Research Article

Nonlinear Vibrations of Multiwalled Carbon Nanotubes under Various Boundary Conditions

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The present work deals with applying the homotopy perturbation method to the problem of the nonlinear oscillations of multiwalled carbon nanotubes embedded in an elastic medium under various boundary conditions. A multiple-beam model is utilized in which the governing equations of each layer are coupled with those of its adjacent ones via the van der Waals interlayer forces. The amplitude-frequency curves for large-amplitude vibrations of single-walled, double-walled, and triple-walled carbon nanotubes are obtained. The influences of some commonly used boundary conditions, changes in material constant of the surrounding elastic medium, and variations of the nanotubes geometrical parameters on the vibration characteristics of multiwalled carbon nanotubes are discussed. The comparison of the generated results with those from the open literature illustrates that the solutions obtained are of very high accuracy and clarifies the capability and the simplicity of the present method. It is worthwhile to say that the results generated are new and can be served as a benchmark for future works.

1. Introduction

In recent years, considerable effort has been devoted to the subject area of nanotechnology. Since nanomaterials possess unique mechanical, physical, and chemical properties, they occupy the chief topic of research in many scientific fields. Nowadays, they are being used as the substantial parts of nanoelectronics, nanodevices, and nanocomposites. Amongst the materials at the scale of nanometers, carbon nanotubes (CNTs) discovered by Iijima [1] in 1991 have attracted intense interest to many research workers. In spite of being too small and having light weight, CNTs have a very large Young's modulus up to 1 TPa in the axial direction. A good review on their properties and industrial applications can be found detailed in [2]. The application of analytical and numerical approaches to the

study of CNTs is suggested recently [3–5]. Since the vibrations of CNTs are of considerable importance in a number of nanomechanical devices such as oscillators, charge detectors, field emission devices, and sensors, many researches have been so far devoted to the problem of the vibration of these nanomaterials [6–11]. However, most of the investigations conducted on the vibration of CNTs have been restricted to the linear regime. Fu et al. [12] studied the nonlinear vibrations of embedded nanotubes using the incremental harmonic balanced method (IHBM). In that work, single-walled nanotubes (SWNTs) and double-walled nanotubes (DWNTs) with simply supported end conditions were considered for the study. Recently, He [5] applied the homotopy perturbation method (HPM) to investigate the nonlinear vibrations of MWNTs using the same beam model as the one used in [12]. In that paper, they also extended Fu's work to the problem of triple-walled nanotubes (TWNTs) and gave the nonlinear amplitude-frequency curves. The present work is an extension of the authors' previous work for free nonlinear oscillations of MWNTs with arbitrary boundary conditions. The HPM as a powerful analytical approach was first introduced by He [13–20] and has been used by many mathematicians and engineers to solve various functional equations. In this method, the solution is considered as the summation of an infinite series which converges rapidly to the exact solution. Some applications of this method are summarized in a review article by He [21]. This simple method has been applied to solve linear and nonlinear equations of heat transfer [22–24], fluid mechanics [25], nonlinear Schrödinger equations [26], nonlinear oscillations [27, 28], some boundary value problems, and other subjects in different disciplines [29]. The recent developments of this perturbation-based method can be found in [30]. The convergence study of HPM for partial differential equations has been recently reported in [31]. The aim of this investigation is to show the effectiveness of HPM and the capability of this simple method for handling the nonlinear oscillations of many vibrating systems. It is shown that the first approximate solution of New HPM admits a remarkable accuracy in comparison with the results obtained from the incremental harmonic balanced method for the amplitude-frequency curves.

2. Basic Equations

Consider a CNT of length L, Young's modulus E, density ρ, cross-sectional area A, and cross-sectional inertia moment I, embedded in an elastic medium with constant k determined by the material constants of the surrounding medium. This model is shown in Figure 1. Assume that u and w are the displacements of the nanotube along x and z directions respectively in terms of the spatial coordinate x and the time variable t.

The free vibration governing equation of this embedded beam-modeled CNT considering the geometric nonlinearity is [12]

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = \left[\frac{EA}{2L} \int_0^L \left(\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right) dx \right] \frac{\partial^2 w}{\partial x^2} + P, \tag{2.1}$$

where P is the transverse load considered as the interaction pressure per unit axial length between the outermost tube and the surrounding medium, which can be described by the Winkler model [32, 33] as
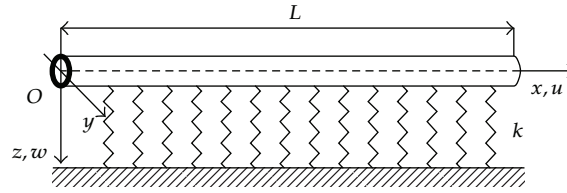
$$P(x,t) = -kw. \tag{2.2}$$

Figure 1: Model of an embedded carbon nanotube.

