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Research Article

Antisynchronization of Nonidentical Fractional-Order Chaotic Systems Using Active Control

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Antisynchronization phenomena are studied in nonidentical fractional-order differential systems. The characteristic feature of antisynchronization is that the sum of relevant state-variables vanishes for sufficiently large value of time variable. Active control method is used first time in the literature to achieve antisynchronization between fractional-order Lorenz and Financial systems, Financial and Chen systems, and Lü and Financial systems. The stability analysis is carried out using classical results. We also provide numerical results to verify the effectiveness of the proposed theory.

1. Introduction

In their pioneering work [1, 2], Pecora and Carroll have shown that chaotic systems can be synchronized by introducing appropriate coupling. The notion of synchronization of chaos has further been explored in secure communications of analog and digital signals [3] and for developing safe and reliable cryptographic systems [4]. For the synchronization of chaotic systems, a variety of approaches have been proposed which include nonlinear feedback [5], adaptive [6, 7], and active controls [8, 9].

Antisynchronization (AS) is a phenomenon in which the state vectors of the synchronized systems have the same amplitude but opposite signs to those of the driving system. Hence the sum of two signals converges to zero when AS appears. Antisynchronization has applications in lasers [10], in periodic oscillators, and in communication systems. Using AS to lasers, one may generate not only drop-outs of the intensity but also short pulses of high intensity, which results in the pulses of special shapes.

Active control method is used to AS for two identical integer order systems by Ho et al. [11] and for nonidentical systems by Li and Zhou [12]. Nonlinear control scheme was used by Li et al. [13] to study AS. Al-Sawalha [14] have reported AS between Chua's system and Nuclear spin generator (NSG) system. Recently AS between Lorenz system, Lü system, and Four-scroll system is investigated by Elabbasy and El-Dessoky [15].

Fractional calculus deals with derivatives and integration of arbitrary order [16–18] and has deep and natural connections with many fields of applied mathematics, engineering, and physics. Fractional calculus has a wide range of applications in control theory [19], viscoelasticity [20], diffusion [21–27], turbulence, electromagnetism, signal processing [28, 29], and bioengineering [30]. Analysis of fractional-order dynamical systems involving Riemann-Liouville as well as Caputo derivatives has been dealt with by present authors [31, 32].

Synchronization of fractional-order chaotic systems was first studied by Deng and Li [33] who carried out synchronization in case of the fractional Lü system. Further they have investigated synchronization of fractional Chen system [34]. Li and Deng have summarized the theory and techniques of synchronization in [35]. The theory for synchronization problems in an ω-symmetrically coupled fractional differential systems have been studied by Zhou and Li [36]. Since then many fractional systems have been investigated by various researchers. A few examples in this regards are Li et al. [37] (Chua system), Wang et al. [38] (Chen system), Wang and Zhang [39] (unified system), Wang and He [40] (unified system), Yu and Li [41] (Rossler hyperchaos system), and Tavazoei and Haeri [42] (Lü system and Chen system). Of late Matouk [43] has synchronized fractional Lü system with fractional Chen system and fractional Chen system with fractional Lorenz system. Hu et al. [44] have synchronized fractional Lorenz and fractional Chen systems. Further Bhalekar and Daftardar-Gejji [45] have investigated the interrelationship between the (fractional) order and synchronization in different chaotic dynamical systems. However, it seems that there are no previous results on AS of two nonidentical fractional-order chaotic systems.

In the present paper, we study the antisynchronization of the following fractional systems using active control method: (i) Lorenz with Financial, (ii) Financial with Chen, and (iii) Lü with Financial.

2. Preliminaries

2.1. Fractional Calculus

Basic definitions and properties of fractional derivative/integrals are given below [16, 17, 46].

Definition 2.1. A real function $f(t),\ t > 0$, is said to be in space $C_\alpha,\ \alpha \in \Re$ if there exists a real number $p\ (> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0, \infty)$.

Definition 2.2. A real function $f(t),\ t > 0$, is said to be in space $C_\alpha^m,\ m \in \mathbb{N} \cup \{0\}$ if $f^{(m)} \in C_\alpha$.

