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Research Article

Positive Solutions of Sturm-Liouville Boundary Value Problems in Presence of Upper and Lower Solutions

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We consider a kind of Sturm-Liouville boundary value problems. Using variational techniques combined with the methods of upper-lower solutions, the existence of at least one positive solution is established. Moreover, the upper solution and the lower solution are presented.

1. Introduction

The Sturm-Liouville boundary value problems (for short, BVPs) have received a lot of attention. Many works have been carried out to discuss the existence of at least one solution or multiple solutions. The methods used therein mainly depend on the Leray-Schauder continuation theorem and the Mawhin continuation theorem. Since it is very difficult to give the corresponding Euler functional for Sturm-Liouville BVPs and verify the existence of critical points for the Euler functional, few people consider the existence of solutions for Sturm-Liouville BVPs by critical point theory and many works considered the existence of solutions for Dirichlet BVPs. For example, by a three-critical-point theorem due to Ricceri [1], Bonanno [2] considered Dirichlet problems. Moreover, Afrouzi and Heidarkhani [3] also considered the existence of three solutions for a kind of Dirichlet BVP. By using an appropriate variational framework, the authors [4] considered the existence of positive solutions for the Dirichlet BVP.

In this paper, using variational methods combined with the methods of upper-lower solutions, we consider the positive solutions of the following BVP:

$$\begin{gathered}
-\left(\phi_p\left(x'(t)\right)\right)' = -a(t)\phi_p(x) + f(t,x), \quad t \in [0,1], \\
\alpha_1 x(0) - \alpha_2 x'(0) = 0, \\
\beta_1 x(1) + \beta_2 x'(1) = 0,
\end{gathered} \tag{1.1}$$

where $p > 1$, $\phi_p(x) = |x|^{p-2}x$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $\alpha_1^2 + \alpha_2^2 > 0$, $\beta_1^2 + \beta_2^2 > 0$.

The paper is organized as follows. In the forthcoming section, we give the Euler functional of BVP(1.1) and some basic lemmas. In Section 3, firstly, we give an upper solution of BVP(1.1), then, by the mountain pass lemma, the lower solution of BVP(1.1) is obtained. At last, we show the existence of at least one positive solution of BVP(1.1) based on the upper solution and the lower solution we obtain.

2. Preliminary

The Sobolev space $W^{1,p}[0,1]$ is defined by

$$W^{1,p}[0,1] = \left\{x : [0,1] \longrightarrow \mathbb{R} \mid x \text{ is absolutely continuous and } x' \in L^p(0,1;\mathbb{R})\right\} \tag{2.1}$$

and is endowed with the norm

$$\|x\| = \left(\int_0^1 |x(t)|^p dt + \int_0^1 \left|x'(t)\right|^p dt\right)^{1/p}. \tag{2.2}$$

Then, $W^{1,p}[0,1]$ is a separable and reflexive Banach space [5].

Lemma 2.1 (see [6]). *There exists a positive constant c_p such that*

$$\left(|x|^{p-2}x - |y|^{p-2}y, x-y\right) \geq \begin{cases} c_p|x-y|^p, & p \geq 2, \\ c_p \dfrac{|x-y|^2}{(|x|+|y|)^{2-p}}, & 1 < p < 2 \end{cases} \tag{2.3}$$

for any $x, y \in R^N$. Here $(x,y) = x \cdot y^T$.

For $x \in C[0,1]$, suppose that $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$, $\|x\|_m = \min_{t\in[0,1]}|x(t)|$.

Lemma 2.2 (see [7]). *If $x \in W^{1,p}[0,1]$, then, $\|x\|_\infty \leq 2\|x\|$.*

Lemma 2.3 (see [8]). *For $x \in X$, let $x^\pm = \max\{\pm x, 0\}$; then, the following properties hold:*

(i) $x \in X \Rightarrow x^+,\ x^- \in X$;

(ii) $x = x^+ - x^-$;

(iii) $\|x^+\|_X \leq \|x\|_X$;

(iv) *if* $(x_n)_{n\in N}$ *uniformly converges to* x *in* $C([0,1])$, *then,* $(x_n^+)_{n\in N}$ *uniformly converges to* x^+;

(v) $\phi_p(x)x^+ = |x^+|^p$, $\phi_p(x)x^- = -|x^-|^p$.