Herein, the negative sign indicates that the pressure P is opposite to the deflection of the outermost tube. Substituting (2.2) into (2.1) we get

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial t^4} + kw = \left[\frac{EA}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx \right] \frac{\partial^2 w}{\partial x^2}. \tag{2.3}$$

For a MWNT with N layers, the pressure at any point between any two adjacent tubes, denoted as the van der Waals forces, can be expressed as

$$F_i = c_i(w_i - w_{i-1}), \tag{2.4}$$

where F_i is the van der Waals force between the ith tube and the $i-1$th tube. c_i is the coefficient of the van der Waals force between the ith tube and the $i-1$th tube defined according to [34]. Therefore, imposing the effects of the van der Waals forces to (2.3) it results in the n coupled nonlinear differential equations corresponding to the transverse nonlinear vibrations of an embedded MWNT with n layers

$$EI_1 \frac{\partial^4 w_1}{\partial x^4} + \rho A_1 \frac{\partial^2 w_1}{\partial t^2} = \left[\frac{EA_1}{2l} \int_0^L \left(\frac{\partial w_1}{\partial x} \right)^2 dx \right] \frac{\partial^2 w_1}{\partial x^2} + c_1(w_2 - w_1),$$

$$EI_2 \frac{\partial^4 w_2}{\partial x^4} + \rho A_2 \frac{\partial^2 w_2}{\partial t^2} = \left[\frac{EA_2}{2l} \int_0^L \left(\frac{\partial w_2}{\partial x} \right)^2 dx \right] \frac{\partial^2 w_2}{\partial x^2} + c_2(w_3 - w_2) - c_1(w_2 - w_1),$$

$$\vdots$$

$$EI_n \frac{\partial^4 w_n}{\partial x^4} + \rho A_n \frac{\partial^2 w_n}{\partial t^2} = \left[\frac{EA_n}{2l} \int_0^L \left(\frac{\partial w_n}{\partial x} \right)^2 dx \right] \frac{\partial^2 w_n}{\partial x^2} - kw_n - c_{n-1}(w_n - w_{n-1}).$$

$$\tag{2.5}$$

Assuming $w_i(x,t) = \varphi(x) W_i(t)$, $i = 1, 2, \ldots, n$, where $\varphi(x)$ is the first eigenmode of the beam satisfying the kinematic boundary conditions [34] and $W_i(t)$ is the time-dependent

deflection parameter of the ith layer of the nanotube and applying the Galerkin method, the governing equations of motion are obtained as follows:

$$EI_1W_1\alpha_1 + \left(\rho A_1\frac{d^2W_1}{dt^2} + c_1W_1 - c_1W_2\right)\alpha_2 - \frac{EA_1}{2l}W_1^3\alpha_3 = 0,$$

$$EI_2W_2\alpha_1 + \left(\rho A_2\frac{d^2W_2}{dt^2} + c_1W_2 - c_1W_1 + c_2W_2 - c_2W_3\right)\alpha_2 - \frac{EA_2}{2l}W_2^3\alpha_3 = 0,$$

$$\vdots$$

$$EI_nW_n\alpha_1 + \left(\rho A_n\frac{\partial^2W_n}{\partial t^2} + c_{n-1}(W_n - W_{n-1}) + kW_n\right) - \frac{EA_n}{2l}W_n^3\alpha_3 = 0.$$

(2.6)

The above equations are the differential equations of motion governing the nonlinear vibrations of CNTs subjected to the following initial conditions:

$$W_i(0) = W_{\max}, \quad \frac{dW_i(0)}{dt} = 0, \quad (i = 1, 2, \ldots, n),$$

(2.7)

where $W_{\max}$ denotes the maximum amplitude of oscillation. In (2.6), α_1, α_2, and α_3 are as follows:

$$\alpha_1 = \int_0^L \left(\frac{d^4\varphi(x)}{dx^4}\right)\varphi(x)dx,$$

$$\alpha_2 = \int_0^L \varphi^2(x)dx,$$

(2.8)

$$\alpha_3 = \int_0^L \left[\frac{d^2\varphi(x)}{dx^2}\int_0^L \left(\frac{d\varphi(x)}{dx}\right)^2 dx\right]\varphi(x)dx.$$

The deflection of the nanotube is subjected to the following boundary conditions.
For a simply supported (S-S) nanotube,

$$w(0,t) = 0, \quad \frac{\partial^2w(0,t)}{\partial x^2} = 0, \quad w(L,t) = 0, \quad \frac{\partial^2w(L,t)}{\partial x^2} = 0.$$
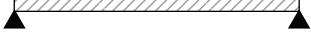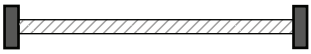
(2.9)

For a clamped-clamped (C-C) nanotube,

$$w(0,t) = 0, \quad \frac{\partial w(0,t)}{\partial x} = 0, \quad w(L,t) = 0, \quad \frac{\partial w(L,t)}{\partial x} = 0.$$

(2.10)

For a clamped-simply supported (C-S) nanotube,

$$w(0,t) = 0, \quad \frac{\partial w(0,t)}{\partial x} = 0, \quad w(L,t) = 0, \quad \frac{\partial^2w(L,t)}{\partial x^2} = 0.$$

(2.11)

The base functions corresponding to the above boundary conditions are given in Table 1.

Table 1: Common boundary conditions for the transverse vibration of beams.

End conditions of beam	Mode shape (normal function)	Value of β
Simply supported 	$\sin\left(\dfrac{\beta x}{L}\right)$	π
Clamped-Clamped 	$\left(\cosh\left(\dfrac{\beta x}{L}\right) - \cos\left(\dfrac{\beta x}{L}\right)\right)$ $-\left(\dfrac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta}\right)\left(\sinh\left(\dfrac{\beta x}{L}\right) - \sin\left(\dfrac{\beta x}{L}\right)\right)$	4.730041
Clamped-Simply supported 	$\left(\cosh\left(\dfrac{\beta x}{L}\right) - \cos\left(\dfrac{\beta x}{L}\right)\right)$ $-\left(\dfrac{\cosh\beta - \cos\beta}{\sinh\beta - \sin\beta}\right)\left(\sinh\left(\dfrac{\beta x}{L}\right) - \sin\left(\dfrac{\beta x}{L}\right)\right)$	3.926602