Definition 2.3. Let $f \in C_\alpha$ and $\alpha \geq -1$, then the (left-sided) Riemann-Liouville integral of order $\mu,\ \mu > 0$ is given by

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad t > 0. \tag{2.1}$$

Definition 2.4. The (left-sided) Caputo fractional derivative of f, $f \in C_{-1}^{m}$, $m \in \mathbb{N} \cup \{0\}$, is defined as

$$\begin{aligned} D^{\mu} f(t) &= \frac{d^m}{dt^m} f(t), \quad \mu = m \\ &= I^{m-\mu} \frac{d^m f(t)}{dt^m}, \quad m-1 < \mu < m, \ m \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Note that for $m-1 < \mu \leq m$, $m \in \mathbb{N}$,

$$\begin{aligned} I^{\mu} D^{\mu} f(t) &= f(t) - \sum_{k=0}^{m-1} \frac{d^k f}{dt^k}(0) \frac{t^k}{k!}, \\ I^{\mu} t^{\nu} &= \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}. \end{aligned} \tag{2.3}$$

2.2. Numerical Method for Solving Fractional Differential Equations

Numerical methods used for solving ODEs have to be modified for solving fractional differential equations (FDEs). A modification of Adams-Bashforth-Moulton algorithm is proposed by Diethelm et al. in [47–49] to solve FDEs.

Consider for $\alpha \in (m-1, m]$ the initial value problem (IVP)

$$\begin{aligned} D^{\alpha} y(t) &= f(t, y(t)), \quad 0 \leq t \leq T, \\ y^{(k)}(0) &= y_0^{(k)}, \quad k = 0, 1, \ldots, m-1. \end{aligned} \tag{2.4}$$

The IVP (2.4) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{2.5}$$

Consider the uniform grid $\{t_n = nh/n = 0, 1, \ldots, N\}$ for some integer N and $h := T/N$. Let $y_h(t_n)$ be approximation to $y(t_n)$. Assume that we have already calculated approximations $y_h(t_j)$, $j = 1, 2, \ldots, n$, and we want to obtain $y_h(t_{n+1})$ by means of the equation

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f\left(t_{n+1}, y_h^P(t_{n+1})\right) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_n(t_j)), \tag{2.6}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha} & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & \text{if } 1 \leq j \leq n, \\ 1 & \text{if } j = n+1. \end{cases} \tag{2.7}$$

The preliminary approximation $y_h^P(t_{n+1})$ is called predictor and is given by

$$y_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_n(t_j)), \tag{2.8}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha - (n-j)^\alpha\right). \tag{2.9}$$

Error in this method is

$$\max_{j=0,1,\ldots,N} \left| y(t_j) - y_h(t_j) \right| = O(h^p), \tag{2.10}$$

where $p = \min(2, 1+\alpha)$.

3. System Description

The fractional-order Lorenz system [50, 51] is described by

$$\begin{aligned} D^\alpha x &= \sigma(y - x), \\ D^\alpha y &= rx - y - xz, \\ D^\alpha z &= xy - \mu z, \end{aligned} \tag{3.1}$$

where $\sigma = 10$ is the Prandtl number, $r = 28$ is the Rayleigh number over the critical Rayleigh number, and $\mu = 8/3$ gives the size of the region approximated by the system. The minimum effective dimension for this system is 2.97 [51].

In [52] Chen proposed the financial system to fractional-order

$$\begin{aligned} D^\alpha x &= z + (y - a)x, \\ D^\alpha y &= 1 - by - x^2, \\ D^\alpha z &= -x - cz, \end{aligned} \tag{3.2}$$

where $a = 3$, $b = 0.1$, and $c = 1$. The minimum effective dimension for which the system exhibits chaos is given by 2.32 [52].

Li and Peng [53] studied chaos in fractional-order Chen system

$$\begin{aligned} D^\alpha x &= a_1(y - x), \\ D^\alpha y &= (c_1 - a_1)x - xz + c_1 y, \\ D^\alpha z &= xy - b_1 z, \end{aligned} \tag{3.3}$$

where $a_1 = 35$, $b_1 = 3$, and $c_1 = 27$. The minimum effective dimension reported is 2.92 [53].