In the following, we state the (C) condition [9].

(C) *Every sequence* $(x_n)_{n\in N} \subset H$ *such that the following conditions hold:*

(i) $(\varphi(x_n))_{n\in N}$ *is bounded;*

(ii) $(1 + \|x_n\|_H)\|\varphi'(x_n)\|_{H^*} \to 0$, $n \to \infty$

has a subsequence which converges strongly in H.

With a similar proof of Lemma 2.5 [8], one has the following lemma.

Lemma 2.4. *If* $x(t) \in W^{1,p}[0,1]$ *is a critical point of the Euler functional*

$$\varphi(x) = \frac{1}{p}\int_0^1 a(t)|x|^p dt + \frac{1}{p}\int_0^1 \left|x'\right|^p dt - \int_0^1 F(t,x)dt + \frac{\alpha_2}{p\alpha_1}\left|\frac{\alpha_1 x(0)}{\alpha_2}\right|^p + \frac{\beta_2}{p\beta_1}\left|\frac{\beta_1 x(1)}{\beta_2}\right|^p, \tag{2.4}$$

then, $x(t)$ *is a solution of BVP* (1.1). *Here,* $F(t,x) = \int_0^x f(t,s)ds$.

Remark 2.5. While $\alpha_2 = 0$, the Euler functional $\varphi(x)$ does not include $(1/p)(\alpha_1/\alpha_2)^{p-1}|x(0)|^p$, while $\beta_2 = 0$, $\varphi(x)$ does not include $(1/p)(\beta_1/\beta_2)^{p-1}|x(1)|^p$. Hence, in order to be convenient, we assume that $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

With little modification to the proof of Theorem 1.4 in [7], we obtain the following.

Remark 2.6. φ is continuously differentiable on $W^{1,p}[0,1]$, and, by computation, one has

$$\begin{aligned}\langle \varphi'(x), y\rangle = {} & \int_0^1 a(t)\phi_p(x)y\,dt + \int_0^1 \phi_p(x')y'\,dt - \int_0^1 f(t,x)y dt + \phi_p\left(\frac{\alpha_1 x(0)}{\alpha_2}\right)y(0) \\ & + \phi_p\left(\frac{\beta_1 x(1)}{\beta_2}\right)y(1), \quad x, y \in W^{1,p}[0,1].\end{aligned} \tag{2.5}$$

Definition 2.7. $u \in W^{1,p}[0,1]$ is an upper solution of BVP (1.1) if it satisfies

$$\begin{gathered} -(\phi_p(u'(t)))' + a(t)\phi_p(u) - f(t,u) \geq 0, \quad t \in [0,1], \\ \alpha_1 u(0) - \alpha_2 u'(0) \geq 0, \qquad \beta_1 u(1) + \beta_2 u'(1) \geq 0. \end{gathered} \tag{2.6}$$

If u is not a solution of BVP(1.1), then, u is a strict upper solution.

Definition 2.8. $v \in W^{1,p}[0,1]$ is a lower solution of BVP(1.1) if it satisfies

$$\begin{gathered} -(\phi_p(v'(t)))' + a(t)\phi_p(v) - f(t,v) \leq 0, \quad t \in [0,1], \\ \alpha_1 v(0) - \alpha_2 v'(0) \leq 0, \qquad \beta_1 v(1) + \beta_2 v'(1) \leq 0. \end{gathered} \tag{2.7}$$

If v is not a solution of BVP(1.1), then, v is a strict lower solution.

Definition 2.9. $x \in W^{1,p}[0,1]$ is said to be a positive solution of BVP(1.1) if $x(t) \geq 0$, $x(t) \not\equiv 0$, $t \in [0,1]$.

3. Existence of Positive Solutions

Choose $x_0 \in W^{1,p}[0,1]$ and $x_0(t) > 0$, $t \in [0,1]$ satisfying $-(\phi_p(x_0'))' = 1$, then, $x_0(t) = c_2 + \int_0^t \phi_q(-s + c_1)ds$ where $(1/p) + (1/q) = 1, c_1, c_2$ are constants. If we choose $c_1 \geq 1$, $c_2 \geq (\alpha_2\phi_q(c_1))/\alpha_1$, $x_0(t)$ satisfies $\alpha_1 x_0(0) - \alpha_2 x_0'(0) \geq 0$, $\beta_1 x_0(1) + \beta_2 x_0'(1) \geq 0$. Moreover, $x_0'(t) = \phi_q(-t + c_1)$ is continuous.

