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Research Article

Time-Periodic Solution of the Weakly Dissipative Camassa-Holm Equation

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This paper is concerned with time-periodic solution of the weakly dissipative Camassa-Holm equation with a periodic boundary condition. The existence and uniqueness of a time periodic solution is presented.

1. Introduction

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

modeling the unidirectional propagation of shallow water waves over a flat bottom, where $u(t, x)$ represents the fluid's free surface above a flat bottom (or equivalently, the fluid velocity at time $t \geq 0$ and in the spatial x direction).

Since the equation was derived physically by Camassa and Holm [1, 2], many researchers have paid extensive attention to it. The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [3, 4]. It has a bi-Hamiltonian structure [5, 6] and is completely integrable [1, 2, 7–11]. It is a reexpression of geodesic flow on the diffeomorphism group of the circle [12] and on the Virasoro group [13]. Its solitary waves are peaked [7], and they are orbitally stable and interact like solitons [14–16]. The peakons capture a characteristic of the traveling waves of greatest height-exact traveling solutions of the governing equations for water waves with a peak at their crest [17–19].

The Cauchy problem of the Camassa-Holm equation has been extensively studied. It has been shown that this equation is locally well posed [20–25] for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$. Moreover, it has global strong solutions modeling permanent waves [20, 24–27] but also blow-up solutions modeling wave breaking [20–28]. On the other hand, it has global weak solutions with initial data $u_0 \in H^1$ [29–35]. Moreover, the initial-boundary value problem for the Camassa-Holm equation on the half-line and on a finite interval was discussed in [36, 37]. It is observed that if u is the solution of the Camassa-Holm equation with the initial data u_0 in $H^1(\mathbb{R})$, we have for all $t > 0$,

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{\sqrt{2}}{2}\|u(t,\cdot)\|_{H^1} \le \frac{\sqrt{2}}{2}\|u_0(\cdot)\|_{H^1}. \tag{1.2}$$

It is worth pointing out that the advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [2, 20, 21].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. Ott and Sudan [38] investigated how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of the KdV equation. Ghidaglia [39] investigated the long-time behavior of solutions to the weakly dissipative KdV equation as a finite-dimensional dynamical system.

The Camassa-Holm equation with dissipative term is

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + L(u) = f(t,x), \quad t > 0,\ x \in \mathbb{R}, \tag{1.3}$$

where $f(t,x)$ is the forcing term, $L(u)$ is a dissipative term, L can be a differential operator or a quasi-differential operator according to different physical situations.

With $f = 0$ and $L(u) = \gamma(1 - \partial_x^2)u$, (1.3) becomes weakly dissipative Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x + \gamma(u - u_{xx}) = 2u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.4}$$

where $\gamma > 0$ is a constant.

The local well-posedness, global existence, and blow-up phenomena of the Cauchy problem of (1.4) on the line [40] and on the circle [41] were studied. A new global existence result and a new blow-up result for strong solutions to this equation with certain profiles are presented recently [42]. We found that the behaviors of (1.4) are similar to the Camassa-Holm equation in a finite interval of time, such as the local well-posedness and the blow-up phenomena, and that there are considerable differences between (1.4) and the Camassa-Holm equation in their long-time behaviors. The global solutions of (1.4) decay to zero as time goes to infinity provided the potential $y_0 = (1 - \partial_x^2)u_0$ is of one sign (see [40, 41]). This long-time behavior is an important feature that the Camassa-Holm equation does not possess. It is well known that the Camassa-Holm equation has peaked traveling wave solutions. But the fact that any global solution of (1.4) decays to zero means that there are no traveling wave solutions of (1.4).

Another difference between (1.4) and the Camassa-Holm equation is that (1.4) does not have the following conservation laws

$$I_1 = \int_S u\,dx, \qquad I_2 = \int_S \left(u^2 + u_x^2\right)dx, \tag{1.5}$$

which play an important role in the study of the Camassa-Holm equation.

Equation (1.4) has the same blow-up rate as the Camassa-Holm equation does when the blow-up occurs [41]. This fact shows that the blow-up rate of the Camassa-Holm equation is not affected by the weakly dissipative term, but the occurrence of blow-up of (1.4) is affected by the dissipative parameter [40, 41].

In the paper, we would like to consider the following weakly dissipative Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \gamma(u - u_{xx}) = f(t,x), \quad t > 0,\ x \in \mathbb{R}, \tag{1.6}$$

$$u(t, x+L) = u(t,x), \quad t > 0,\ x \in \mathbb{R}, \tag{1.7}$$

$$u(t+\omega, x) = u(t,x), \quad t > 0,\ x \in \mathbb{R}, \tag{1.8}$$

where $\gamma(1 - \partial_x^2)u$ is the weakly dissipative term, $\gamma > 0$ is a constant, and the forcing term f is ω-periodic in time t and L-periodic in spatial x. Without loss of generality, we assume further $\int_\Omega f(t,x)dx = 0$, where $\Omega = [0, L]$. When system is periodically dependent on time t, we want to know whether there exists time-periodic solution with the same period for the system. In many nonlinear evolution equations, the study of time-periodic solution has attracted considerable interest (e.g., [43–45]). In this paper, we will prove that (1.6)–(1.8) have a solution by using the Galerkin method [46], and Leray-Schauder fixed point theorem [44].