3. Application of HPM

3.1. Applying a New HPM for Nonlinear Vibrations of an SWNT

For an SWNT, the nonlinear vibration governing equation is given by (2.6) with $n = 1$ as follows:

$$\frac{d^2 W}{dt^2} + \left(\frac{EI}{\rho A}\frac{\alpha_1}{\alpha_2} + \frac{k}{\rho A}\right)W - \frac{E}{2\rho L}\frac{\alpha_3}{\alpha_2}W^3 = 0. \tag{3.1}$$

Under the transformations $r = \sqrt{I/A}$, $\tau = \omega t$, and $a = W/r$, the above equation can be transformed to the following nonlinear equation:

$$\omega^2 \frac{d^2 a}{d\tau^2} + f_1 a - f_2 a^3 = 0, \tag{3.2}$$

with f_1 and f_2 defined as

$$f_1 = \frac{EI}{\rho A}\frac{\alpha_1}{\alpha_2} + \frac{k}{\rho A} = \omega_b^2, \qquad f_2 = \frac{EI}{2\rho AL}\frac{\alpha_3}{\alpha_2}. \tag{3.3}$$

In (3.2), $\sqrt{f_1} = \omega_b$ is the linear, free vibration frequency, and ω is an unknown angular frequency to be determined. Now, we apply a New HPM to seek the solution of (3.1). To this aim, we construct the following homotopy with ω_0 as the initial approximation for the angular frequency:

$$(1-p)\omega_0^2\left(\frac{d^2 a}{d\tau^2} + a\right) + p\left(\omega^2\frac{d^2 a}{d\tau^2} + f_1 a - f_2 a^3\right) = 0. \tag{3.4}$$

Here p is an embedding parameter, $a = a(\tau, p)$, and $\omega = \omega(p)$. Clearly, when $p = 0$, (3.4) yields the linear harmonic

$$\frac{d^2 a}{d\tau^2} + a = 0, \qquad a(0) = X, \qquad \frac{da(0)}{d\tau} = 0, \tag{3.5}$$

and, for $p = 1$, it results the nonlinear equation (3.1). As embedding parameter p travels from 0 to 1, the solutions $a = a(\tau, p)$ and $\omega = \omega(p)$ of the homotopy (3.4) change from their initial approximations $a_0(\tau)$ and ω_0 to the required solutions $a(\tau)$ and ω of (3.1).

Suppose the solution of (3.2) to be in the following form:

$$a(\tau) = a_0(\tau) + p a_1(\tau) + p^2 a_2(\tau) + \cdots,$$
$$\omega = \omega_0 + p\omega_1 + p^2 \omega_2 + \cdots. \tag{3.6}$$

Substituting the above relations into the homotopy equation (3.4) and rearranging the coefficients of the terms with identical powers of p, we have a series of linear differential equations

$$p^0 : \frac{d^2 a_0}{d\tau^2} + a_0 = 0, \qquad a_0(0) = X, \qquad \frac{da_0(0)}{d\tau} = 0,$$

$$p^1 : \omega_0^2 \left(\frac{d^2 a_1}{d\tau^2} + a_1 \right) - \omega_0^2 a_0 + f_1 a_0 - f_2 a_0^3 = 0, \qquad a_1(0) = 0, \qquad \frac{da_1(0)}{d\tau} = 0,$$

$$p^2 : \omega_0^2 \left(\frac{d^2 a_2}{d\tau^2} + a_2 \right) - \omega_0^2 a_1 + 2\omega_0 \omega_1 \frac{d^2 a_0}{d\tau^2} + f_1 a_1 - 3 f_2 a_0^2 a_1 = 0, \tag{3.7}$$

$$a_2(0) = 0, \qquad \frac{da_2(0)}{d\tau} = 0,$$

$$\vdots$$

The solution of the initial zeroth approximation is simply given by

$$a_0(\tau) = X \cos \tau, \tag{3.8}$$

putting this into the first approximation equation, we get

$$\omega_0^2 \frac{d^2 a_1}{d\tau^2} + \omega_0^2 a_1 - \omega_0^2 X \cos \tau + f_1 X \cos \tau - f_2 X^3 \cos^3 \tau = 0. \tag{3.9}$$

Expanding the trigonometric function $\cos^3 \tau$ as $\cos^3 \tau = (3/4)\cos\tau + (1/4)\cos 3\tau$, and letting the coefficient of $\cos\tau$ to be zero in order to eliminate the secular terms, we arrive at

$$w_0 = \sqrt{f_1 - \frac{3}{4}f_2 X^2}. \tag{3.10}$$

Therefore, the ratio of the nonlinear frequency, w_0, to the linear frequency, w_b, becomes

$$\Psi = \sqrt{1 - \frac{3}{4}\left(\frac{f_2}{f_1}\right)X^2}. \tag{3.11}$$

The solution of (3.9) can be easily achieved as

$$a_1(\tau) = \frac{f_2 X^3}{32 f_1 - 24 f_2 X^2}(\cos\tau - \cos 3\tau). \tag{3.12}$$

Thus, the first approximate solution of (3.2) can be written as follows:

$$a(\tau) = a_0(\tau) + a_1(\tau) = X\cos\tau + \frac{f_2 X^3}{32 f_1 - 24 f_2 X^2}(\cos\tau - \cos 3\tau). \tag{3.13}$$