Lemma 3.1. *Assume*

(A_1) $f(t,x) \in C([0,1] \times [0,+\infty))$, $\overline{\lim}_{x\to+\infty}(f(t,x)/\phi_p(x)) < a(t)$, $t \in [0,1]$,

is satisfied; then, $\overline{x} = a_0^{1/(p-1)} x_0$ *is a strict upper solution of BVP* (1.1). *Here* $a_0 > 1$ *is some positive constant.*

Proof. From (A_1), there exists a constant $N > 0$ such that

$$\frac{f(t,x)}{\phi_p(x)} < a(t), \quad x > N. \tag{3.1}$$

Hence,

$$f(t,x) < a(t)\phi_p(x) + a_0, \quad t \in [0,1], \tag{3.2}$$

holds for $x \geq 0$ and some large positive constant $a_0 > 1$. Then,

$$\begin{aligned} f\left(t, a_0^{1/(p-1)} x_0\right) &< a(t)\phi_p\left(a_0^{1/(p-1)} x_0\right) + a_0 \\ &= a(t)\phi_p\left(a_0^{1/(p-1)} x_0\right) - \left(\phi_p\left(a_0^{1/(p-1)} x_0'\right)\right)', \quad t \in [0,1], \end{aligned} \tag{3.3}$$

that is, $-(\phi_p(\overline{x}'))' + a(t)\phi_p(\overline{x}) - f(t,\overline{x}) > 0$, $t \in [0,1]$. Obviously, $\alpha_1\overline{x}(0) - \alpha_2\overline{x}'(0) \geq 0$, $\beta_1\overline{x}(1) + \beta_2\overline{x}'(1) \geq 0$. Therefore, from Definition 2.7, one has that $\overline{x} = a_0^{1/(p-1)} x_0$ is a strict upper solution of BVP (1.1). □

In the following, we assume the following conditions.

(A_2) There exist $\delta > 0$ and $g(x) : [0,+\infty) \to [0,+\infty)$ satisfying $g(mx) \geq m^{p-1}g(x)$ for $m < 1$, $g(0) \neq 0$, $f(t,x) > g(x)$ for $x \in (0,\delta]$, $t \in [0,1]$.

(A_3) There exists $\mu > p$ such that $\mu G(x) \leq g(x)x$, $x \geq 0$, $G(x) = \int_0^x g(s)ds$.

Consider the auxiliary BVP

$$\begin{gathered} -\left(\phi_p\left(x'(t)\right)\right)' = -a(t)\phi_p(x) + g\left(x^+\right), \quad t \in [0,1], \\ \alpha_1 x(0) - \alpha_2 x'(0) = 0, \\ \beta_1 x(1) + \beta_2 x'(1) = 0. \end{gathered} \tag{3.4}$$

Obviously, the corresponding Euler functional of BVP(3.4) is

$$\varphi_+(x) = \frac{1}{p}\int_0^1 a(t)|x|^p dt + \frac{1}{p}\int_0^1 \left|x'\right|^p dt - \int_0^1 \left(G(x^+(t)) - g(0)x^-\right)dt + \frac{\alpha_2}{p\alpha_1}\left|\frac{\alpha_1 x(0)}{\alpha_2}\right|^p + \frac{\beta_2}{p\beta_1}\left|\frac{\beta_1 x(1)}{\beta_2}\right|^p. \tag{3.5}$$

Obviously, φ_+ is continuously differentiable on $W^{1,p}[0,1]$, and, by computation, one has

$$\langle\varphi'(x), y\rangle = \int_0^1 a(t)\phi_p(x)y\,dt + \int_0^1 \phi_p(x')y'dt - \int_0^1 g(x^+)y\,dt + \phi_p\left(\frac{\alpha_1 x(0)}{\alpha_2}\right)y(0) + \phi_p\left(\frac{\beta_1 x(1)}{\beta_2}\right)y(1), \quad x, y \in W^{1,p}[0,1]. \tag{3.6}$$