Our paper is organized as follows. In Section 2, we give some notations and definition of some space used in this paper. In Section 3, we prove the existence of the approximate solution and give uniform a priori estimates needed where we prove the convergence of a sequence of the approximate solution. Section 4 is devoted to the study of the existence and uniqueness of time-periodic solution for (1.6)–(1.8).

2. Preliminaries

Before starting our work, it is appropriate to introduce some notations and inequalities that will be used in the paper.

Let X be a Banach space, we denote by $C^k(\omega; X)$ the set of ω-periodic X-valued measurable functions on $\mathbb{R}^1$ with continuous derivatives up to order k. The norm in the space $C^k(\omega; X)$ is $\|u\|_{C^k(\omega;X)} = \sup_{0\le t\le\omega}\{\sum_{i=0}^k \|D_t^i u\|_X\}$.

For $1 \le p \le \infty$, the space $L^p(\omega; X)$ is the set of ω-periodic X-valued measurable functions on $\mathbb{R}$ such that

$$\|u\|_{L^p(\omega;X)} = \begin{cases} \left(\displaystyle\int_0^\omega \|u\|_X^p dt\right)^{1/p} < \infty, & 1 \le p < \infty \\ \displaystyle\sup_{0\le t\le\omega} \|u\|_X < \infty, & p = \infty. \end{cases} \tag{2.1}$$

The space $W^{k,p}(\omega;X)$ denote the set of functions which belong to $L^p(\omega;X)$ together with their derivatives up to order k, and we write $W^{k,2}(\omega;X) = H^k(\omega;X)$ in particular when X is a Hilbert space.

$L^p(\Omega)$ and $H^m(\Omega)$ are classical Sobolev spaces. For simplicity, we write $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$ as $p \neq 2$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$.

The following inequalities (see [47]) will be used in the proofs later

$$\|u\|_\infty \leq k_1\|u\|_{H^1}. \tag{2.2}$$

$$\left\|D^j u\right\|_p \leq k_2\|u\|_{H^m}^{\theta}\|u\|^{1-\theta}, \tag{2.3}$$

where $D^j u = (\partial^j u)/(\partial x^j)$, $1/p = j + \theta(1/2 - m) + (1-\theta)(1/2)$ as $0 \leq j < m$, $j/m \leq \theta \leq 1$.

$$\|u\| \leq k_3\|u_x\|, \int_\Omega u(x)dx = 0. \tag{2.4}$$

3. A Priori Estimates

In this section, we first prove that (1.6)–(1.8) have a sequence of approximate solutions $\{u_n\}_{n=1}^\infty$, then give a prior, estimates about $\{u_n\}_{n=1}^\infty$.

We denote the unbounded linear operator $Au = -u_{xx}$ on $X = L^2 \cap \{u \mid u(x+L) = u(x), \int_\Omega u\,dx = 0\}$, then the set of all linearly independent eigenvectors $\{w_j\}_{j=0}^\infty$ of A, that is, $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$, is an orthonormal basis of $L^2(\Omega)$. For any n and a group of function $\{a_{jn}(t)\}_{j=1}^n$, where $a_{jn}(t)(j = 1,2,\ldots,n) \in C^1(\omega;\mathbb{R})$, the function $u_n = \sum_{j=1}^n a_{jn}(t)w_j \in C^1(\omega;H_n)$ is called an approximate solution to (1.6)–(1.8) if it satisfies the equation as follows:

$$\left(u_{nt} - u_{nxxt} + \gamma(u_n - u_{nxx}), w_j\right) = \left(Nu_n + f, w_j\right), \quad j = 1,\ldots,n, \tag{3.1}$$

where $Nu_n = -3u_nu_{nx} + 2u_{nx}u_{nxx} + u_nu_{nxxx}$ and $H_n = \operatorname{span}\{w_1, w_2, \ldots, w_n\}$. By the classical theory of ordinary differential equations, for any fixed $v_n(t) = \sum_{j=1}^n b_{jn}(t)w_j \in C^1(\omega;H_n)$, the equation $(u_{nt} - u_{nxxt} + \gamma(u_n - u_{nxx}), w_j) = (Nv_n + f, w_j)$, $j = 1,\ldots,n$ has a unique ω-periodic solution u_n and the mapping $F : v_n \to u_n$ is continuous and compact in $C^1(\omega;H_n)$. Hence by Leray-Schauder fixed point theorem, we want to prove the existence of an approximate solution only to show $\sup_{0\leq t\leq\omega}\|u_n\|^2 \leq c$ for all possible solution of (3.1) replaced by $\lambda Nu_n (0 \leq \lambda \leq 1)$ instead of nonlinear term Nu_n, where c is a constant which only depends on L, ε, ω, γ, and f.

Lemma 3.1. *If $f \in C^1(\omega;H^{-1}(\Omega))$, then*

$$\sup_{0\leq t\leq\omega}\left(\|u_n\|^2 + \|u_{nx}\|^2\right) \leq c_1, \tag{3.2}$$

where c_1 is a constant which only depends on L, ω, ε, γ, k_3, and f, $M = \sup_{0\leq t\leq\omega}\{\|f(t,x)\|_{H^{-1}(\Omega)}^2\}$ and $d_1 = \min\{2\gamma, 2\gamma - \varepsilon\} > 0$.

