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Research Article

Periodic Solutions for a Class of n-th Order Functional Differential Equations

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We study the existence of periodic solutions for n-th order functional differential equations $x^{(n)}(t) = \sum_{i=0}^{n-1} b_i[x^{(i)}(t)]^k + f(x(t-\tau(t))) + p(t)$. Some new results on the existence of periodic solutions of the equations are obtained. Our approach is based on the coincidence degree theory of Mawhin.

1. Introduction

In this paper, we are concerned with the existence of periodic solutions of the following n-th order functional differential equations:

$$x^{(n)}(t) = \sum_{i=0}^{n-1} b_i\left[x^{(i)}(t)\right]^k + f(x(t-\tau(t))) + p(t), \tag{1.1}$$

where $b_i,\ i = 0,1,\dots,n-1$ are constants, k is a positive odd, $f \in C^1(R,R)$ for $\forall x \in R$, $p \in C(R,R)$ with $p(t+T) = p(t)$.

In recent years, there are many papers studying the existence of periodic solutions of first-, second- or third-order differential equations [1–12]. For example, in [5], Zhang and Wang studied the following differential equations:

$$x'''(t) + ax''^{2k-1}(t) + bx'^{2k-1}(t) + cx^{2k-1}(t) + g\big(t,\ x(t-\tau_1), x'(t-\tau_2)\big) = p(t). \tag{1.2}$$

The authors established the existence of periodic solutions of (1.2) under some conditions on a, b, c, and $2k-1$.

In [13–24], periodic solutions for n, $2n$, and $2n+1$ th order differential equations were discussed. For example, in [22, 24], Pan et al. studied the existence of periodic solutions of higher order differential equations of the form

$$x^{(n)}(t) = \sum_{i=1}^{n-1} b_i x^{(i)}(t) + f(t, x(t),\ x(t-\tau_1(t)), \dots, x(t-\tau_m(t))) + p(t). \tag{1.3}$$

The authors obtained the results based on the damping terms $x^{(i)}(t)$ and the delay $\tau_i(t)$.

In present paper, by using Mawhin's continuation theorem, we will establish some theorems on the existence of periodic solutions of (1.1). The results are related to not only b_i and $f(t,x)$ but also the positive odd k. In addition, we give an example to illustrate our new results.

2. Some Lemmas

We investigate the theorems based on the following lemmas.

Lemma 2.1 (see [17]). *Let $n_1 > 1$, $\alpha \in [0, +\infty)$ be constants, $s \in C(R,R)$ with $s(t+T) = s(t)$, and $s(t) \in [-\alpha, \alpha]$, for all $t \in [0,T]$. Then for $\forall x \in C^1(R,R)$ with $x(t+T) = x(t)$, one has*

$$\int_0^T |x(t) - x(t-s(t))|^{n_1} dt \le 2\alpha^{n_1} \int_0^T \left|x'(t)\right|^{n_1} dt. \tag{2.1}$$

Lemma 2.2. *Let $k \ge 1$, $\alpha \in [0, +\infty)$ be constants, $s \in C(R,R)$ with $s(t+T) = s(t)$, and $s(t) \in [-\alpha, \alpha]$, for all $t \in [0,T]$. Then for $\forall x \in C^1(R,R)$ with $x(t+T) = x(t)$, one has*

$$\int_0^T \left|x^k(t) - x^k(t-s(t))\right|^{(k+1)/k} dt \le 2\alpha^{(k+1)/k} k^{1/k} \left[(k-1)\int_0^T |x(t)|^{k+1} dt + \int_0^T \left|x'(t)\right|^{k+1} dt\right]. \tag{2.2}$$

Proof. Let $F(t) = x^k(t)$. By Lemma 2.2, one has

$$\begin{aligned}
\int_0^T \left|x^k(t) - x^k(t-s(t))\right|^{(k+1)/k} dt &= \int_0^T |F(t) - F(t-s(t))|^{(k+1)/k} dt \\
&\le 2\alpha^{(k+1)/k} \int_0^T \left|F'(t)\right|^{(k+1)/k} dt \\
&= 2\alpha^{(k+1)/k} \int_0^T \left|kx^{k-1}(t)x'(t)\right|^{(k+1)/k} dt \\
&= 2\alpha^{(k+1)/k} k^{(k+1)/k} \int_0^T |x(t)|^{((k-1)(k+1))/k} \left|x'(t)\right|^{(k+1)/k} dt.
\end{aligned} \tag{2.3}$$

By inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \quad x \geq 0,\ y \geq 0,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{2.4}$$

one has

$$|x(t)|^{((k-1)(k+1))/k} \left|x'(t)\right|^{(k+1)/k} \leq \frac{(k-1)|x(t)|^{k+1}}{k} + \frac{|x'(t)|^{k+1}}{k}. \tag{2.5}$$

Thus we obtain

$$\int_0^T \left|x^k(t) - x^k(t - s(t))\right|^{(k+1)/k} dt \leq 2\alpha^{(k+1)/k} k^{1/k} \left[(k-1)\int_0^T |x(t)|^{k+1} dt + \int_0^T \left|x'(t)\right|^{k+1} dt\right]. \tag{2.6}$$

□

Lemma 2.3. *If $k \geq 1$ is an integer, $x \in C^n(R, R)$, and $x(t+T) = x(t)$, then*

$$\left(\int_0^T \left|x'(t)\right|^k dt\right)^{1/k} \leq T\left(\int_0^T \left|x''(t)\right|^k dt\right)^{1/k} \leq \cdots \leq T^{n-1}\left(\int_0^T \left|x^{(n)}(t)\right|^k dt\right)^{1/k}. \tag{2.7}$$

The proof of Lemma 2.3 is easy, here we omit it.

We first introduce Mawhin's continuation theorem.

Let X and Y be Banach spaces, $L : D(L) \subset X \to Y$ are a Fredholm operator of index zero, here $D(L)$ denotes the domain of L. $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q. \tag{2.8}$$

It follows that

$$L|_{D(L)\cap \operatorname{Ker} P} : D(L) \cap \operatorname{Ker} P \longrightarrow \operatorname{Im} L \tag{2.9}$$

is invertible, we denote the inverse of that map by K_p. Let Ω be an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.4 (see [25]). *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(ii) $QNx \neq 0$, *for all* $x \in \partial\Omega \cap \operatorname{Ker} L$;

(iii) $\deg\{QNx, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,

then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap D(L)$.

Now, we define $Y = \{x \in C(R,R) \mid x(t+T) = x(t)\}$ with the norm $|x|_\infty = \max_{t\in[0,T]}\{|x(t)|\}$ and $X = \{x \in C^{n-1}(R,R) \mid x(t+T) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x'|_\infty, \dots, |x^{(n-1)}|_\infty\}$. It is easy to see that X, Y are two Banach spaces. We also define the operators L and N as follows:

$$\begin{gathered} L : D(L) \subset X \longrightarrow Y, \quad Lx = x^{(n)}, \ D(L) = \{x \mid x \in C^n(R,R),\ x(t+T) = x(t)\}, \\ N : X \longrightarrow Y, \quad Nx = -\sum_{i=1}^{n-1} b_i\left[x^{(i)}(t)\right]^k - f(t, x(t-\tau(t))) + p(t). \end{gathered} \tag{2.10}$$

It is easy to see that (1.1) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we see that $\ker L = R$, $\dim(\ker L) = 1$, $\operatorname{Im} L = \{y \mid y \in Y,\ \int_0^T y(s)ds = 0\}$ is closed, and $\dim(Y \backslash \operatorname{Im} L) = 1$, one has $\operatorname{codim}(\operatorname{Im} L) = \dim(\ker L)$. So L is a Fredholm operator with index zero. Let

$$P : X \longrightarrow \ker L, \quad Px = x(0), \quad Q : Y \longrightarrow Y \setminus \operatorname{Im} L, \quad Qy = \frac{1}{T}\int_0^T y(t)dt, \tag{2.11}$$

and let

$$L|_{D(L)\cap \operatorname{Ker} P} : D(L) \cap \operatorname{Ker}\ P \longrightarrow \operatorname{Im} L. \tag{2.12}$$

Then $L|_{D(L)\cap \operatorname{Ker} P}$ has a unique continuous inverse K_p. One can easily find that N is L-compact in $\overline{\Omega}$, where $\overline{\Omega}$ is an open bounded subset of X.