3.2. Applying a New HPM for Nonlinear Vibrations of a DWNT

For a DWNT, the nonlinear vibration governing equation is given by (2.6) with $n = 2$ as follows:

$$\frac{d^2 W_1}{dt^2} + \left(\frac{EI_1}{\rho A_1}\frac{\alpha_1}{\alpha_2} + \frac{c_1}{\rho A_1}\right)W_1 - \frac{E}{2\rho L}\frac{\alpha_3}{\alpha_2}W_1^3 - \frac{c_1}{\rho A_1}W_2 = 0,$$

$$\frac{d^2 W_2}{dt^2} + \left(\frac{EI_2}{\rho A_2}\frac{\alpha_1}{\alpha_2} + \frac{k}{\rho A_2} + \frac{c_1}{\rho A_2}\right)W_2 - \frac{E}{2\rho L}\frac{\alpha_3}{\alpha_2}W_2^3 - \frac{c_1}{\rho A_2}W_1 = 0. \tag{3.14}$$

Substituting the following dimensionless parameters

$$r = \sqrt{\frac{I_1}{A_1}}, \qquad a_1 = \frac{W_1}{r}, \qquad a_2 = \frac{W_2}{r}, \qquad \tau = wt. \tag{3.15}$$

Equation (3.14) can be transformed to the following dimensionless nonlinear system of equations:

$$\omega^2 \frac{d^2 a_1}{dt^2} + f_1' a_1 - f_2' a_1^3 - f_3' a_2 = 0,$$

$$\omega^2 \frac{d^2 a_2}{dt^2} + f_4' a_2 - f_5' a_2^3 - f_6' a_1 = 0,$$

$$(3.16)$$

with f_1' to f_6' defined as

$$f_1' = \left(\frac{EI_1}{\rho A_1} \frac{\alpha_1}{\alpha_2} + \frac{c_1}{\rho A_1} \right), \qquad f_1' = \frac{EI_1}{2\rho A_1 L} \frac{\alpha_3}{\alpha_2}, \qquad f_3 = \frac{c_1}{\rho A_1},$$

$$f_4' = \left(\frac{EI_2}{\rho A_2} \frac{\alpha_1}{\alpha_2} + \frac{k}{\rho A_2} + \frac{c_1}{\rho A_2} \right), \qquad f_5' = \frac{EI_1}{2\rho A_1 L} \frac{\alpha_3}{\alpha_2}, \qquad f_6' = \frac{c_1}{\rho A_2}.$$

$$(3.17)$$

In a similar manner as above, one constructs a homotopy on (3.16) as follows:

$$(1-p)\omega_0^2 \left(\frac{d^2 a_1}{d\tau^2} + a_1 \right) + p \left(\omega^2 \frac{d^2 a_1}{d\tau^2} + f_1' a_1 - f_2' a_1^3 - f_3' a_2 \right) = 0,$$

$$(1-p)\omega_0^2 \left(\frac{d^2 a_2}{d\tau^2} + a_2 \right) + p \left(\omega^2 \frac{d^2 a_2}{dt^2} + f_4' a_2 - f_5' a_2^3 - f_6' a_1 \right) = 0.$$

$$(3.18)$$

Here, the solutions $a_1 = a_1(\tau, p)$, $a_2 = a_2(\tau, p)$, and $\omega = \omega(p)$ of the homotopy (3.18) change from their initial approximations $a_{10}(\tau)$, $a_{20}(\tau)$ and ω_0 to the required solutions $a_1(\tau)$, $a_2(\tau)$ and ω of (3.16) as the embedding parameter p travels from 0 to 1.

Suppose the solution of (3.18) to be in the following form

$$a_1(\tau) = a_{10}(\tau) + p a_{11}(\tau) + p^2 a_{12}(\tau) + \cdots,$$

$$a_2(\tau) = a_{20}(\tau) + p a_{21}(\tau) + p^2 a_{22}(\tau) + \cdots,$$

$$(3.19)$$

$$\omega = \omega_0 + p\omega_1 + p^2 \omega_2 + \cdots.$$

Substituting the above relations into the homotopy equation (3.18) and rearranging the coefficients of the terms with identical powers of p, we have a series of linear differential equations

$$
p^0 : \begin{cases} \dfrac{d^2 a_{10}}{d\tau^2} + a_{10} = 0, \quad a_{10}(0) = X_1, \quad \dfrac{da_{10}(0)}{d\tau} = 0, \\[3mm] \dfrac{d^2 a_{20}}{d\tau^2} + a_{20} = 0, \quad a_{20}(0) = X_2, \quad \dfrac{da_{20}(0)}{d\tau} = 0, \end{cases}
$$

$$
p^1 : \begin{cases} \omega_0^2\left(\dfrac{d^2 a_{11}}{d\tau^2} + a_{11}\right) - \omega_0^2 a_{10} + f_1' a_{10} - f_2' a_{10}^3 - f_3' a_{20} = 0, \quad a_{11}(0) = 0, \quad \dfrac{da_{11}(0)}{d\tau} = 0, \\[3mm] \omega_0^2\left(\dfrac{d^2 a_{21}}{d\tau^2} + a_{21}\right) - \omega_0^2 a_{20} + f_4' a_{20} - f_5' a_{20}^3 - f_6' a_{10} = 0, \quad a_{21}(0) = 0, \quad \dfrac{da_{21}(0)}{d\tau} = 0, \end{cases}
$$