Proof. Multiplying (3.1) by $a_{jn}(t)$ and summing up over j from 1 to n, we obtain

$$(u_{nt} - u_{nxxt} + \gamma(u_n - u_{nxx}), u_n) = (Nu_n + f, u_n). \tag{3.3}$$

Then, we can get

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2 + \|u_{nx}\|^2\right) + \gamma\left(\|u_n\|^2 + \|u_{nx}\|^2\right) = (Nu_n + f, u_n). \tag{3.4}$$

Notice that $-3\int_\Omega u_n^2 u_{nx} dx = 0$, $2\int_\Omega u_n u_{nx} u_{nxx} dx + \int_\Omega u_n^2 u_{nxxx} dx = 0$.

From Young's inequality, we have $\int_\Omega f u_n dx \leq (\varepsilon/2)\|u_{nx}\|^2 + k_3^2\ M/2\varepsilon$, where $\varepsilon > 0$ is a constant.

According to the above relations, we can derive from (3.4) that

$$\frac{d}{dt}\left(\|u_n\|^2 + \|u_{nx}\|^2\right) + d_1\left(\|u_n\|^2 + \|u_{nx}\|^2\right) \leq \frac{k_3^2 M}{\varepsilon}, \tag{3.5}$$

where $d_1 = \min\{2\gamma, 2\gamma - \varepsilon\} > 0$.

Considering the time periodicity of u_n and integrating (3.5) over $[0, \omega]$, we get

$$d_1 \int_0^\omega \left(\|u_n\|^2 + \|u_{nx}\|^2\right) dt \leq \frac{\omega k_3^2 M}{\varepsilon}. \tag{3.6}$$

Hence, there exists $t^* \in [0, \omega)$ such that $\|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 \leq k_3^2 M / d_1 \varepsilon$.

From (3.5), we have $(d/dt)(\|u_n\|^2 + \|u_{nx}\|^2) \leq k_3^2 M/\varepsilon$.

Integrating the above inequality with respect to t from t^* to $t \in [t^*,\ t^* + \omega]$, we deduce that

$$\|u_n(t)\|^2 + \|u_{nx}(t)\|^2 \leq \|u_n(t^*)\|^2 + \|u_{nx}(t^*)\|^2 + \frac{\omega k_3^2 M}{\varepsilon} \leq \left(\frac{1}{d_1} + \omega\right)\frac{k_3^2 M}{\varepsilon}. \tag{3.7}$$

Hence, we infer

$$\sup_{0 \leq t \leq \omega}\left(\|u_n\|^2 + \|u_{nx}\|^2\right) \leq \left(\frac{1}{d_1} + \omega\right)\frac{k_3^2 M}{\varepsilon} \triangleq c_1, \tag{3.8}$$

which concludes our proof. □

From Lemma 3.1 and Leray-Schauder fixed point theorem, (3.1) has solution $\{u_n\}_{n=1}^\infty$, which is also a sequence of approximate solutions of (1.6)–(1.8). In order to obtain the convergence of sequence $\{u_n\}_{n=1}^\infty$, we need to give a priori estimates for the high-order derivers of $\{u_n\}_{n=1}^\infty$.

Lemma 3.2. *If $f \in C^1(\omega; H^{-1}(\Omega))$, then*

$$\sup_{0\le t\le\omega}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) \le c_2, \tag{3.9}$$

where c_2 is a constant which only depends on L, ω, ε, γ, λ_n, k_1, k_2, k_3, and f, $M = \sup_{0\le t\le\omega}\{\|f(t,x)\|^2_{H^{-1}(\Omega)}\}$ and $d_2 = \min\{2\gamma - (13/2)\varepsilon\lambda_n, 2\gamma - (21/2)\varepsilon\} > 0$.

Proof. Multiplying (3.1) by $-\lambda_j a_{jn}(t)$ and summing up over j from 1 to n, we have

$$\left(u_{nt} - u_{nxxt} + \gamma(u_n - u_{nxx}), u_{nxx}\right) = \left(Nu_n + f, u_{nxx}\right). \tag{3.10}$$

The above equation yields

$$-\frac{1}{2}\frac{d}{dt}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) - \gamma\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) = \left(Nu_n + f, u_{nxx}\right). \tag{3.11}$$

From Young's inequality, we have

$$\left|\int_\Omega f u_{nxx} dx\right| \le \varepsilon\|u_{nxxx}\|^2 + \frac{k_3^2 M}{4\varepsilon}, \tag{3.12}$$

where $\varepsilon > 0$ is a constant.