3. Main Result

Theorem 3.1. *Suppose $n = 2m+1$, $m > 0$ an integer and the following conditions hold:*

(H_1) *The function f satisfies*

$$\lim_{x\to\infty}\left|\frac{f(t,x)}{x^k}\right| \le \gamma, \tag{3.1}$$

$$\left|f(t,x) - f(t,y)\right| \le \beta\left|x^k - y^k\right|, \tag{3.2}$$

where $\gamma \ge 0$.

(H_2)

$$|b_0| > \gamma + \theta_2. \tag{3.3}$$

(H_3) *There is a positive integer $0 < s \le m$ such that*

$$\begin{gathered} b_{2s} \ne 0, \quad \textit{if } s = m, \\ b_{2s} \ne 0, \quad b_{2s+i} = 0, \quad i = 1, 2, \dots, 2m-2s, \ \textit{if } 0 < s < m. \end{gathered} \tag{3.4}$$

(H_4)

$$A_2(2s,k)+\theta_1 T^{(2s-1)k}+\frac{(\gamma+\theta_2)\left(A_1(2s,k)+\theta_1 T^{(2s-1)k}\right)}{|b_0|-\gamma-\theta_2}$$
$$+k|b_0|T^{2s}\left[\frac{A_1(2s,k)+\theta_1 T^{(2s-1)k}}{|b_0|-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{2s}|,\quad \text{if } 1<s\le m, \tag{3.5}$$
$$\theta_1 T^k+\frac{(\gamma+\theta_2)\left(A_1(2,k)+\theta_1 T^k\right)}{|b_0|-\gamma-\theta_2}+k|b_0|T^2\left[\frac{A_1(2,k)+\theta_1 T^k}{|b_0|-\gamma-\theta_2}\right]^{(k-1)/k}<|b_2|,\quad \text{if } s=1,$$

where $A_1(s,k)=\sum_{i=1}^{s}|b_i|T^{(s-i)k}$, $A_2(s,k)=\sum_{i=1}^{s-2}|b_i|T^{(s-i)k}$, $\theta_1=2^{k/(k+1)}\beta|\tau(t)|_\infty k^{1/(k+1)}$, $\theta_2=2^{k/(k+1)}\beta|\tau(t)|_\infty k^{1/(k+1)}(k-1)^{k/(k+1)}$. *Then* (1.1) *has at least one* T*-periodic solution.*

Proof. Consider the equation

$$Lx=\lambda Nx,\quad \lambda\in(0,1), \tag{3.6}$$

where L and N are defined by (2.10). Let

$$\Omega_1=\left\{x\in\frac{D(L)}{\operatorname{Ker} L}, Lx=\lambda Nx\quad \text{for some } \lambda\in(0,1)\right\}. \tag{3.7}$$

For $x\in\Omega_1$, one has

$$x^{(n)}(t)=\lambda\sum_{i=0}^{2s}b_i\left[x^{(i)}(t)\right]^k+\lambda f(t,x(t-\tau(t)))+\lambda p(t),\quad \lambda\in(0,1). \tag{3.8}$$

Multiplying both sides of (3.8) by $x(t)$, and integrating them on $[0,T]$, one has for $\lambda\in(0,1)$

$$\int_0^T x^{(n)}(t)x(t)dt=\lambda\sum_{i=0}^{2s}b_i\int_0^T\left[x^{(i)}(t)\right]^k x(t)dt$$
$$+\lambda\int_0^T f(t,x(t-\tau(t)))x(t)dt+\lambda\int_0^T p(t)x(t)dt. \tag{3.9}$$

Since for any positive integer i,

$$\int_0^T x^{(2i-1)}(t)x(t)dt=0, \tag{3.10}$$

and in view of $n = 2m + 1$ and k is odd, it follows from (3.3) and (3.9) that

$$\begin{aligned}
&|b_0| \int_0^T |x(t)|^{k+1} dt \\
&\quad \le \sum_{i=1}^{2s} |b_i| \int_0^T \left|x^{(i)}(t)\right|^k |x(t)| dt + \int_0^T \left|f(t, x(t-\tau(t)))\right| |x(t)| dt + \int_0^T \left|p(t)\right| |x(t)| dt \\
&\quad \le \sum_{i=1}^{2s} |b_i| \int_0^T \left|x^{(i)}(t)\right|^k |x(t)| dt + \int_0^T \left|f(t, x(t))\right| |x(t)| dt \\
&\qquad + \int_0^T \left|f(t,x) - f(t, x(t-\tau(t)))\right| |x(t)| dt + \int_0^T \left|p(t)\right| |x(t)| dt.
\end{aligned} \tag{3.11}$$