Fractional-order Lü system is the lowest-order chaotic system among all the chaotic systems reported in the literature [54]. The minimum effective dimension reported is 0.30. The system is given by

$$\begin{aligned} D^{\alpha}x &= a_2(y-x), \\ D^{\alpha}y &= c_2 y - xz, \\ D^{\alpha}z &= xy - b_2 z, \end{aligned} \tag{3.4}$$

where $a_2 = 35$, $b_2 = 3$, and $c_2 = 28$.

4. Antisynchronization between Fractional-Order Lorenz and Financial System

In this section, we study the antisynchronization between Lorenz and Financial systems. Assuming that the Lorenz system drives the Financial system, we define the drive (master) and response (slave) systems as follows:

$$\begin{aligned} D^{\alpha}x_1 &= \sigma(y_1 - x_1), \\ D^{\alpha}y_1 &= rx_1 - y_1 - x_1 z_1, \\ D^{\alpha}z_1 &= x_1 y_1 - \mu z_1, \end{aligned} \tag{4.1}$$

$$\begin{aligned} D^{\alpha}x_2 &= z_2 + (y_2 - a)x_2 + u_1(t), \\ D^{\alpha}y_2 &= 1 - by_2 - x_2^2 + u_2(t), \\ D^{\alpha}z_2 &= -x_2 - cz_2 + u_3(t). \end{aligned} \tag{4.2}$$

The unknown terms u_1, u_2, u_3 in (4.2) are active control functions to be determined. Define the error functions as

$$e_1 = x_1 + x_2, \qquad e_2 = y_1 + y_2, \qquad e_3 = z_1 + z_2. \tag{4.3}$$

Equation (4.3) together with (4.1) and (4.2) yields the error system

$$\begin{aligned} D^{\alpha}e_1 &= (a-\sigma)x_1 + \sigma y_1 + x_1 y_1 - z_1 - ae_1 - y_1 e_1 - x_1 e_2 + e_1 e_2 + e_3 + u_1(t), \\ D^{\alpha}e_2 &= 1 + rx_1 - x_1^2 + (b-1)y_1 - x_1 z_1 + 2x_1 e_1 - e_1^2 - be_2 + u_2(t), \\ D^{\alpha}e_3 &= x_1 + (c-\mu)z_1 + x_1 y_1 - e_1 - ce_3 + u_3(t). \end{aligned} \tag{4.4}$$

We define active control functions $u_i(t)$ as

$$\begin{aligned} u_1(t) &= V_1(t) - (a-\sigma)x_1 - \sigma y_1 - x_1 y_1 + z_1 + y_1 e_1 + x_1 e_2 - e_1 e_2, \\ u_2(t) &= V_2(t) - 1 - r x_1 + x_1^2 - (b-1) y_1 + x_1 z_1 - 2 x_1 e_1 + e_1^2, \\ u_3(t) &= V_3(t) - x_1 - (c-\mu) z_1 - x_1 y_1. \end{aligned} \tag{4.5}$$

The terms $V_i(t)$ are linear functions of the error terms $e_i(t)$. With the choice of $u_i(t)$ given by (4.5), the error system (4.5) becomes

$$\begin{aligned} D^\alpha e_1 &= -a e_1 - e_3 + V_1(t), \\ D^\alpha e_2 &= -b e_2 + V_2(t), \\ D^\alpha e_3 &= -e_1 - c e_3 + V_3(t). \end{aligned} \tag{4.6}$$

The control terms $V_i(t)$ are chosen so that the system (4.6) becomes stable. There is not a unique choice for such functions. We choose

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = A \begin{pmatrix} e_1 \\ \mathrm{e}_2 \\ e_3 \end{pmatrix}, \tag{4.7}$$

where A is a 3×3 real matrix, chosen so that for all eigenvalues λ_i of the system (4.6) the condition

$$\left|\arg(\lambda_i)\right| > \frac{\alpha\pi}{2} \tag{4.8}$$

is satisfied. (The stability condition (4.8) is discussed in the literature [55–57]). If we choose

$$A = \begin{pmatrix} a-1 & 0 & -1 \\ 0 & -1+b & 0 \\ 1 & 0 & c-1 \end{pmatrix}, \tag{4.9}$$

then the eigenvalues of the linear system (4.6) are -1, -1, and -1. Hence the condition (4.8) is satisfied for $\alpha < 2$. Since we consider only the values $\alpha \leq 1$, we get the required antisynchronization.