From (2.2), (3.8), and Young's inequality, we can deduce that

$$\begin{aligned}
\left|\int_\Omega u_n u_{nx} u_{nxx} dx\right| &\le \|u_n\|_\infty \int_\Omega |u_{nx} u_{nxx}| dx \\
&\le k_1\|u_n\|_{H^1}\int_\Omega |u_{nx} u_{nxx}| dx \\
&\le k_1 c_1^{1/2}\left(\frac{\varepsilon}{k_1 c_1^{1/2}}\|u_{nxx}\|^2 + \frac{k_1 c_1^{1/2}}{4\varepsilon}\|u_{nx}\|^2\right) \\
&\le \varepsilon\|u_{nxx}\|^2 + \frac{k_1^2 c_1^2}{4\varepsilon}.
\end{aligned} \tag{3.13}$$

From (2.3), (3.8), Cauchy-Schwarz inequality, Young's inequality, and Lemma 3.1, we get

$$\begin{aligned}\left|\int_{\Omega} u_n u_{nxx} u_{nxxx} dx\right| &= \left|-\frac{1}{2}\int_{\Omega} u_{nx} u_{nxx}^2 dx\right| \le \frac{1}{2}\|u_{nx}\|\|u_{nxx}\|_4^2 \\ &\le \frac{1}{2} c_1^{1/2} k_2^2 \|u_n\|^{1/2} \|u_n\|_{H^3}^{3/2} \\ &\le \frac{3}{4}\varepsilon \|u_n\|_{H^3}^2 + \frac{c_1^3 k_2^8}{64\varepsilon^3} \\ &\le \frac{3}{4}\varepsilon\|u_{nxx}\|^2 + \frac{3}{4}\varepsilon\|u_{nxxx}\|^2 + \frac{3}{4}\varepsilon c_1 + \frac{c_1^3 k_2^8}{64\varepsilon^3},\end{aligned} \tag{3.14}$$

$$\|u_{nxxx}\|^2 = \int_{\Omega}\left|\Big(\sum_{j=1}^{n} a_{jn}(t) w_j\Big)_{xxx}\right|^2 dx = \int_{\Omega}\left|\Big(\sum_{j=1}^{n} \lambda_j a_{jn}(t) w_j\Big)_{x}\right|^2 dx \le \lambda_n \|u_{nx}\|^2. \tag{3.15}$$

Taking (3.11)–(3.15) into account, we can infer that

$$\frac{d}{dt}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) + d_2\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) \le \frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3}, \tag{3.16}$$

where $d_2 = \min\{2\gamma - (13/2)\varepsilon\lambda_n, 2\gamma - (21/2)\varepsilon\} > 0$.

Integrating (3.16) about t from 0 to ω and considering the time periodicity of u_n, we get

$$d_2 \int_0^{\omega}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) dt \le \left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3}\right)\omega. \tag{3.17}$$

Hence, there exists $t^* \in [0,\omega)$ such that

$$\|u_{nx}(t^*)\|^2 + \|u_{nxx}(t^*)\|^2 \le \frac{1}{d_2}\left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3}\right). \tag{3.18}$$

From (3.16), we have

$$\|u_{nx}(t)\|^2 + \|u_{nxx}(t)\|^2 \le \|u_{nx}(t^*)\|^2 + \|u_{nxx}(t^*)\|^2 + \left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3}\right)\omega. \tag{3.19}$$

Then we can obtain

$$\sup_{0\le t\le\omega}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) \le \left(\frac{1}{d_2} + \omega\right)\left(\frac{k_3^2 M}{2\varepsilon} + \frac{3k_1^2 c_1^2}{2\varepsilon} + \frac{9\varepsilon c_1}{2} + \frac{3c_1^3 k_2^8}{32\varepsilon^3}\right) \triangleq c_2, \tag{3.20}$$

which concludes our proof. □

In the following, we continue to establish a priori estimates for high-order derivers of the approximate solution $\{u_n\}_{n=1}^{\infty}$ by an inductive argument.

Lemma 3.3. *For any $k \geq 0$, if $f \in C^1(\omega; H^{k-1}(\Omega))$, then*

$$\sup_{0\leq t\leq\omega}\left(\left\|D^{k+1}u_n\right\|^2 + \left\|D^{k+2}u_n\right\|^2\right) \leq c, \tag{3.21}$$

where c is a constant which only depends on L, ω, ε, γ, λ_n, k, k_1, k_2, k_3, f and $d_3 = \{2\gamma - 16\varepsilon\lambda_n, 2\gamma - 14\varepsilon\} > 0$.

Proof. By Lemma 3.2, we know the conclusion of Lemma 3.3 holds for $k = 0$.

Assume that for $k \leq m-1 (m \geq 2)$ the conclusion of Lemma 3.3 holds, we want to prove that the same statement holds for $k = m$ also.

Multiplying (3.1) by $(-1)^{m+1}\lambda_j^{m+1}a_{jn}(t)$ and summing up over j from 1 to n, we have

$$\begin{aligned}(-1)^{m+1}\frac{1}{2}\frac{d}{dt}\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right) + (-1)^{m+1}\gamma\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right)& \\ = \left(Nu_n + f, D^{2(m+1)}u_n\right).&\end{aligned} \tag{3.22}$$

Follow the same methods discussed in Lemma 3.2, we have

$$\left|\int_\Omega fD^{2(m+1)}u_n dx\right| = \left|\int_\Omega D^{m-1}fD^{m+3}u_n dx\right| \leq \varepsilon\left\|D^{m+3}u_n\right\|^2 + \frac{1}{4\varepsilon}\left\|D^{m-1}f\right\|^2. \tag{3.23}$$