By using Hölder inequality and Lemma 2.1, from (3.11), we obtain

$$\begin{aligned}
&|b_0| \int_0^T |x(t)|^{k+1} dt \\
&\quad \le \left(\int_0^T |x(t)|^{k+1} dt\right)^{1/(k+1)} \left[\sum_{i=1}^{2s} |b_i| \left(\int_0^T \left|x^{(i)}(t)\right|^{k+1} dt\right)^{k/(k+1)} \right. \\
&\qquad + \left(\int_0^T \left|f(t, x(t))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \\
&\qquad + \left(\int_0^T \left|f(t,x) - f(t, x(t-\tau(t)))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \\
&\qquad \left. + \left(\int_0^T \left|p(t)\right|^{(k+1)/k} dt\right)^{k/(k+1)} \right] \\
&\quad \le \left(\int_0^T |x(t)|^{k+1} dt\right)^{1/(k+1)} \left[\sum_{i=1}^{2s} |b_i| T^{(2s-i)k} \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} \right. \\
&\qquad + \left(\int_0^T \left|f(t, x(t))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \\
&\qquad + \left(\int_0^T \left|f(t,x) - f(t, x(t-\tau(t)))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \\
&\qquad \left. + \left|p(t)\right|_\infty T^{k/(k+1)} \right].
\end{aligned} \tag{3.12}$$

So

$$
\begin{aligned}
|b_0|\left(\int_0^T |x(t)|^{k+1}dt\right)^{k/(k+1)} \\
\leq A_1(2s,k)\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1}dt\right)^{k/(k+1)} + \left(\int_0^T \left|f(t,x(t))\right|^{(k+1)/k}dt\right)^{k/(k+1)} \\
+\left(\int_0^T \left|f(t,x(t)) - f(t,x(t-\tau(t)))\right|^{(k+1)/k}dt\right)^{k/(k+1)} + u_1,
\end{aligned}
\tag{3.13}
$$

where u_1 is a positive constant. Choosing a constant $\varepsilon > 0$ such that

$$
\gamma + \varepsilon + \theta_2 < |b_0|,
\tag{3.14}
$$

$$
\begin{aligned}
A_2(2s,k) + \theta_1 T^{(2s-1)k} + \frac{(\gamma+\varepsilon+\theta_2)\left(A_1(2s,k)+\theta_1 T^{(2s-1)k}\right)}{|b_0| - (\gamma+\varepsilon) - \theta_2} \\
+k|b_0|T^{2s}\left[\frac{A_1(2s,k)+\theta_1 T^{(2s-1)k}}{|b_0| - (\gamma+\varepsilon) - \theta_2}\right]^{(k-1)/k} < |b_{2s}|, \quad \text{if } 1 < s \leq m,
\end{aligned}
$$

$$
\theta_1 T^k + \frac{(\gamma+\varepsilon+\theta_2)\left(A_1(2,k)+\theta_1 T^k\right)}{|b_0| - (\gamma+\varepsilon) - \theta_2} + k|b_0|T^2\left[\frac{A_1(2,k)+\theta_1 T^k}{|b_0| - (\gamma+\varepsilon) - \theta_2}\right]^{(k-1)/k} < |b_2|, \quad \text{if } s = 1,
\tag{3.15}
$$

for the above constant $\varepsilon > 0$, we see from (3.1) that there is a constant $\delta > 0$ such that

$$
\left|f(t,x(t))\right| < (\gamma+\varepsilon)|x(t)|^k, \quad \text{for } |x(t)| > \delta, \ t \in [0,T].
\tag{3.16}
$$

Denote

$$
\Delta_1 = \{t \in [0,T] : |x(t)| \leq \delta\}, \qquad \Delta_2 = \{t \in [0,T] : |x(t)| > \delta\}.
\tag{3.17}
$$