4.1. Simulation and Results

Parameters of the Lorenz system are taken as $\sigma = 10,\ r = 28,\ \mu = 8/3$ and Financial system as $a = 3,\ b = 0.1,\ c = 1$. The fractional-order α is taken to be 0.99 for which both the systems are chaotic. The initial conditions for drive and response system are $x_1(0) = 10,\ y_1(0) = 5,$

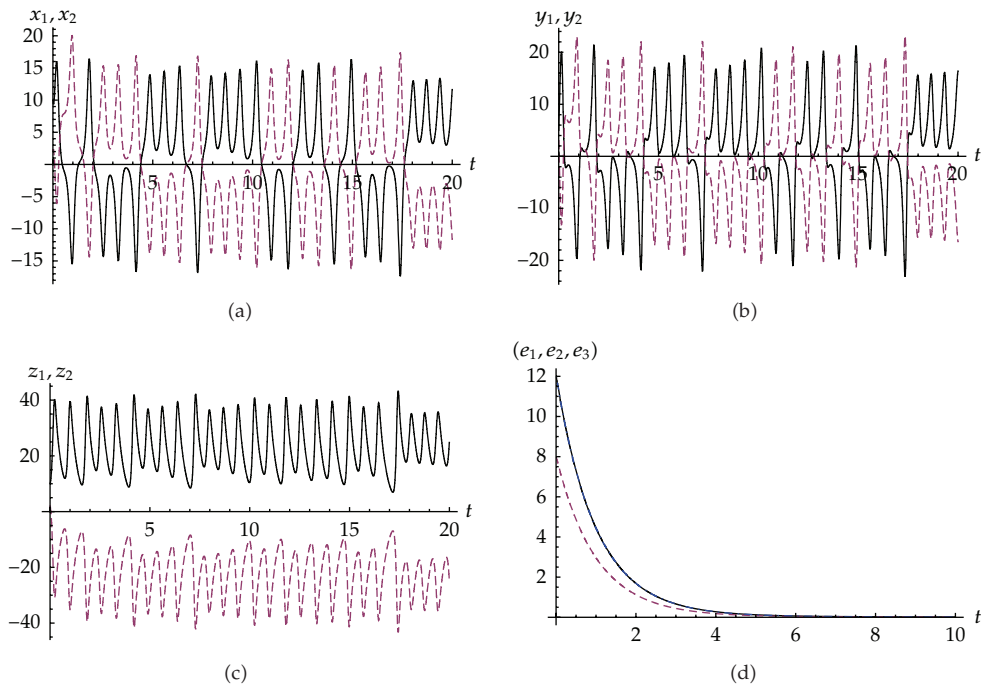

Figure 1: (a) Signals x_1, x_2, (b) Signals y_1, y_2, (c) Signals z_1, z_2, and (d) Error system.

$z_1(0) = 10$ and $x_2(0) = 2$, $y_2(0) = 3$, $z_2(0) = 2$, respectively. Initial conditions for the error system are thus $e_1(0) = 12$, $e_2(0) = 8$, and $e_3(0) = 12$. Figures 1(a)–1(c) show the antisynchronization between Lorenz and Financial system; the response system is given by dashed line. The errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line) and $e_3(t)$ (dot-dashed line) in the antisynchronization are shown in Figure 1(d).

5. Antisynchronization between Financial and Chen Systems of Fractional Order

Assuming that Chen system is antisynchronized with Financial system; define the drive system as

$$\begin{aligned} D^{\alpha}x_1 &= z_1 + (y_1 - a)x_1, \\ D^{\alpha}y_1 &= 1 - by_1 - x_1^2, \\ D^{\alpha}z_1 &= -x_1 - cz_1 \end{aligned} \tag{5.1}$$

and the response system as

$$\begin{aligned} D^{\alpha}x_2 &= a_1(y_2 - x_2) + u_4, \\ D^{\alpha}y_2 &= (c_1 - a_1)x_2 - x_2z_2 + c_1y_2 + u_5, \\ D^{\alpha}z_2 &= x_2y_2 - b_1z_2 + u_6. \end{aligned} \tag{5.2}$$