Lemma 3.2. *If $x(t) \in W^{1,p}[0,1]$ is a solution of BVP* (3.4)*, then, $x(t) \geq 0$.*

Proof. Let $x(t) \in W^{1,p}[0,1]$ be a solution of the BVP (3.4). If there exists a subset $E_0 \subset [0,1]$, meas $E_0 \neq 0$, $x(t) \equiv 0$ for $t \in E_0$, then from the BVP (3.4), one has $g(0) \equiv 0$ for $t \in E_0$ which contradicts with the assumptions. Moreover, x^- is an absolutely continuous function on $[0,1]$, and so the fundamental theorem of calculus ensures the existence of a set $E_1 \subset [0,1]$ such that $\text{meas}([0,1] \setminus E_1) = 0$ and x^- is differentiable on E_1, $(x^-)' \in L^1[0,1]$,

$$\begin{aligned}
0 &= \int_0^1 \left((\phi_p(x'(t)))' - a(t)\phi_p(x(t)) + g(x^+)\right)x^- dt \\
&\geq x^-(1)\phi_p(x'(1)) - x^-(0)\phi_p(x'(0)) - \int_{E_1} \phi_p(x'(t))(x^-)'dt - \int_0^1 a(t)\phi_p(x(t))x^- dt \\
&\geq x^-(1)\phi_p(x'(1)) - x^-(0)\phi_p(x'(0)) + \int_{E_1} \left|(x^-)'\right|^p dt + \int_0^1 a(t)\left|x^-\right|^p dt \\
&\geq x^-(1)\phi_p\left(-\frac{\beta_1 x(1)}{\beta_2}\right) - x^-(0)\phi_p\left(\frac{\alpha_1 x(0)}{\alpha_2}\right) + \min\{\|a\|_m, 1\}\left\|x^-\right\|^p \\
&\geq \min\{\|a\|_m, 1\}\left\|x^-\right\|^p.
\end{aligned} \tag{3.7}$$

Therefore, for a.e. $t \in [0,1]$, $x^- = 0$. Since $x(t)$ is absolutely continuous on $[0,1]$, then, $x(t) \geq 0$ for $t \in [0,1]$. □

Lemma 3.3. *Assume that (A_2), (A_3) hold; then, BVP*(3.4) *has a solution x_1, that is, BVP*(3.4) *has a positive solution x_1.*

Proof. Assume that $(x_n)_{n\in N} \subset W^{1,p}[0,1]$ satisfies (i) and (ii) of the (C) condition; then,

$$\left|\varphi_+(x_n)\right| \leq c_1, \qquad \left\|\varphi'_+(x_n)\right\|(1 + \|x_n\|) \leq \varepsilon_n. \tag{3.8}$$

Here, c_1 is some positive constant and $\varepsilon_n \to 0$, $n \to \infty$.

First, we show that $(x_n^-)_{n\geq 1} \subset W^{1,p}[0,1]$ is bounded. Indeed, from (3.8), we have

$$\left|\langle \varphi'_+(x_n), u\rangle\right| \leq \varepsilon_n, \quad u \in W^{1,p}[0,1]. \tag{3.9}$$

Choose $u = -x_n^-$; then,

$$\begin{aligned}\left|\langle \varphi'_+(x_n), -x_n^-\rangle\right| = {}& \int_0^1 a(t)|x_n^-|^p dt + \frac{1}{p}\int_0^1 \left|(x_n^-)'\right|^p dt + \phi_p\left(\frac{\alpha_1}{\alpha_2}\right)|x_n^-(0)|^p \\ & + \phi_p\left(\frac{\beta_1}{\beta_2}\right)|x_n^-(1)|^p + \int_0^1 g(x_n^+)x_n^- dt.\end{aligned} \tag{3.10}$$

Hence, $(x_n^-)_{n\in N}$ is bounded.