From the conclusion of Lemma 3.3 for $k \leq m-1$, (2.2), (2.4) and Young's inequality, we can deduce that

$$\begin{aligned}\left|\int_\Omega u_n u_{nx} D^{2(m+1)}u_n dx\right| &= \left|\int_\Omega\left(\sum_{i=0}^{m+1}C_{m+1}^i D^i u_n D^{m+1-i}u_{nx}\right)D^{m+1}u_n dx\right| \\ &\leq \int_\Omega\left|u_n D^{m+2}u_n D^{m+1}u_n\right|dx \\ &\quad + \int_\Omega\left|\left(\sum_{i=1}^{m+1}C_{m+1}^i D^i u_n D^{m+1-i}u_{nx}\right)D^{m+1}u_n\right|dx \\ &\leq \varepsilon\left\|D^{m+2}u_n\right\|^2 + c(\varepsilon,k_1,k_3)\left\|D^{m+1}u_n\right\|^2 + c(m,k_1) \\ &\leq \varepsilon\left\|D^{m+2}u_n\right\|^2 + c(\varepsilon,k_1,k_3,m).\end{aligned} \tag{3.24}$$

Similarly, we can also deduce that

$$
\begin{aligned}
\left|\int_{\Omega} u_n u_{nxxx} D^{2(m+1)} u_n dx\right| &= \left|\int_{\Omega}\left(\sum_{i=0}^{m} C_m^i D^i u_n D^{m-i} u_{nxxx}\right) D^{m+2} u_n dx\right| \\
&\leq \int_{\Omega}\left|u_n D^{m+3} u_n D^{m+2} u_n\right| dx + \int_{\Omega}\left|C_m^1 D u_n \left(D^{m+2} u_n\right)^2\right| dx \\
&\quad + \int_{\Omega}\left|C_m^2 D^2 u_n D^{m+1} u_n D^{m+2} u_n\right| dx \\
&\quad + \int_{\Omega}\left|\left(\sum_{i=3}^{m} C_m^i D^i u_n D^{m+3-i} u_n\right) D^{m+2} u_n\right| dx \\
&\leq \|u_n\|_{\infty} \int_{\Omega}\left|D^{m+3} u_n D^{m+2} u_n\right| dx + m\|D u_n\|_{\infty} \int_{\Omega}\left|D^{m+2} u_n\right|^2 dx \\
&\quad + C_m^2 \left\|D^2 u_n\right\|_{\infty} \int_{\Omega}\left|D^{m+1} u_n D^{m+2} u_n\right| dx \\
&\quad + \sum_{i=3}^{m} C_m^i \left\|D^i u_n\right\|_{\infty} \left\|D^{m+3-i} u_n\right\|_{\infty} \int_{\Omega}\left|D^{m+2} u_n\right| dx \\
&\leq c(k_1, k_3, m)\left(\int_{\Omega}\left|D^{m+3} u_n D^{m+2} u_n\right| dx + \int_{\Omega}\left|D^{m+2} u_n\right|^2 dx \right. \\
&\quad \left. + \int_{\Omega}\left|D^{m+1} u_n D^{m+2} u_n\right| dx + \int_{\Omega}\left|D^{m+2} u_n\right| dx\right).
\end{aligned}
\tag{3.25}
$$

From the conclusion of Lemma 3.3 for $k \leq m-1$, Young's inequality and (2.3), we have

$$
\begin{aligned}
c(k_1, k_3, m)\int_{\Omega}\left|D^{m+3} u_n D^{m+2} u_n\right| dx &\leq \varepsilon\left\|D^{m+3} u_n\right\|^2 + c(k_1, k_3, m, \varepsilon)\left\|D^{m+2} u_n\right\|^2 \\
&\leq \varepsilon\left\|D^{m+3} u_n\right\|^2 + c(k_1, k_2, k_3, m, \varepsilon) \\
&\quad \times \|u_n\|_{H^{m+3}}^{2(m+2)/m+3} \|u_n\|^{2(1-(m+2)/(m+3))} \\
&\leq \varepsilon\left\|D^{m+3} u_n\right\|^2 + \varepsilon\|u_n\|_{H^{m+3}}^2 + c(k_1, k_2, k_3, m, \varepsilon)\|u_n\|^2 \\
&\leq 2\varepsilon\left\|D^{m+3} u_n\right\|^2 + \varepsilon\left\|D^{m+2} u_n\right\|^2 + c(k_1, k_2, k_3, m, \varepsilon), \\
c(k_1, k_3, m)\int_{\Omega}\left|\left(D^{m+2} u_n\right)^2\right| dx &= c(k_1, k_3, m)\left\|D^{m+2} u_n\right\|^2 \\
&\leq c(k_1, k_2, k_3, m)\|u_n\|_{H^{m+3}}^{2(m+2)/(m+3)} \|u_n\|^{2(1-(m+2)/(m+3))} \\
&\leq \varepsilon\left\|D^{m+3} u_n\right\|^2 + \varepsilon\left\|D^{m+2} u_n\right\|^2 + c(k_1, k_2, k_3, m, \varepsilon),
\end{aligned}
$$

$$\begin{aligned}
c(k_1,k_3,m)\int_\Omega \left|D^{m+1}u_n D^{m+2}u_n\right|dx &\le \varepsilon\left\|D^{m+2}u_n\right\|^2 + c(k_1,k_3,m,\varepsilon)\left\|D^{m+1}u_n\right\|^2 \\
&\le \varepsilon\left\|D^{m+2}u_n\right\|^2 + c(k_1,k_3,m,\varepsilon), \\
c(k_1,k_3,m)\int_\Omega \left|D^{m+2}u_n\right|dx &\le \varepsilon\left\|D^{m+2}u_n\right\|^2 + c(k_1,k_3,m,\varepsilon,L).
\end{aligned} \tag{3.26}$$