Let $e_1 = x_1 + x_2,\ e_2 = y_1 + y_2$, and $e_3 = z_1 + z_2$ be error functions. For antisynchronization, it is essential that the errors $e_i \to 0$ as $t \to \infty$. Note that

$$\begin{aligned} D^\alpha e_1 &= (a_1 - a)x_1 - a_1 y_1 + x_1 y_1 + z_1 - a_1 e_1 + a_1 e_2 + u_4(t), \\ D^\alpha e_2 &= 1 + (a_1 - c_1)x_1 - x_1^2 - (b + c_1)y_1 - x_1 z_1 \\ &\quad + (c_1 - a_1)e_1 + z_1 e_1 + c_1 e_2 + x_1 e_3 - e_1 e_3 + u_5(t), \\ D^\alpha e_3 &= -x_1 + x_1 y_1 + (b_1 - c)z_1 - y_1 e_1 - x_1 e_2 + e_1 e_2 - b_1 e_3 + u_6(t). \end{aligned} \tag{5.3}$$

The control functions are chosen as

$$\begin{aligned} u_4 &= V_4 - (a_1 - a)x_1 + a_1 y_1 - x_1 y_1 - z_1, \\ u_5 &= V_5 - 1 - (a_1 - c_1)x_1 + x_1^2 + (b + c_1)y_1 + x_1 z_1 - z_1 e_1 - x_1 e_3 + e_1 e_3, \\ u_6 &= V_6 + x_1 - x_1 y_1 - (b_1 - \mathrm{c})z_1 + y_1 e_1 + x_1 e_2 - e_1 e_2. \end{aligned} \tag{5.4}$$

The linear functions $V_4,\ V_5,\ V_6$ are given by

$$\begin{aligned} V_4 &= (a_1 - 1)e_1 - a_1 e_2, \\ V_5 &= -(a_1 - c_1)e_1 - (c_1 + 1)e_2, \\ V_6 &= (b_1 - 1)e_3. \end{aligned} \tag{5.5}$$

With the values given in (5.4) and (5.5), the error system (5.3) becomes

$$\begin{pmatrix} D^\alpha e_1 \\ D^\alpha e_2 \\ D^\alpha e_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{5.6}$$

It can be observed that the coefficient matrix of the error system (5.6) has eigenvalues -1, $-1,\ -1$. So the system is stable and antisynchronization is achieved.

5.1. Simulations and Results

We take parameters for fractional-order Chen system as $a_1 = 35,\ b_1 = 3,\ c_1 = 27$. Parameters for the Financial system are same as given in Section 4.1. Experiments are done for fixed value of fractional-order $\alpha = 0.95$, which is same for drive and response system (5.1) and (5.2). The initial conditions for the systems (5.1) and (5.2) are $x_1(0) = 2,\ y_1(0) = 3,\ z_1(0) = 2$ and $x_2(0) = 10,\ y_2(0) = 25,\ z_2(0) = 36$, respectively. For the error system (5.6), the initial conditions turns out to be $e_1(0) = 12,\ e_2(0) = 28,\ e_3(0) = 38$. The simulation results are summarized in Figure 2. Antisynchronization between fractional Financial and Chen system is shown in Figure 2(a) (signals $x_1,\ x_2$), Figure 2(b) (signals $y_1,\ y_2$), and Figure 2(c) (signals $z_1,\ z_2$). Note that the drive systems are shown by solid lines, whereas response systems are