Moreover,

$$\begin{aligned}\langle \varphi'_+(x_n), x_n^+\rangle = {}& \int_0^1 a(t)\phi_p(x_n)x_n^+ dt + \int_0^1 \phi_p(x'_n)(x_n^+)' dt - \int_0^1 g(x_n^+)x_n^+ dt + \phi_p\left(\frac{\alpha_1 x_n(0)}{\alpha_2}\right)x_n^+(0) \\ & + \phi_p\left(\frac{\beta_1 x_n(1)}{\beta_2}\right)x_n^+(1) \\ = {}& \int_0^1 a(t)|x_n^+|^p dt + \int_0^1 \left|(x_n^+)'\right|^p dt - \int_0^1 g(x_n^+)x_n^+ dt + \phi_p\left(\frac{\alpha_1}{\alpha_2}\right)|x_n^+(0)|^p \\ & + \phi_p\left(\frac{\beta_1}{\beta_2}\right)|x_n^+(0)|^p.\end{aligned} \tag{3.11}$$

For large n,

$$\begin{aligned}(\mu+1)c_1 = c_1 + \mu c_1 \geq {}& \mu\varphi_+(x_n) - \langle \varphi'_+(x_n), x_n^+\rangle \\ = {}& \frac{\mu}{p}\int_0^1 a(t)|x_n|^p dt + \frac{\mu}{p}\int_0^1 |x'_n|^p dt - \mu\int_0^1 G(x_n^+(t))dt + \mu\int_0^1 g(0)x_n^-(t)dt \\ & + \frac{\mu\alpha_2}{p\alpha_1}\left|\frac{\alpha_1 x_n(0)}{\alpha_2}\right|^p + \frac{\mu\beta_2}{p\beta_1}\left|\frac{\beta_1 x_n(1)}{\beta_2}\right|^p - \int_0^1 a(t)|x_n^+|^p dt - \int_0^1 \left|(x_n^+)'\right|^p dt \\ & + \int_0^1 g(x_n^+)x_n^+(t)dt - \phi_p\left(\frac{\alpha_1}{\alpha_2}\right)|x_n^+(0)|^p - \phi_p\left(\frac{\beta_1}{\beta_2}\right)|x_n^+(1)|^p \\ \geq {}& \frac{\mu}{p}\int_0^1 a(t)|x_n|^p dt + \frac{\mu}{p}\int_0^1 |x'_n|^p dt - \int_0^1 a(t)|x_n^+|^p dt - \int_0^1 \left|(x_n^+)'\right|^p dt \\ \geq {}& \left(\frac{\mu}{p}-1\right)\int_0^1 a(t)|x_n^+|^p dt + \left(\frac{\mu}{p}-1\right)\int_0^1 \left|(x_n^+)'\right|^p dt \\ \geq {}& \left(\frac{\mu}{p}-1\right)\min\{\|a\|_m, 1\}\|x_n^+\|^p.\end{aligned} \tag{3.12}$$

Hence, $(x_n^+)_{n\in N}$ is bounded; then, $(x_n)_{n\in N}$ is uniformly bounded in $W^{1,p}[0,1]$. By the compactness of the embedding $W^{1,p}[0,1]C[0,1]$, the sequence $(x_n)_{n\in N}$ has a subsequence, again denoted by $(x_n)_{n\in N}$ for convenience, such that

$$\begin{aligned} x_n &\rightharpoonup x \quad \text{weakly in } W^{1,p}[0,1], \\ x_n &\longrightarrow x \quad \text{strongly in } C[0,1]. \end{aligned} \tag{3.13}$$

Moreover,

$$\begin{aligned} &\langle \varphi'_+(x_n) - \varphi'_+(x_m), x_n - x_m \rangle \\ &\quad = \int_0^1 (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m)dt \\ &\qquad + \int_0^1 a(t)(\phi_p(x_n) - \phi_p(x_m))(x_n - x_m)dt \\ &\qquad - \int_0^1 (g(x_n^+) - g(x_m^+))(x_n - x_m)dt \\ &\qquad + \left(\phi_p\left(\frac{\alpha_1 x_n(0)}{\alpha_2}\right) - \phi_p\left(\frac{\alpha_1 x_m(0)}{\alpha_2}\right)\right)(x_n(0) - x_m(0)) \\ &\qquad + \left(\phi_p\left(\frac{\beta_1 x_n(1)}{\beta_2}\right) - \phi_p\left(\frac{\beta_1 x_m(1)}{\beta_2}\right)\right)(x_n(1) - x_m(1)). \end{aligned} \tag{3.14}$$

