conducive to the boundary layer separation of the bubble wakes, which reduces the bubble resistance and increases the bubble velocity. However, it is uncertain whether this reduction can cause the bubble velocity to jump several times. Therefore, one of the future research directions would be to investigate the relationship between the deformation and the velocity jump discontinuity by observing the rise of the hollow light plastic balls and bubbles in viscoelastic fluids.

For solid balls, no shape deformation and no velocity jump have been observed, whereas a negative wake has been observed. On the other hand, for drops, a negative wake and shape deformation has been observed, but no velocity jump has been observed. This could be due to the high density of the solid balls and drops, where the viscoelastic thrust is negligible compared to gravity, and thus a velocity jump cannot occur.

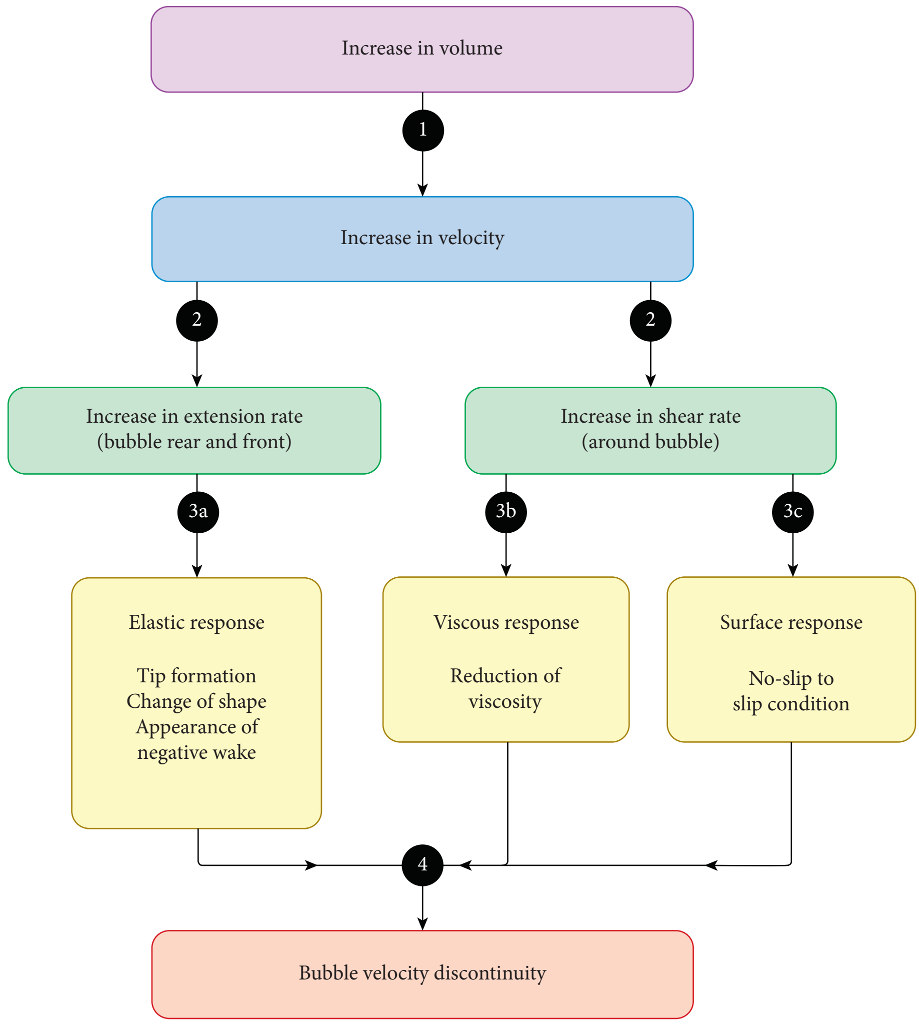


FIGURE 11: Schematic showing the mechanism for the occurrence of the bubble velocity jump [74].

The relationship among these three peculiar phenomena is shown in Figure 12. Here, the solid arrow indicates that if A exists, B must exist, whereas the dashed arrow indicates that if A exists, B might not necessarily exist. Thus, Figure 12 shows that if the velocity jump phenomenon occurs, then a shape cusp and negative wake must be observed. However, the observation of a shape cusp and negative wake cannot guarantee the occurrence of a velocity jump. In addition, deformation and negative wake cannot be mutually assured.

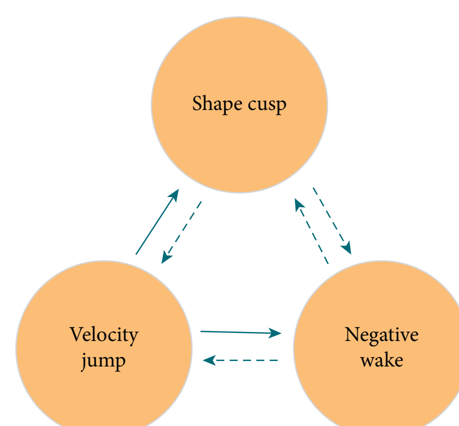


FIGURE 12: The relationship among the three peculiar phenomena of viscoelastic fluids discussed in this review.