Combining (3.25) and the above inequality, we can get

$$\left|\int_\Omega u_n u_{nxxx} D^{2(m+1)}u_n dx\right| \le 3\varepsilon\left\|D^{m+3}u_n\right\|^2 + 4\varepsilon\left\|D^{m+2}u_n\right\|^2 + c(k_1,k_2,k_3,m,\varepsilon,L). \tag{3.27}$$

Similarly,

$$\begin{aligned}
\left|\int_\Omega u_{nx}u_{nxx}D^{2(m+1)}u_n dx\right| \le& \int_\Omega \left|u_{nx}D^{m+1}u_n D^{m+3}u_n\right|dx \\
&+ \int_\Omega \left|\left(\sum_{i=1}^{m-1} C_{m-1}^i D^{i+1}u_n D^{m+1-i}u_n\right)D^{m+3}u_n\right|dx \\
\le& \|u_{nx}\|_\infty \int_\Omega \left|D^{m+1}u_n D^{m+3}u_n\right|dx \\
&+ \sum_{i=1}^{m-1} C_{m-1}^i \left\|D^{i+1}u_n\right\|_\infty \left\|D^{m+1-i}u_n\right\|_\infty \int_\Omega \left|D^{m+3}u_n\right|dx \\
\le& 2\varepsilon\left\|D^{m+3}u_n\right\|^2 + c(m,k_1,\varepsilon,L).
\end{aligned} \tag{3.28}$$

Taking (3.22)–(3.24) and (3.27)-(3.28) into account, we can deduce that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right) + 2\gamma\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right) \\
&\quad\le 16\varepsilon\left\|D^{m+3}u_n\right\|^2 + 14\varepsilon\left\|D^{m+2}u_n\right\|^2 + c(k_1,k_2,k_3,m,\varepsilon,f,L).
\end{aligned} \tag{3.29}$$

From the above relation, we can infer

$$\frac{d}{dt}\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right) + d_3\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right) \le c(k_1,k_2,k_3,m,\varepsilon,f,L), \tag{3.30}$$

where $d_3 = \{2\gamma - 16\varepsilon\lambda_n, 2\gamma - 14\varepsilon\} > 0$.

Integrating (3.30) about t from 0 to ω, there exists $t^* \in [0, \omega)$ such that

$$\left\|D^{m+1}u_n(t^*)\right\|^2 + \left\|D^{m+2}u_n(t^*)\right\|^2 \leq \frac{c(k_1, k_2, k_3, m, \varepsilon, f, L)}{d_3}. \tag{3.31}$$

From (3.30), we have

$$\frac{d}{dt}\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right) \leq c(k_1, k_2, k_3, m, \varepsilon, f, L). \tag{3.32}$$

Integrating the above inequality from t^* to $t \in [t^*,\ t^*+\omega]$ and with (3.31), we can easily obtain

$$\sup_{0\leq t\leq\omega}\left(\left\|D^{m+1}u_n\right\|^2 + \left\|D^{m+2}u_n\right\|^2\right) \leq \left(\frac{1}{d_3} + \omega\right)c(k_1, k_2, k_3, m, \varepsilon, f, L) \triangleq c. \tag{3.33}$$

The proof is completed. □

Lemma 3.4. *For any $k \geq 0$, if $f \in C^1(\omega; H^{k+1}(\Omega))$, then*

$$\sup_{0\leq t\leq\omega}\left(\left\|D^k u_{nt}\right\|^2 + \left\|D^{k+1}u_{nt}\right\|^2\right) \leq c \tag{3.34}$$

where c is a constant which only depends on L, ω, ε, γ, λ_n, k, k_1, k_2, k_3, and f.

Proof. We first prove the conclusion of Lemma 3.4 holds for $k = 0$. Multiplying (3.1) by $a'_{jn}(t)$ and summing up over j from 1 to n, we have

$$\|u_{nt}\|^2 + \|u_{nxt}\|^2 = (Nu_n + f - \gamma(u_n - u_{nxx}), u_{nt}). \tag{3.35}$$

By Lemma 3.3, if $f \in C^1(\omega; H^1(\Omega))$, then we have $\|u_n\|^2_{H^4} \leq c$. Hence,

$$\left|(Nu_n + f - \gamma(u_n - u_{nxx}), u_{nt})\right| \leq \left\|Nu_n + f - \gamma(u_n - u_{nxx})\right\|\|u_{nt}\| \leq c\|u_{nt}\|. \tag{3.36}$$

Therefore, from (3.35) and (3.36), it is easy to know that

$$\sup_{0\leq t\leq\omega}\left(\|u_{nt}\|^2 + \|u_{nxt}\|^2\right) \leq c. \tag{3.37}$$

Assume that the conclusion of Lemma 3.4 holds for $k \leq m (m \geq 1)$, we want to prove that the conclusion of Lemma 3.4 also holds for $k = m + 1$.