Since

$$
\begin{aligned}
\int_0^T \left|f(t,x(t))\right|^{(k+1)/k}dt &\leq \int_{\Delta_1} \left|f(t,x(t))\right|^{(k+1)/k}dt + \int_{\Delta_2} \left|f(t,x(t))\right|^{(k+1)/k}dt \\
&\leq (f_\delta)^{(k+1)/k}T + (\gamma+\varepsilon)^{(k+1)/k}\int_0^T |x(t)|^{k+1}dt \\
&= (f_\delta)^{(k+1)/k}T + (\gamma+\varepsilon)^{(k+1)/k}\int_0^T |x(t)|^{k+1}dt,
\end{aligned}
\tag{3.18}
$$

using inequality

$$(a+b)^l \le a^l + b^l \quad \text{for } a \ge 0,\ b \ge 0,\ 0 \le l \le 1, \tag{3.19}$$

it follows from (3.18) that

$$\left(\int_0^T \left|f(t,x(t))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \le f_\delta T^{k/(k+1)} + (\gamma+\varepsilon)\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)}. \tag{3.20}$$

From (3.2) and by Lemma 2.2, one has

$$\begin{aligned}
&\left(\int_0^T \left|f(t,x(t)) - f(t,x(t-\tau(t)))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \\
&\quad\le \beta\left[\int_0^T \left|x^k(t) - x^k(t-\tau(t))\right|^{(k+1)/k} dt\right]^{k/(k+1)} \\
&\quad\le 2^{k/(k+1)}\beta|\tau(t)|_\infty k^{1/(k+1)}\left[(k-1)\int_0^T |x(t)|^{k+1} dt + \int_0^T \left|x'(t)\right|^{k+1} dt\right]^{k/(k+1)} \\
&\quad\le 2^{k/(k+1)}\beta|\tau(t)|_\infty k^{1/(k+1)}\left[(k-1)^{k/(k+1)}\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)}\right. \\
&\qquad\qquad \left. + \left(\int_0^T \left|x'(t)\right|^{k+1} dt\right)^{k/(k+1)}\right] \\
&\quad\le 2^{k/(k+1)}\beta|\tau(t)|_\infty k^{1/(k+1)}(k-1)^{k/(k+1)}\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)} \\
&\qquad + 2^{k/(k+1)}\beta|\tau(t)|_\infty k^{1/(k+1)} T^{(2s-1)k}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} \\
&\quad= \theta_2\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)} + \theta_1 T^{(2s-1)k}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)}.
\end{aligned} \tag{3.21}$$

Substituting the above formula into (3.13), one has

$$
\begin{aligned}
&\left[|b_0| - (\gamma + \varepsilon) - \theta_2\right]\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)} \\
&\quad \le \left[A_1(2s, k) + \theta_1 T^{(2s-1)k}\right]\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} + u_2,
\end{aligned}
\tag{3.22}
$$

where u_2 is a positive constant. That is

$$
\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)} \le \frac{A_1(2s, k) + \theta_1 T^{(2s-1)k}}{|b_0| - (\gamma + \varepsilon) - \theta_2}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} + u_3, \tag{3.23}
$$

where u_3 is a positive constant. □

On the other hand, multiplying both sides of (3.8) by $x^{(2s)}(t)$, and integrating on $[0, T]$, one has

$$
\begin{aligned}
&\int_0^T x^{(n)}(t) x^{(2s)}(t) dt \\
&\quad = \sum_{i=0}^{2s} b_i \int_0^T \left[x^{(i)}(t)\right]^k x^{(2s)}(t) dt + \int_0^T f(t, x(t - \tau(t))) x^{(2s)}(t) dt + \int_0^T p(t) x^{(2s)}(t) dt.
\end{aligned}
\tag{3.24}
$$

If $1 < s \le m$, since

$$
\int_0^T x^{(2m+1)}(t) x^{(2s)}(t) dt = 0, \qquad \int_0^T \left[x^{(2s-1)}(t)\right]^k x^{(2s)}(t) dt = 0, \tag{3.25}
$$

$$
\int_0^T [x(t)]^k x^{(2s)}(t) dt = -k \int_0^T [x(t)]^{k-1} x^{(2s-1)}(t) x'(t) dt, \tag{3.26}
$$