$$\vdots$$

$$(3.20)$$

The solution of the initial zeroth approximation is simply given by

$$
a_{10}(\tau) = X_1 \cos\tau,
$$
$$
a_{20}(\tau) = X_2 \cos\tau.
$$

$$(3.21)$$

Substituting these into the first approximation together with eliminating the coefficient of $\cos\tau$ in the above system to avoid the secular terms it results in the following nonlinear system which can be easily solved using a simple mathematical algorithm such as Newton-Raphson technique:

$$
-X_1 \omega_0^2 + f_1' X_1 - \frac{3}{4} f_2' X_1^3 - f_3' X_2 = 0,
$$
$$
-X_2 \omega_0^2 + f_4' X_2 - \frac{3}{4} f_5' X_2^3 - f_6' X_1 = 0.
$$

$$(3.22)$$

3.3. Applying a New HPM for Nonlinear Vibrations of a TWNT

For a TWNT, the nonlinear vibration governing equation is given by (2.6) with $n = 3$ as follows:

$$
\frac{d^2 W_1}{dt^2} + \left(\frac{EI_1}{\rho A_1}\frac{\alpha_1}{\alpha_2} + \frac{c_1}{\rho A_1}\right) W_1 - \frac{E}{2\rho L}\frac{\alpha_3}{\alpha_2} W_1^3 - \frac{c_1}{\rho A_1} W_2 = 0,
$$

$$
\frac{d^2 W_2}{dt^2} + \left(\frac{EI_2}{\rho A_2}\frac{\alpha_1}{\alpha_2} + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2}\right) W_2 - \frac{E}{2\rho L}\frac{\alpha_3}{\alpha_2} W_2^3 - \frac{c_1}{\rho A_2} W - \frac{c_2}{\rho A_2} W_2 = 0,
$$

$$
\frac{d^2 W_3}{dt^2} + \left(\frac{EI_3}{\rho A_3}\frac{\alpha_1}{\alpha_2} + \frac{c_2}{\rho A_3} + \frac{k}{\rho A_3}\right) W_3 - \frac{E}{2\rho L}\frac{\alpha_3}{\alpha_2} W_3^3 - \frac{c_2}{\rho A_3} W_2 = 0.
$$

$$(3.23)$$

Substituting the following dimensionless parameters:

$$r = \sqrt{\frac{I_1}{A_1}}, \qquad a_1 = \frac{W_1}{r}, \qquad a_2 = \frac{W_2}{r}, \qquad a_3 = \frac{W_3}{r}, \qquad \tau = \omega t. \qquad (3.24)$$

Equation (3.3) can be transformed to the following dimensionless nonlinear system of equations:

$$\omega^2 \frac{d^2 a_1}{dt^2} + f_1'' a_1 - f_2'' a_1^3 - f_3'' a_2 = 0,$$

$$\omega^2 \frac{d^2 a_2}{dt^2} + f_4'' a_2 - f_5'' a_2^3 - f_6'' a_1 - f_7'' a_3 = 0, \qquad (3.25)$$

$$\omega^2 \frac{d^2 a_3}{dt^2} + f_8'' a_3 - f_9'' a_3^3 - f_{10}'' a_2 = 0,$$

with f_1'' to f_{10}'' defined as

$$f_1'' = \left(\frac{EI_1}{\rho A_1} \frac{\alpha_1}{\alpha_2} + \frac{c_1}{\rho A_1} \right), \qquad f_2'' = \frac{EI_1}{2\rho A_1 L} \frac{\alpha_3}{\alpha_2}, \qquad f_3'' = \frac{c_1}{\rho A_1},$$

$$f_4'' = \left(\frac{EI_2}{\rho A_2} \frac{\alpha_1}{\alpha_2} + \frac{c_1}{\rho A_2} + \frac{c_2}{\rho A_2} \right), \qquad f_5'' = \frac{EI_1}{2\rho A_1 L} \frac{\alpha_3}{\alpha_2}, \qquad f_6'' = \frac{c_1}{\rho A_2}, \qquad f_7'' = \frac{c_2}{\rho A_2},$$

$$f_8'' = \left(\frac{EI_3}{\rho A_3} \frac{\alpha_1}{\alpha_2} + \frac{c_2}{\rho A_3} + \frac{k}{\rho A_3} \right), \qquad f_9'' = \frac{EI_1}{2\rho A_1 L} \frac{\alpha_3}{\alpha_2}, \qquad f_{10}'' = \frac{c_2}{\rho A_3}.$$

$$(3.26)$$

Likewise, we construct a homotopy on (3.25) as follows:

$$(1-p)\omega_0^2 \left(\frac{d^2 a_1}{d\tau^2} + a_1 \right) + p \left(\omega^2 \frac{d^2 a_1}{d\tau^2} + f_1'' a_1 - f_2'' a_1^3 - f_3'' a_2 \right) = 0,$$

$$(1-p)\omega_0^2 \left(\frac{d^2 a_2}{d\tau^2} + a_2 \right) + p \left(\omega^2 \frac{d^2 a_2}{dt^2} + f_4'' a_2 - f_5'' a_2^3 - f_6'' a_1 - f_7'' a_3 \right) = 0, \qquad (3.27)$$

$$(1-p)\omega_0^2 \left(\frac{d^2 a_3}{d\tau^2} + a_3 \right) + p \left(\omega^2 \frac{d^2 a_3}{dt^2} + f_8'' a_3 - f_9'' a_3^3 - f_{10}'' a_2 \right) = 0.$$