Since $x_n(t) \to x(t)$ in $C[0,1]$, then, $(\phi_p(\alpha_1 x_n(0)/\alpha_2) - \phi_p(\alpha_1 x_m(0)/\alpha_2))(x_n(0) - x_m(0)) \to 0$, $(\phi_p(\beta_1 x_n(1)/\beta_2) - \phi_p(\beta_1 x_m(1)/\beta_2))(x_n(1) - x_m(1)) \to 0$, $(g(x_n^+) - g(x_m^+))(x_n - x_m) \to 0$, $\int_0^1 (x_n(t) - x_m(t))dt \to 0$, $n, m \to \infty$. Moreover,

$$\begin{aligned} \left| \int_0^1 a(t)(\phi_p(x_n) - \phi_p(x_m))(x_n - x_m)dt \right| &\le \|a\|_\infty \|x_n - x_m\|_\infty \int_0^1 (\phi_p(x_n) - \phi_p(x_m))dt \\ &\longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \end{aligned} \tag{3.15}$$

From $|\langle \varphi'(x_n) - \varphi'(x_m), x_n - x_m \rangle| \le (\|\varphi'(x_n)\| + \|\varphi'(x_m)\|) \cdot (\|x_n\| + \|x_m\|)$ and $\|x_n\| + \|x_m\|$ is bounded in $W^{1,p}[0,1]$, $\|\varphi'(x_n)\| \to 0$, $\|\varphi'(x_m)\| \to 0$, $m, n \to \infty$, and one has $\langle \varphi'(x_n) - \varphi'(x_m), x_n - x_m \rangle \to 0$. Hence,

$$\int_0^1 (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m)dt \longrightarrow 0, \quad n, m \longrightarrow \infty. \tag{3.16}$$

If $p \ge 2$, from Lemma 2.1, there exists a positive constant c_p such that

$$\int_0^1 (\phi_p(x'_n) - \phi_p(x'_m))(x'_n - x'_m)dt \ge c_p \int_0^1 |x'_n - x'_m|^p dt. \tag{3.17}$$

If $p < 2$, by Lemma 2.1, the Hölder inequality, and the boundedness of $(x_n)_{n\in N}$ in $W^{1,p}[0,1]$, one has

$$\begin{aligned}
\int_0^1 \left|x_n' - x_m'\right|^p dt &= \int_0^1 \frac{|x_n' - x_m'|^p}{(|x_n'| + |x_m'|)^{p(2-p)/2}} \left(\left|x_n'\right| + \left|x_m'\right|\right)^{p(2-p)/2} dt \\
&\le \left(\int_0^1 \frac{|x_n' - x_m'|^2}{(|x_n'| + |x_m'|)^{2-p}} dt\right)^{p/2} \left(\int_0^1 \left(\left|x_n'\right| + \left|x_m'\right|\right)^p dt\right)^{(2-p)/2} \\
&\le c_p^{-p/2} \left(\int_0^1 \left(\phi_p(x_n') - \phi_p(x_m')\right)\left(x_n' - x_m'\right) dt\right)^{p/2} \\
&\quad \times 2^{(p-1)(2-p)/2} \left(\int_0^1 \left(\left|x_n'\right|^p + \left|x_m'\right|^p\right) dt\right)^{(2-p)/2} \\
&\le c_p^{-p/2} \left(\int_0^1 \left(\phi_p(x_n') - \phi_p(x_m')\right)\left(x_n' - x_m'\right) dt\right)^{p/2} 2^{((p-1)(2-p))/2} \\
&\quad \times \left(\|x_n\|^p + \|x_m\|^p\right)^{(2-p)/2}.
\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we have $\int_0^1 |x_n' - x_m'|^p dt \to 0$ as $n, m \to \infty$. Then, $\|x_n - x_m\| \to 0$, that is, $(x_n)_{n\in N}$ is a Cauchy sequence in $W^{1,p}[0,1]$. By the completeness of $W^{1,p}[0,1]$, we have $x_n \to x$ in $W^{1,p}[0,1]$. From the discussion above, $\varphi(x)$ satisfies the (C) condition.