by using Hölder inequality and Lemma 2.1, from (3.23), one has

$$
\begin{aligned}
&|b_{2s}|\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt \\
&\quad\le \int_0^T \left|x^{(2s)}(t)\right|\left[\sum_{i=1}^{2s-2}|b_i|\left|x^{(i)}(t)\right|^k + \left|f(t,x(t-\tau(t)))\right| + \left|p(t)\right|\right]dt \\
&\qquad + k|b_0|\int_0^T |x(t)|^{k-1}\left|x^{(2s-1)}(t)\right|\left|x'(t)\right|dt \\
&\quad\le \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{1/(k+1)} \left[\sum_{i=1}^{2s-2}|b_i|T^{(2s-i)k}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)}\right. \\
&\qquad\qquad + \left(\int_0^T \left|f(t,x(t))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \\
&\qquad\qquad + \left(\int_0^T \left|f(t,x(t)) - f(t,x(t-\tau))\right|^{(k+1)/k} dt\right)^{k/(k+1)} \\
&\qquad\qquad \left. + \left|p(t)\right|_\infty T^{k/(k+1)}\right] \\
&\qquad + k|b_0|\left|x'(t)\right|_\infty \int_0^T |x(t)|^{k-1}\left|x^{(2s-1)}(t)\right|dt.
\end{aligned}
\tag{3.27}
$$

Since $x(0) = x(T)$, there exists $\xi \in [0,T]$ such that $x'(\xi) = 0$. So for $t \in [0,T]$

$$
x'(t) = x'(\xi) + \int_\xi^t x''(\sigma)d\sigma. \tag{3.28}
$$

Using Hölder inequality and Lemma 2.1, one has

$$
\begin{aligned}
\left|x'(t)\right|_\infty &\le \int_0^T \left|x''(t)\right|dt \le T^{k/(k+1)}\left(\int_0^T \left|x''(t)\right|^{k+1}dt\right)^{1/(k+1)} \\
&\le T^{2s-1-(1/(k+1))}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1}dt\right)^{1/(k+1)}.
\end{aligned}
\tag{3.29}
$$

Using inequality

$$
\left(\frac{1}{T}\int_0^T \left||x(t)|^r\right|\right)^{1/r} \le \left(\frac{1}{T}\int_0^T \left||x(t)|^l\right|\right)^{1/l} \quad \text{for } 0 \le r \le l,\ \forall x \in R. \tag{3.30}
$$

and applying Hölder inequality and by Lemma 2.1, we obtain

$$\begin{aligned}
\int_0^T |x(t)|^{k-1}\left| x^{(2s-1)}(t)\right| dt &\le \left(\int_0^T |x(t)|^k dt\right)^{(k-1)/k} \left(\int_0^T \left|x^{(2s-1)}(t)\right|^k dt\right)^{1/k} \\
&\le T^{1/(k+1)}\left(\int_0^T |x(t)|^{k+1} dt\right)^{(k-1)/(k+1)} \left(\int_0^T \left|x^{(2s-1)}(t)\right|^{k+1} dt\right)^{1/(k+1)} \\
&\le T^{1\,+\,1/(k+1)}\left(\int_0^T |x(t)|^{k+1} dt\right)^{(k-1)/(k+1)} \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{1/(k+1)}.
\end{aligned} \tag{3.31}$$

Substituting the above formula, (3.20), (3.27), and (3.30) into (3.26), one has

$$\begin{aligned}
&|b_{2s}|\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt \\
&\quad\le \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{1/(k+1)} \Bigg\{ \left[A_2(2s,k) + \theta_1 T^{(2s-1)k}\right]\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} \\
&\qquad + \left[(\gamma+\varepsilon)+\theta_2\right]\left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)} \\
&\qquad + \left(|p(t)|_\infty + f_\delta\right)T^{k/(k+1)} \Bigg\} \\
&\quad + k|b_0|T^{2s}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{2/(k+1)} \left(\int_0^T \left||x(t)|^{k+1}\right| dt\right)^{(k-1)/(k+1)}.
\end{aligned} \tag{3.32}$$