Suppose the solution of (3.27) to be in the following form:

$$a_1(\tau) = a_{10}(\tau) + pa_{11}(\tau) + p^2 a_{12}(\tau) + \cdots,$$

$$a_2(\tau) = a_{20}(\tau) + pa_{21}(\tau) + p^2 a_{22}(\tau) + \cdots,$$

$$a_3(\tau) = a_{30}(\tau) + pa_{31}(\tau) + p^2 a_{32}(\tau) + \cdots, \tag{3.28}$$

$$\omega = \omega_0 + p\omega_1 + p^2 \omega_2 + \cdots.$$

Taking $a_{10} = X_1 \cos\tau$, $a_{20} = X_2 \cos\tau$, and $a_{30} = X_3 \cos\tau$, as the initial guesses for a_1, a_2, and a_3 then eliminating secular terms as above, the first approximate solution appears in the following nonlinear system:

$$-X_1\omega_0^2 + f_1''X_1 - \frac{3}{4}f_2''X_1^3 - f_3''X_2 = 0,$$

$$-X_2\omega_0^2 + f_4''X_2 - \frac{3}{4}f_5''X_2^3 - f_6''X_1 - f_7''X_3 = 0, \tag{3.29}$$

$$-X_3\omega_0^2 + f_8''X_3 - \frac{3}{4}f_9''X_2^3 - f_{10}''X_2 = 0.$$

Using the former solving process, the nonlinear amplitude frequency response curves can be obtained for TWNTs.

4. Discussion of Results

The material and geometric parameters of a CNT are taken as $E = 1.1\,\text{TPa}$, $\rho = 1300\,\text{kg/m}^3$, $l = 45\,\text{nm}$, the outer diameter $d_1 = 3\,\text{nm}$, and the thickness of the each layer $0.68\,\text{nm}$. The amplitude-frequency response curves for a TWNT for several spring constants k are shown in Figure 2 in which Ψ is the ratio of the fundamental nonlinear frequency, ω to the fundamental linear frequency ω_b, and X is the maximum vibration amplitude. The fundamental linear frequency can be simply calculated by setting $W_i = W_{\text{max}_i} \cos\omega_b t$, $i = 1, 2, \ldots, n$, into nonlinear vibration governing equations of MWNTs without considering its nonlinear terms. Then, by setting the determinant of the achieved matrix equal to zero, the frequency characteristic equation will be obtained. The fundamental linear vibration frequency of MWNT is the lowest root of the resulting characteristic equation as illustrated in [35]. As can be seen from Figure 2, in contrast to linear systems, the nonlinear frequency is a function of amplitude so that the larger the amplitude, the more pronounced the discrepancy between the linear and nonlinear frequencies becomes. It is also seen that as the spring constant k increases, the nonlinear frequencies tend to approach the linear ones especially when k exceeds the value $k = 10^7\,\text{N/m}^2$. It should be noted that the results are in an excellent agreement with those obtained via incremental harmonic balance method (IHBM) according to the formulations presented in [12]. Figure 3 shows the nonlinear to linear frequency ratio versus nondimensional amplitude ratio for three different boundary conditions and $k = 10^7\,\text{n/m}^2$. Similar figures for DWNT and TWNT are illustrated in Figures 4 and 5. In Figures 4 and 5, the values of the coefficients of the van der Waals forces are $c_1 = c_2 = 0.3 \times 10^{12}\,\text{N/m}^2$ and $k = 10^7\,\text{N/m}^2$.
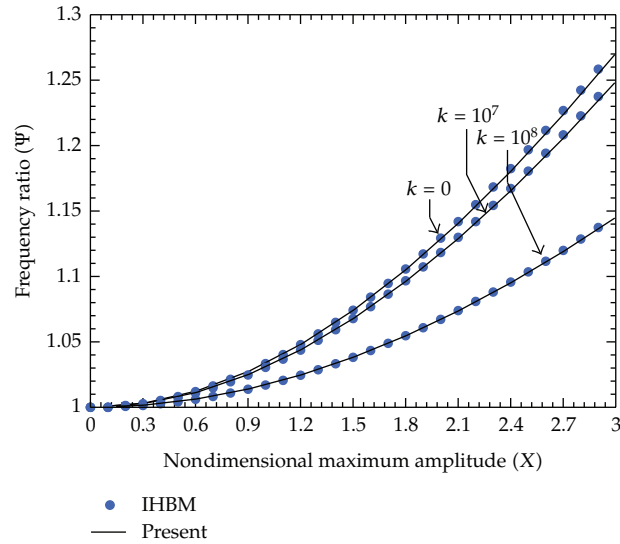
Figure 2: Effect of spring constant k on nonlinear amplitude frequency response curves of TWNT.
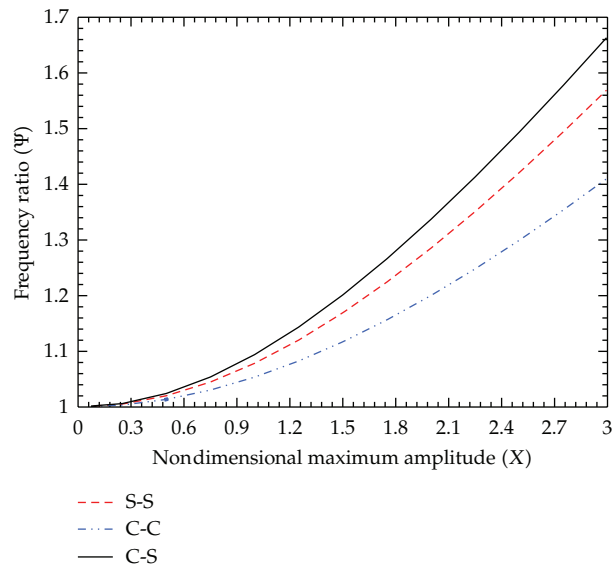

Figure 3: Nondimensional frequency ratio against nondimensional maximum amplitude of SWNTs under various boundary conditions ($k = 10^7\,\mathrm{N/m^2}$).