Multiplying (3.1) by $(-1)^{m+1}\lambda_j^{m+1}a'_{jn}(t)$ and summing up over j from 1 to n, we have

$$(-1)^{m+1}\left(\left\|D^{m+1}u_{nt}\right\|^2 + \left\|D^{m+2}u_{nt}\right\|^2\right) = \left(Nu_n + f - \gamma(u_n - u_{nxx}), D^{2(m+1)}u_{nt}\right). \tag{3.38}$$

By Lemma 3.3, if $f \in C^1(\omega; H^{m+2}(\Omega))$, then $\|D^k u_n\|^2 \le c$ for $k \le m+5$. Hence,

$$\begin{aligned}\left|\left(Nu_n + f - \gamma(u_n - u_{nxx}), D^{2(m+1)}u_{nt}\right)\right| &\le \left\|D^{m+1}\left[Nu_n + f - \gamma(u_n - u_{nxx})\right]\right\|\left\|D^{m+1}u_{nt}\right\| \\ &\le c\left\|D^{m+1}u_{nt}\right\|.\end{aligned} \tag{3.39}$$

Taking (3.38) and (3.39) into account, it follows

$$\sup_{0\le t\le\omega}\left(\left\|D^{m+1}u_{nt}\right\|^2 + \left\|D^{m+2}u_{nt}\right\|^2\right) \le c. \tag{3.40}$$

This completes the proof of Lemma 3.4 by an inductive argument. □

4. Existence and Uniqueness of Time-Periodic Solution

We have proved that (1.6)–(1.8) have a sequence of approximate solutions $\{u_n\}_{n=1}^{\infty}$. In this section, we want to prove that the sequence converges and the limit is a solution of (1.6)–(1.8).

By Lemmas 3.1–3.4 and standard compactness arguments, we conclude that there is a subsequence which we denote also by $\{u_n\}$ such that for any $K \ge 0$, if $f \in C^1(\omega; H^{k+1}(\Omega))$, we have

$$\begin{gathered} u_n(t) \longrightarrow u(t), \text{ weakly } * \text{ in } L^\infty\left(\omega; H^{k+4}(\Omega)\right), \\ u_n(t) \longrightarrow u(t), \text{ strongly in } L^\infty\left(\omega; H^{k+3}(\Omega)\right), \\ u_{nt}(t) \longrightarrow u_t(t), \text{ weakly } * \text{ in } L^\infty\left(\omega; H^{k+1}(\Omega)\right), \\ u_{nt}(t) \longrightarrow u_t(t), \text{ strongly in } L^\infty\left(\omega; H^{k}(\Omega)\right). \end{gathered} \tag{4.1}$$

From the above lemmas, we know that the nonlinear terms are well defined

$$\begin{aligned}\|u_n u_{nx} - u u_x\| &\le \|u_n(u_{nx} - u_x)\| + \|u_x(u_n - u)\| \\ &\le \|u_n\|_\infty \|u_{nx} - u_x\| + \|u_x\|_\infty \|u_n - u\| \longrightarrow 0,\end{aligned} \tag{4.2}$$

as $n \to \infty$, uniformly in t,

$$\begin{aligned}\|u_{nx} u_{nxx} - u_x u_{xx}\| &\le \|u_{nx}(u_{nxx} - u_{xx})\| + \|u_{xx}(u_{nx} - u_x)\| \\ &\le \|u_{nx}\|_\infty \|u_{nxx} - u_{xx}\| + \|u_{xx}\|_\infty \|u_{nx} - u_x\| \longrightarrow 0,\end{aligned} \tag{4.3}$$

as $n \longrightarrow \infty$, uniformly in t,

$$\begin{aligned}\|u_n u_{nxxx} - u u_{xxx}\| &\le \|u_n(u_{nxxx} - u_{xxx})\| + \|u_{xxx}(u_n - u)\| \\ &\le \|u_n\|_\infty \|u_{nxxx} - u_{xxx}\| + \|u_n - u\|_\infty \|u_{xxx}\| \\ &\le \|u_n\|_\infty \|u_{nxxx} - u_{xxx}\| + k_1 \|u_n - u\|_{H^1} \|u_{xxx}\| \longrightarrow 0,\end{aligned} \tag{4.4}$$

as $n \longrightarrow \infty$, uniformly in t.

Consequently, it follows that

$$\left(u_t - u_{xxt} + \gamma(u - u_{xx}), \eta\right) = \left(Nu + f, \eta\right), \eta \in L^2_{\text{per}}. \tag{4.5}$$

Thanks to the estimates obtained in the previous section, we have

$$u_t - u_{xxt} + \gamma(u - u_{xx}) = Nu + f, \tag{4.6}$$

a.e. on $\mathbb{R}^1 \times \Omega$.