For $t > 0$, $x > 0$, one has

$$\frac{d(G(t^{-1}x)t^{\mu})}{dt} = t^{\mu-1}\left(\mu G\left(t^{-1}x\right) - t^{-1}xg\left(t^{-1}x\right)\right) \le 0, \tag{3.19}$$

that is, $G(t^{-1}x)t^{\mu}$ is nonincreasing in t. Assume that $M = \max_{x\in[0,1]} G(x)$,

$$G(x^+) \le G\left(\frac{x^+}{|x|}\right)|x|^{\mu} \le M|x|^{\mu}, \quad 0 < |x| \le 1. \tag{3.20}$$

Hence,

$$\begin{aligned}
\varphi_+(x) &\ge \frac{1}{p}\int_0^1 a(t)|x|^p dt + \frac{1}{p}\int_0^1 \left|x'\right|^p dt - \int_0^1 G(x^+) dt \\
&\ge \frac{1}{p}\min\{\|a\|_m, 1\}\|x\|^p - M\|x\|^{\mu}.
\end{aligned} \tag{3.21}$$

Obviously, there exists $\rho > 0$ such that, for $\|x\| = \rho$, $(1/p)\min\{|a|_m, 1\}\rho^p - M\rho^{\mu} = \alpha > 0$.

On the other hand, $G(x) \geq G(1)x^{\mu}$ for $x > 1$; then, by Lemma 2.2

$$\begin{aligned}\varphi_+(x) \leq \frac{1}{p}\max\{\|a\|_\infty, 1\}\|x\|^p - G(1)\int_0^1 |x|^\mu dt + \int_0^1 g(0)x^- dt \\ + \frac{2^{p+1}}{p}\left(\phi_p\left(\frac{\alpha_1}{\alpha_2}\right) + \phi_p\left(\frac{\beta_1}{\beta_2}\right)\right)\|x\|^p.\end{aligned} \tag{3.22}$$

Let e be some large positive constant. Since $\mu > p$, $\varphi_+(e) < 0$. Moreover, $\varphi_+(0) = 0$. From the mountain pass lemma [10], φ_+ possesses a critical value $c \geq \alpha$, that is, there exists x_1 such that $\varphi'_+(x_1) = 0$, $\varphi_+(x_1) = c \geq \alpha > 0$. Then, from Lemma 2.4, one has that BVP(3.4) has a positive solution x_1 and $x_1 \not\equiv 0$, $t \in [0,1]$. □

Lemma 3.4. *Assume that (A_2), (A_3) hold; then, BVP*(1.1) *has a strict lower solution $\underline{x} = \beta x_1$ where β is some positive constant and x_1 is the positive solution of BVP*(3.4) *one obtains that in Lemma 3.3.*

Proof. Assume $\beta \in (0,1]$ is small enough such that $\beta x_1 \in (0,\delta]$ and $\overline{x}(t) - \beta x_1(t) \geq 0$, $\overline{x}(t) - \beta x_1(t) \not\equiv 0$, $t \in [0,1]$. Then,

$$\begin{aligned}-\left(\phi_p\left(\underline{x}'\right)\right)' = -\beta^{p-1}\left(\phi_p\left(x_1'\right)\right)' = -\beta^{p-1}a(t)\phi_p(x_1) + \beta^{p-1}g(x_1) \leq -a(t)\phi_p(\underline{x}) + g(\underline{x}) \\ < -a(t)\phi_p(\underline{x}) + f(t,\underline{x}).\end{aligned} \tag{3.23}$$

Moreover, $\alpha_1\underline{x}(0) - \alpha_2\underline{x}'(0) = 0$, $\beta_1\underline{x}(1) + \beta_2\underline{x}'(1) = 0$. Hence, $\underline{x}$ is a strictly lower solution of BVP(1.1) and $\underline{x} \leq \overline{x}$, $\underline{x} \not\equiv \overline{x}$, $t \in [0,1]$. □

Theorem 3.5. *Assume that (A_1)–(A_3) hold; then, BVP*(1.1) *has a positive solution x^* and $\underline{x} \leq x^* \leq \overline{x}$.*