Then, one has

$$\begin{aligned}
&\left[|b_{2s}| - A_2(2s,k) - \theta_1 T^{(2s-1)k}\right]\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} \\
&\quad\le k|b_0|T^{2s}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{1/(k+1)} \left(\int_0^T \left||x(t)|^{k+1}\right| dt\right)^{(k-1)/(k+1)} \\
&\qquad + \left[(\gamma+\varepsilon)+\theta_2\right]\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} + u_4,
\end{aligned} \tag{3.33}$$

where u_4 is a positive constant. Using inequality

$$(a+b)^l \le a^l + b^l \quad \text{for } a \ge 0,\ b \ge 0,\ 0 \le l \le 1, \tag{3.34}$$

it follows from (3.23) that

$$\begin{aligned} &\left(\int_0^T |x(t)|^{k+1} dt\right)^{(k-1)/(k+1)} \\ &\quad \le \left[\frac{A_1(2s,k)+\theta_1 T^{(2s-1)k}}{|b_0|-(\gamma+\varepsilon)-\theta_2}\right]^{(k-1)/k} \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{(k-1)/(k+1)} + u_5, \end{aligned} \tag{3.35}$$

where u_5 is a positive constant. Substituting the above formula and (3.23) into (3.33), one has

$$\begin{aligned} &\left\{ |b_{2s}| - A_2(2s,k) - \theta_1 T^{(2s-1)k} - \frac{(\gamma+\varepsilon+\theta_2)\left(A_1(2s,k)+\theta_1 T^{(2s-1)k}\right)}{|b_0|-(\gamma+\varepsilon)-\theta_2} \right. \\ &\qquad \left. - k|b_0| T^{2s} \left[\frac{A_1(2s,k)+\theta_1 T^{(2s-1)k}}{|b_0|-(\gamma+\varepsilon)-\theta_2}\right]^{(k-1)/k} \right\} \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} \\ &\quad \le u_5 k |b_0| T^{2s} \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{1/(k+1)} + u_6, \end{aligned} \tag{3.36}$$

where u_6 is a positive constant.

If $s = 1$, since $\int_0^T [x'(t)]^k x''(t) dt = 0, \int_0^T [x(t)]^k x''(t) dt = -k\int_0^T [x(t)]^{k-1} [x'(t)]^2 dt$, from (3.24), one has

$$\begin{aligned} &b_2 \int_0^T \left[x''(t)\right]^{k+1} dt \\ &\quad = -k b_0 \int_0^T [x(t)]^{k-1} \left[x'(t)\right]^2 dt - \int_0^T f(t, x(t-\tau)) x''(t) dt + \int_0^T p(t) x''(t) dt. \end{aligned} \tag{3.37}$$

Applying the above method, one has

$$\begin{aligned} &\left\{ |b_2| - \theta_1 T^k - \frac{(\gamma+\varepsilon+\theta_2)\left(A_1(2,k)+\theta_1 T^k\right)}{|b_0|-(\gamma+\varepsilon)-\theta_2} - k|b_0| T^2 \left[\frac{A_1(2,k)+\theta_1 T^k}{|b_0|-(\gamma+\varepsilon)-\theta_2}\right]^{(k-1)/k} \right\} \\ &\quad \times \left(\int_0^T \left|x''(t)\right|^{k+1} dt\right)^{k/(k+1)} \le u_7 k |b_0| T^2 \left(\int_0^T \left|x''(t)\right|^{k+1} dt\right)^{1/(k+1)} + u_8, \end{aligned} \tag{3.38}$$

where u_7, u_8 is a positive constant. Hence there is a constant $M_1, M_2 > 0$ such that

$$\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt \le M_1, \tag{3.39}$$

$$\int_0^T |x(t)|^{k+1} dt \le M_2. \tag{3.40}$$

From (3.5), using Hölder inequality and Lemma 2.1, one has

$$\begin{aligned}
\int_0^T \left|x^{(n)}(t)\right| dt &\le \sum_{i=0}^{2s} |b_i| \int_0^T \left|x^{(i)}(t)\right|^k dt + \int_0^T \left|f(t, x(t))\right| dt \\
&\quad + \int_0^T \left|f(t, x(t)) - f(t, x(t-\tau(t)))\right| dt + \int_0^T \left|p(t)\right| dt \\
&\le \left[\sum_{i=1}^{2s} |b_i| T^{(2s-i)k+1/(k+1)} + \theta_1 T^{(2s-1)k+1/(k+1)}\right] \left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{k/(k+1)} \\
&\quad + \left[|b_0| + (\gamma + \varepsilon) + \theta_2\right] T^{1/(k+1)} \left(\int_0^T |x(t)|^{k+1} dt\right)^{k/(k+1)} + \left(\left|p(t)\right|_\infty + f_\delta\right) T \\
&\le \left[\sum_{i=1}^{2s} |b_i| T^{(2s-i)k+1/(k+1)} + \theta_1 T^{(2s-1)k+1/(k+1)}\right] (M_1)^{k/(k+1)} \\
&\quad + |b_0| + (\gamma + \varepsilon) + \theta_2 (M_2)^{k/(k+1)} + \left(\left|p(t)\right|_\infty + f_\delta\right) T = M,
\end{aligned} \tag{3.41}$$