So we obtain that the existence of time periodic solution for (1.6)–(1.8), which is the following theorem.

Theorem 4.1. *Given $f \in C^1(\omega; H^{k+1}(\Omega))$, $k \ge 0$, there exists a time periodic solution $u(t,x)$ to* (1.6)–(1.8)*, such that $u(t,x) \in L^\infty(\omega; H^{k+4}(\Omega)) \cap W^{1,\infty}(\omega; H^k(\Omega))$.*

Under the assumption of Theorem 4.1, we are unable to prove the uniqueness of the solution for (1.6)–(1.8). But if we assume that M is sufficiently small, then the result can be obtained.

Theorem 4.2. *Suppose that the assumption in Theorem 4.1 holds. If M is sufficiently small, then the time periodic solution of* (1.6)–(1.8) *in Theorem 4.1 is unique.*

Proof. Let u and $\overline{u}$ be any two time periodic solutions of (1.6)–(1.8). With $v = u - \overline{u}$, we can get from (1.6) that

$$v_t - v_{xxt} + \gamma(v - v_{xx}) = Nu - N\overline{u}. \tag{4.7}$$

Taking the inner product of (4.7) with v, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + \|v_x\|^2\right) + \gamma\left(\|v\|^2 + \|v_x\|^2\right) = (Nu - N\overline{u}, v). \tag{4.8}$$

Since,

$$\begin{aligned}
|(-3uu_x+3\overline{u}\overline{u}_x,v)| &\le |(-3uv_x,v)|+|(-3\overline{u}_x v,v)|\\
&\le \frac{3}{2}\|u\|_\infty\left(\|v\|^2+\|v_x\|^2\right)+3\|\overline{u}_x\|_\infty\|v\|^2\\
&\le \left(\frac{3k_1}{2}c_1^{1/2}+3k_1c_2^{1/2}\right)\|v\|^2+\frac{3k_1}{2}c_1^{1/2}\|v_x\|^2,
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
|(2u_xu_{xx}-2\overline{u}_x\overline{u}_{xx},v)| &\le |(2u_xv_{xx},v)|+|(2\overline{u}_{xx}v_x,v)|\\
&\le \|u_{xx}\|_\infty\left(\|v\|^2+\|v_x\|^2\right)+2\|u_x\|_\infty\|v_x\|^2+\|\overline{u}_{xx}\|_\infty\left(\|v\|^2+\|v_x\|^2\right)\\
&\le 2k_1(c_2+c)^{1/2}\|v\|^2+\left[2k_1(c_2+c)^{1/2}+2k_1c_2^{1/2}\right]\|v_x\|^2,
\end{aligned}\tag{4.10}$$

$$\begin{aligned}
|(uu_{xxx}-\overline{u}\overline{u}_{xxx},v)| &\le |(uv_{xxx},v)|+|(\overline{u}_{xxx}v,v)|\\
&\le \int_\Omega |u_{xx}vv_x|dx+\frac{3}{2}\int_\Omega \left|u_xv_x^2\right|dx+2\int_\Omega |\overline{u}_{xx}vv_x|dx\\
&\le \frac{1}{2}\|u_{xx}\|_\infty\left(\|v\|^2+\|v_x\|^2\right)+\frac{3}{2}\|u_x\|_\infty\|v_x\|^2+\|\overline{u}_{xx}\|_\infty\left(\|v\|^2+\|v_x\|^2\right)\\
&\le \frac{3k_1}{2}(c_2+c)^{1/2}\|v\|^2+\left[\frac{3k_1}{2}(c_2+c)^{1/2}+\frac{3k_1}{2}c_2^{1/2}\right]\|v_x\|^2.
\end{aligned}\tag{4.11}$$

Hence, if M is sufficient small such that $2\gamma \ge 3k_1c_1^{1/2}+6k_1c_2^{1/2}+7k_1(c_2+c)^{1/2}$, $2\gamma \ge 3k_1c_1^{1/2}+7k_1c_2^{1/2}+7k_1(c_2+c)^{1/2}$, then it follows from (4.8)–(4.11), we get

$$\frac{d}{dt}\left(\|v\|^2+\|v_x\|^2\right)+\rho\left(\|v\|^2+\|v_x\|^2\right)\le 0,\tag{4.12}$$

where $\rho \ge 0$ is suitable constant.

Applying Gronwall's inequality, we derive that

$$\left(\|v(t)\|^2+\|v_x(t)\|^2\right)\le\left(\|v(0)\|^2+\|v_x(0)\|^2\right)e^{-\rho t},\quad \text{for any } t\ge 0.\tag{4.13}$$

Since v is ω-periodic in t, then for any positive integer m we have

$$\|v(t)\|^2+\|v_x(t)\|^2=\|v(t+m\omega)\|^2+\|v_x(t+m\omega)\|^2.\tag{4.14}$$

Then we can infer that

$$\left(\|v(t)\|^2+\|v_x(t)\|^2\right)\le\left(\|v(0)\|^2+\|v_x(0)\|^2\right)e^{-\rho(t+m\omega)}.\tag{4.15}$$

It follows from $v(0)=v_x(0)=0$ that $u(t,x)=\overline{u}(t,x)$, which completes the proof of Theorem 4.2. □

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