Proof. Let $I = [\underline{x}, \overline{x}] = \{x \in W^{1,p}[0,1] \mid \underline{x} \leq x \leq \overline{x}\}$. Make a truncation function of $f(t,x)$ as

$$\overline{f}(t,x) = \begin{cases} f(t,\overline{x}), & x > \overline{x}, \\ f(t,x), & \underline{x} \leq x \leq \overline{x}, \\ f(t,\underline{x}), & x < \underline{x}, \end{cases} \tag{3.24}$$

and assume that $\overline{F}(t,x) = \int_0^x \overline{f}(t,s)ds$. Consider the following BVP:

$$\begin{aligned}-\left(\phi_p\left(x'(t)\right)\right)' = -a(t)\phi_p(x) + \overline{f}(t,x), \quad t \in [0,1], \\ \alpha_1 x(0) - \alpha_2 x'(0) = 0, \qquad \beta_1 x(1) + \beta_2 x'(1) = 0.\end{aligned} \tag{3.25}$$

The corresponding Euler functional of BVP(3.25) is

$$\overline{\varphi}(x) = \frac{1}{p}\int_0^1 a(t)|x|^p dt + \frac{1}{p}\int_0^1 \left|x'\right|^p dt - \int_0^1 \overline{F}(t,x)dt + \frac{\alpha_2}{p\alpha_1}\left|\frac{\alpha_1 x(0)}{\alpha_2}\right|^p + \frac{\beta_2}{p\beta_1}\left|\frac{\beta_1 x(1)}{\beta_2}\right|^p. \tag{3.26}$$

It is obvious that $\overline{\varphi}(x)$ is weakly lower semicontinuous. Since $\overline{x}$ and $\underline{x}$ are continuous on $[0,1]$, $\overline{F}(t,x)$ is continuous, $\overline{\varphi}(x)$ is coercive. Hence, $\overline{\varphi}(x)$ can attain its infimum in $W^{1,p}[0,1]$. Without loss of generality, we may assume that $\overline{\varphi}(x)$ attains its infimum in x^*. In the following, we show that x^* is a solution of BVP(1.1).

Assume that $x^* - \underline{x}$ has a negative minimum, and let $t_0 = \sup\{t \in [0,1] \mid (x^* - \underline{x})(t) = \min_{s\in[0,1]}((x^* - \underline{x})(s))\}$.

If $t_0 = 0$, then,

$$0 \le x^*(0)' - \underline{x}(0)' \le \frac{\alpha_1}{\alpha_2}\left(x^*(0) - \underline{x}(0)\right) < 0, \tag{3.27}$$

which reaches a contradiction. Similarly, $t_0 \neq 1$.

If $t_0 \in (0,1)$, there exist an open interval I_0 and $t_1 \in I_0$ with $t_1 < t_0$, $x^*(t) < \underline{x}(t)$, $t \in I_0$, $x^*(t_1)' < \underline{x}(t_1)'$. Hence,

$$\begin{aligned}
0 > \phi_p\left((x^*)'(t_1)\right) - \phi_p\left(\underline{x}'(t_1)\right) &= \int_{t_0}^{t_1}\left[\left(\phi_p\left((x^*)'(s)\right)\right)' - \left(\phi_p\left(\underline{x}'(s)\right)\right)'\right]ds \\
&\ge \int_{t_1}^{t_0}\left[-a(s)\phi_p(x^*(s)) + \overline{f}(s,x^*(s)) + a(s)\phi_p\left(\underline{x}(s)\right) - f\left(s,\underline{x}(s)\right)\right]ds \\
&= \int_{t_1}^{t_0} a(s)\left[\phi_p\left(\underline{x}(s)\right) - \phi_p(x^*(s))\right]ds > 0.
\end{aligned} \tag{3.28}$$

From the discussion above, one has $x^*(t) \ge \underline{x}(t)$. Similarly, $x^*(t) \le \overline{x}(t)$. Then, $x^* \in I$. Since $\underline{x}(t)$ and $\overline{x}(t)$ are the strictly lower and upper solutions of BVP(1.1), respectively, $x^*(t)\overline{x}(t)$, $x^*(t)\underline{x}(t)$. Therefore, we obtain a positive solution of BVP (1.1). □

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