5.2. The Mathematical Description of the Velocity Jump. The phenomenon of velocity jump is mainly related to the elastic effect (shape change and negative wake), the surface tension effect (surface clearness), and the viscous effect (drag reduction). Thus, many criteria for the occurrence of the velocity jump have been proposed. The first criterion of the classical bubble size was proposed by Bond [78] for predicting the critical bubble size of the transition from no-slip to shear-free surfaces:

$$R_C = \left(\frac{\sigma}{g\Delta\rho}\right)^{1/2}, \tag{5}$$

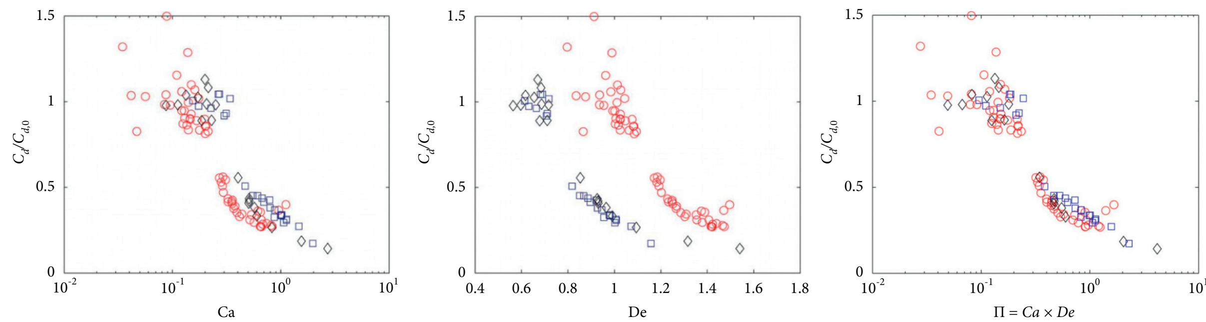


Figure 13: Normalized drag coefficient as a function of Ca, De, and Π [31].

where R_C is the critical radius of the bubble. This criterion is simple and does not include any rheological property. Therefore, Acharya et al. [48] confirmed that this criterion is consistent with a large number of experimental results but cannot account for the velocity jump discontinuity of bubbles observed in viscoelastic fluids. Rodrigue et al. [69] proposed the following criterion for the bubble velocity jump discontinuity:

$$\frac{\rho g R_C}{\sigma} = 7\left(\frac{\Delta\sigma}{\sigma}\right)^{1/2}\left(\frac{N_1 R_C}{\sigma}\right)^{2/5}, \tag{6}$$

where the first normal stress difference (N_1) was calculated using a power-law model.

Liu et al. [30] presented a correlation of the criteria for the velocity jump based on the capillary number as follows:

$$\text{Ca} = \frac{\mu_0 u}{\sigma} \approx 1, \tag{7}$$

where u is the bubble velocity and μ_0 is the zero-shear viscosity of the liquid phase, which can be obtained from the Carreau model without viscoelasticity. Thus, Rodrigue et al. [70] subsequently proposed a more general criterion:

$$\alpha = \frac{\text{Ca De}}{\text{Ma}}, \tag{8}$$

where De is the Deborah number and Ma is the Marangoni number. The related dimensionless numbers are defined as follows:

$$\text{De} = \frac{a}{2m}\left(\frac{u}{R}\right)^{b-n}, \tag{9}$$

where m and n are the parameters of the power-law model of shear stress and a and b are the parameters of the power-law of the first normal stress difference.

$$\text{Ma} = \frac{\Delta\sigma/R}{m(u/R)^n}. \tag{10}$$

It has been observed that the bubble velocity jump occurs when $\alpha > 1$. Here, α represents the balance between the forces acting on the bubble, namely, elastic, viscous, surface tension, and surface tension gradients. Rodrigue and Kee [79] showed that the bubble velocity jump could be eliminated by the surfactant concentrations over the critical association concentration (CAC) and thus proposed another correlation:

$$\frac{\text{De}}{\text{Ma}} = \frac{0.081}{\text{Ca}^{5/3}}. \tag{11}$$

Soto et al. [31] concluded that when the normal stress can be ignored, the bubble is spherical, and the drag coefficient is equal to 16/Re. However, a velocity discontinuity occurs when the normal stress appears in the liquid. They determined that the values of De or Ca, for which the discontinuity occurs, are not unique and depend on the concentration of the liquid phase. Thus, a dimensionless group representing the ratio of the viscous force to the surface tension was proposed as follows:

$$\Pi = \frac{N_1 d}{\sigma} = 4\text{Ca} \times \text{De}. \tag{12}$$

The bubble velocity discontinuity occurs at a critical value of Π ($\Pi \approx 0.25$) for all the liquids tested, as shown in Figure 13.