Figure 6 illustrates the nonlinear frequency variation against length to radius ratio of a TWNT shell under different boundary conditions for $k = 10^7\,\mathrm{N/m^2}$ and $X = 2$. It can be observed that with the increase of the aspect ratio of the nanotubes, the nonlinear vibration frequencies of MWNTs decrease. As is expected, the double clamped CNT has the highest natural frequency among the selected boundary conditions. A comparison between the nondimensional amplitude ratio of the first layer of a C-C SWNT and its linear counterpart is shown in Figure 7 for $k = 10^7\,\mathrm{N/m^2}$. To demonstrate the accuracy of the obtained analytical
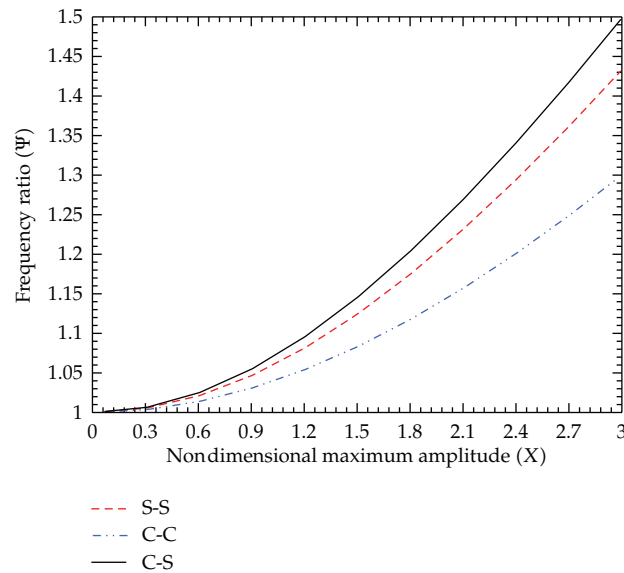
Figure 4: Nondimensional frequency ratio against nondimensional maximum amplitude of DWNTs under various boundary conditions ($k = 10^7 \, \text{N/m}^2$).
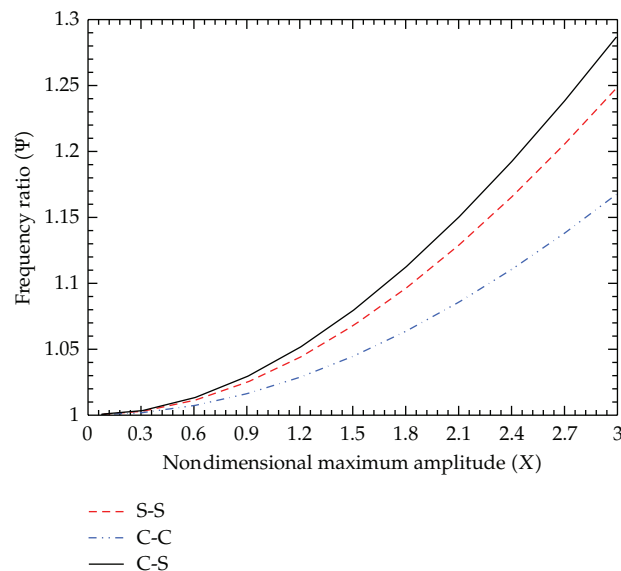

Figure 5: Nondimensional frequency ratio against nondimensional maximum amplitude of TWNTs under various boundary conditions ($k = 10^7 \, \text{N/m}^2$).

results, the authors also calculate the variation of nondimensional amplitude ratio versus the linear period of vibration for the CNT center using fourth-order Runge-Kutta method. As can be seen in the figure, the results obtained using the HPM have a good agreement with numerical results. Using (3.21) and Table 1, the amplitude-space-time diagram of an SWNT
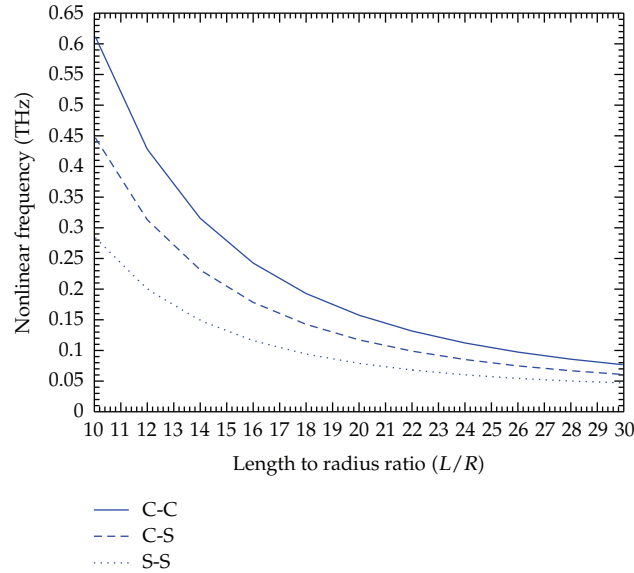
Figure 6: Fundamental nonlinear frequency against length to the outermost diameter ratio of a TWNT under different boundary conditions ($k = 10^7 \, \text{N/m}^2$ and $X = 2$).
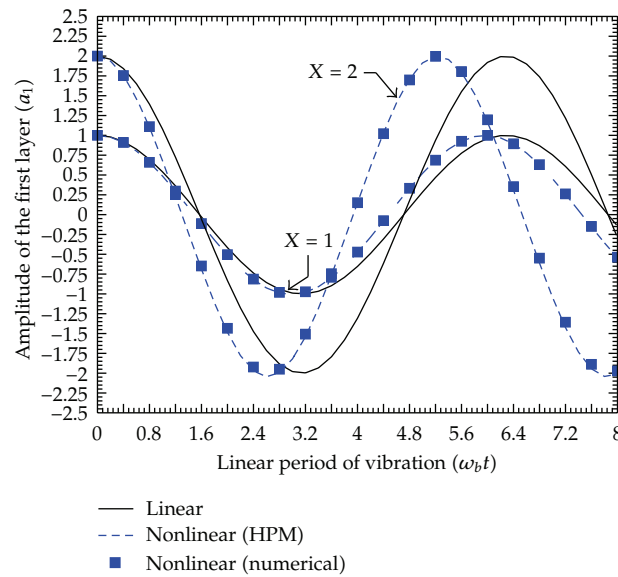