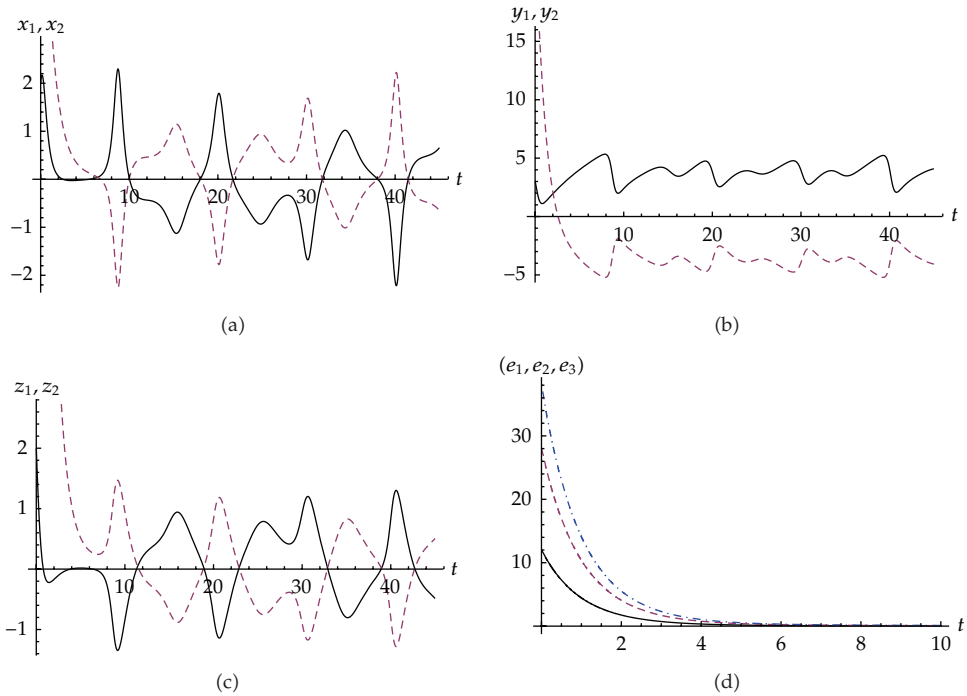

Figure 2: (a) Signals x_1, x_2, (b) Signals y_1, y_2, (c) Signals z_1, z_2, and (d) Error system.

shown by dashed lines. The errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line), and $e_3(t)$ (dot-dashed line) in the antisynchronization are shown in Figure 2(d).

6. Antisynchronization between Fractional Lü and Financial System

In this case, consider Lü system as the drive system

$$\begin{aligned} D^{\alpha} x_1 &= a_2(y_1 - x_1), \\ D^{\alpha} y_1 &= c_2 y_1 - x_1 z_1, \\ D^{\alpha} z_1 &= x_1 y_1 - b_2 z_1, \end{aligned} \tag{6.1}$$

and the response system as the Financial system

$$\begin{aligned} D^{\alpha} x_2 &= z_2 + (y_2 - a) x_2 + u_7, \\ D^{\alpha} y_2 &= 1 - b y_2 - x_2^2 + u_8, \\ D^{\alpha} z_2 &= -x_2 - c z_2 + u_9. \end{aligned} \tag{6.2}$$

Let $e_1 = x_1+x_2,\ e_2 = y_1+y_2$, and $e_3 = z_1+z_2$ be error functions. For antisynchronization, it is essential that the errors $e_i \to 0$ as $t \to \infty$. To achieve this one should choose the control terms $u_7,\ u_8,\ u_9$ properly. The error system thus becomes

$$\begin{aligned} D^\alpha e_1 &= (a-a_2)x_1 + a_2y_1 + x_1y_1 - z_1 - ae_1 - y_1e_1 - x_1e_2 + e_1e_2 + e_3 + u_7, \\ D^\alpha e_2 &= 1 - x_1^2 + (b+c_2)y_1 - x_1z_1 + 2x_1e_1 - e_1^2 - be_2 + u_8, \\ D^\alpha e_3 &= x_1 + x_1y_1 + (c-b_2)z_1 - e_1 - ce_3 + u_9. \end{aligned} \tag{6.3}$$

The control functions are chosen as

$$\begin{aligned} u_7 &= V_7 - (a-a_2)x_1 - a_2y_1 - x_1y_1 + z_1 + y_1e_1 + x_1e_2 - e_1e_2, \\ u_8 &= V_8 - 1 + x_1^2 - (b+c_2)y_1 + x_1z_1 - 2x_1e_1 + e_1^2, \\ u_9 &= V_9 - x_1 - x_1y_1 - (c-b_2)z_1. \end{aligned} \tag{6.4}$$

The linear functions $V_7,\ V_8,\ V_9$ are given by

$$\begin{aligned} V_7 &= (a-1)e_1 - e_3, \\ V_8 &= (-1+b)e_2, \\ V_9 &= e_1 + (c-1)e_3. \end{aligned} \tag{6.5}$$

With the values given in (6.4) and (6.5), the error system (6.3) becomes

$$\begin{pmatrix} D^\alpha e_1 \\ D^\alpha e_2 \\ D^\alpha e_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{6.6}$$

It can be observed that the coefficient matrix of the error system (6.6) has eigenvalues -1, $-1,\ -1$. So the system is stable and antisynchronization is achieved.