For the abovementioned correlations, the critical bubble volume can be calculated by the known bubble velocity and vice versa. However, these correlations assume that the relaxation behavior acts on the surface of the bubble through shear flow, and the relaxation behavior against elongational deformation has been ignored. Thus, Pilz and Brenn [80] proposed a correlation for the occurrence of the rising bubble velocity jump discontinuity for a critical bubble volume. By considering the relaxation behavior against elongational deformation, the correlation can be defined as

$$E\ddot{o}_c = 5.3688\frac{(\Pi_1 - 9.8938)^{0.9211}}{\text{Mo}^{0.3282}}, \tag{13}$$

where $E\ddot{o}_c$ is the critical Eötvös number, Mo is the Morton number, and Π_1 is a new dimensional number that introduces the extensional relaxation time. The Morton number and Π_1 numbers are defined as

$$\text{Mo} = \frac{g\mu_0}{\rho\sigma^3},$$
$$\Pi_1 = \frac{g^{1/3}\lambda\rho^{1/4}}{\sigma^{1/4}}. \tag{14}$$

Vélez-Cordero et al. [62] used equation (13) to calculate the critical diameter of bubbles rising in the Boger-type fluids with a deviation of 15%. However, the correlations of dimensionless numbers, such as the Deborah number, contain only the first normal stress, and the viscoelastic fluids exhibit not only elastic stress relaxation but also viscous stress relaxation (memory effect). Therefore, it is necessary to establish a generalized and universal correlation. In addition, most mathematical models for bubbles constitute a correlation of the drag coefficient in the Newtonian fluids and the inelastic non-Newtonian fluids. Nevertheless, the drag coefficient correlations for bubbles in the viscoelastic fluids are scarce due to the occurrence of the velocity jump discontinuity phenomenon.

6. Conclusion and Prospects

In this paper, the motion behavior of a single bubble in a viscoelastic fluid was reviewed. The origin of the mechanisms for three peculiar phenomena, namely, the teardrop shape with a concave cusp, the negative wake, and the velocity jump discontinuity, were discussed based on the experimental and numerical studies available in the literature.

(1) For bubble volumes below the critical volume, the bubble shape has been observed to be spherical for the smallest volume and becomes prolate with increasing bubble volume. For bubble volumes above the critical volume, the bubble shows a teardrop shape with a concave cusp. The change in the bubble shape is mainly related to the viscoelasticity of the fluid. However, the mechanisms for the two-dimensional cusp, tip streaming, "blade-edge" tip, "fish-bone" tip, and tail splitting into threads in certain special viscoelastic fluids are still not clearly understood.

(2) The mathematical correlations of the bubble shape are mainly for Newtonian fluids or inelastic non-Newtonian fluids, and mathematical models for bubble shapes in viscoelastic fluids are scarce.

(3) The origin of a negative wake might require the synergistic effect of three mechanisms, namely, the viscoelasticity and thixotropy of the liquid phase and sufficiently large bubble size. However, no studies have been conducted for studying the three aspects simultaneously, and no mathematical models exist to describe the phenomenon. Thus, further experimental and theoretical investigations need to be conducted to reveal the internal mechanism of the negative wake phenomenon.

(4) Viscoelasticity is the most critical factor for understanding the phenomenon of the bubble velocity jump discontinuity. The cusp of the bubbles and the surface modifications might be playing ancillary roles. A negative wake does not cause the velocity jump discontinuity, but there has been little progress in studying this phenomenon, especially in understanding the mechanism influencing the cusp of bubbles. Therefore, it is necessary to conduct comparative studies on the rise of hollow spheres and bubbles in viscoelastic fluids.

(5) The mathematical models for the phenomenon of velocity jump are based on Eo, Mo, De, Ca, etc., which fall short in terms of the complexity of viscoelastic fluids. Furthermore, correlations of the bubble drag coefficient in viscoelastic fluids are also scarce.

(6) In actual industrial processes, the bubble motion in viscoelastic fluids is generally accompanied by conditions of high pressure and high temperature. However, most of the results reported in the literature are under normal temperature and pressure. Therefore, it is necessary to conduct basic research under actual industrial conditions.

Data Availability

No data were used in the study.

Conflicts of Interest

The author declares that there are no known conflicts of interest or personal relationships that could have appeared to influence the work reported in this paper.

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