Figure 7: Variation of the nondimensional amplitude ratio versus the linear period of a C-C SWNT ($k = 10^7 \, \text{N/m}^2$).

is plotted for S-S and C-C end conditions in Figures 8 and 9. The parameters used in these figures are $k = 10^7 \, \text{N/m}^2$ and $X = 2$.

5. Conclusions

The HPM has been successfully used to investigate the nonlinear vibration analysis of multiwalled carbon nanotubes with arbitrary end conditions. The generated results obtained
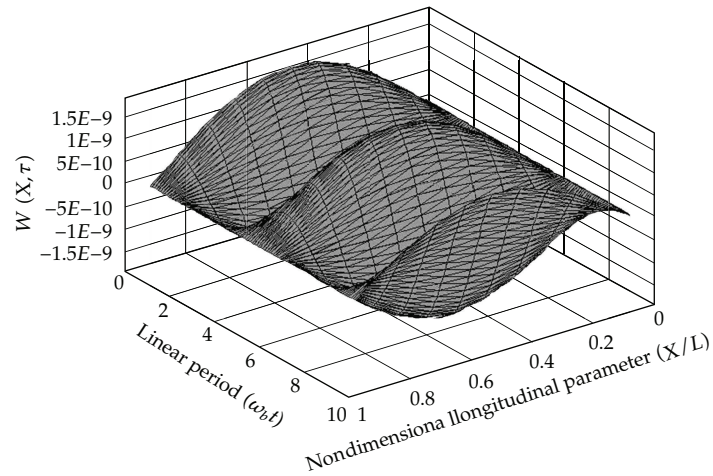
Figure 8: Amplitude-time-space diagram of an S-S SWNT ($k = 10^7 \, \text{N/m}^2$ and $X = 2$).
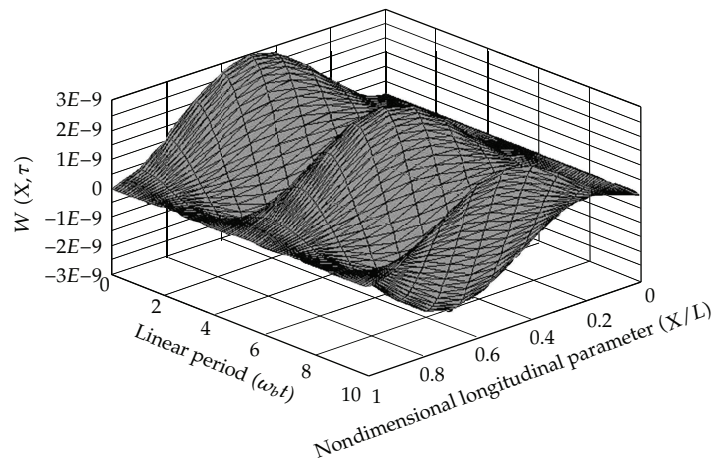

Figure 9: Amplitude-time-space diagram of a C-C SWNT ($k = 10^7 \, \text{N/m}^2$ and $X = 2$).
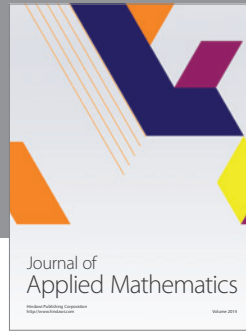
have been compared with those available in open literature, and excellent correlation has been achieved. The significant dependency of this oscillation to the surrounding elastic medium is observed. The nonlinear vibration frequency of nanotubes rises rapidly with increasing the amplitude especially when the stiffness of the medium is relatively small. For larger values of the medium stiffnesses (say $k > 10^8 \, \text{N/m}^2$), the nonlinear vibration tends to the linear regime. It has been shown that with the increase of the aspect ratio of the nanotubes, the nonlinear vibration frequencies of MWNTs decrease. Also, as one travels through the end conditions of S-S, to fully clamped, denoted by C-C, respectively, the influence of the boundary conditions is shown to increase the natural frequencies. This effect is more significant for lower values of length to outermost diameter ratios. However, for higher values, the effect of the changes in the boundary conditions diminishes so that the corresponding natural frequencies are found to coincide.

The proposed method can be readily extended to the multiwalled CNTs with the number of walls more than three. It is worthwhile to mention that HPM is straightforward and powerful, and it is a promising technique for solving strong nonlinear partial differential equations.

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