6.1. Simulations and Results

Parameters for the Lü system are $a_2 = 35,\ b_2 = 3,\ c_2 = 28$, whereas parameters for Financial system are unaltered. The initial conditions for drive system are $x_1(0) = 0.2,\ y_1(0) = 0,\ z_1(0) = 0.5$, whereas the initial conditions for response system are $x_2(0) = 2,\ y_2(0) = 3,\ z_2(0) = 2$. Hence the initial conditions for the error system (6.6) are $e_1(0) = 2.2,\ e_2(0) = 3,\ e_3(0) = 2.5$. We perform the numerical simulations for fractional order α, namely, 0.91 of the drive system (6.1) and response system (6.2). Figures 3(a), 3(b), and 3(c) show antisynchronization between fractional Lü and Financial system for $\alpha = 0.91$. Figure 3(d) shows the errors $e_1(t)$ (solid line), $e_2(t)$ (dashed line), and $e_3(t)$ (dot-dashed line) in the antisynchronization for $\alpha = 0.91$.

Mathematica 7 has been used for computations in the present paper.

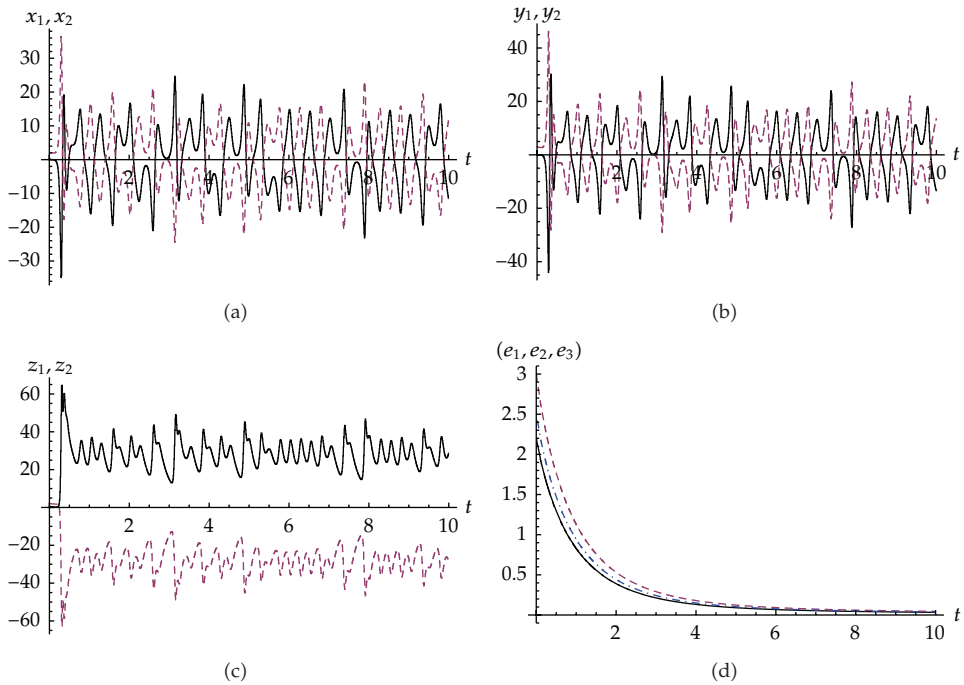

Figure 3: (a) $\alpha = 0.91$, Signals x_1, x_2, (b) $\alpha = 0.91$, Signals y_1, y_2, (c) $\alpha = 0.91$, Signals z_1, z_2, and (d) $\alpha = 0.91$, Error system.

7. Conclusions

Antisynchronization of nonidentical fractional-order chaotic systems has been done first time in the literature using active control. The fractional Financial system is controlled by fractional Lorenz system, the fractional Chen system is controlled by fractional Financial system, and the fractional Financial system is controlled by fractional Lü system.

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