where M is a positive constant. We claim that

$$\left|x^{(i)}(t)\right| \le T^{n-i-1} \int_0^T \left|x^{(n)}(t)\right| dt, \quad i = 1, 2, \ldots, n-1. \tag{3.42}$$

In fact, noting that $x^{(n-2)}(0) = x^{(n-2)}(T)$, there must be a constant $\xi_1 \in [0, T]$ such that $x^{(n-1)}(\xi_1) = 0$, we obtain

$$\left|x^{(n-1)}(t)\right| = \left|x^{(n-1)}(\xi_1) + \int_{\xi_1}^t x^{(n)}(s) ds\right| \le \left|x^{(n-1)}(\xi_1)\right| + \int_0^T \left|x^{(n)}(t)\right| dt = \int_0^T \left|x^{(n)}(t)\right| dt. \tag{3.43}$$

Similarly, since $x^{(n-3)}(0) = x^{(n-3)}(T)$, there must be a constant $\xi_2 \in [0, T]$ such that $x^{(n-2)}(\xi_2) = 0$, from (3.43) we get

$$\left|x^{(n-2)}(t)\right| = \left|x^{(n-2)}(\xi_2) + \int_{\xi_2}^t x^{(n-1)}(s) ds\right| \le \int_0^T \left|x^{(n-1)}(t)\right| dt \le T \int_0^T \left|x^{(n)}(t)\right| dt. \tag{3.44}$$

By induction, we conclude that (3.42) holds. Furthermore, one has

$$\left|x^{(i)}(t)\right|_{\infty} \leq T^{n-i-1}\int_0^T \left|x^{(n)}(t)\right| dt \leq T^{n-i-1}M, \quad i = 1, 2, \ldots, n-1. \tag{3.45}$$

It follows from (3.39) that there exists a $\xi \in [0, T]$ such that $|x(\xi)| \leq M_2^{1/(k+1)}$. Applying Lemma 2.1, we get

$$\begin{aligned}
|x(t)|_{\infty} &\leq x(\xi) + \int_{\xi}^{t} x'(t)dt \leq M_2^{1/(k+1)} \\
&\quad + T^{k/(k+1)}\left(\int_0^T \left|x'(t)\right|^{k+1} dt\right)^{1/(k+1)} \\
&\leq M_2^{1/(k+1)} + T^{2s-1+(k/(k+1))}\left(\int_0^T \left|x^{(2s)}(t)\right|^{k+1} dt\right)^{1/(k+1)} \\
&= M_2^{1/(k+1)} + T^{2s-1+(k/(k+1))}M_1^{1/(k+1)}.
\end{aligned} \tag{3.46}$$

It follows that there is a constant $A > 0$ such that $\|x\| \leq A$. Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in \operatorname{Ker} L, QNx = 0\}$. Suppose $x \in \Omega_2$, then $x(t) = d \in R$ and satisfies

$$QNx = \frac{1}{T}\int_0^T \left[-b_0 d^k - f(t, d) + p(t)\right] dt = 0. \tag{3.47}$$

We will prove that there exists a constant $B > 0$ such that $|d| \leq B$. If $|d| \leq \delta$, taking $\delta = B$, we get $|d| \leq B$. If $|d| > \delta$, from (3.47), one has

$$\begin{aligned}
|b_0||d|^k &= \left|\frac{1}{T}\int_0^T [-f(t, d) + p(t)] dt\right| \\
&\leq \frac{1}{T}\int_0^T |f(t, d)| dt + |p(t)|_{\infty} \leq (\gamma + \varepsilon)|d|^k + |p(t)|_{\infty}.
\end{aligned} \tag{3.48}$$

Thus

$$|d| \leq \left[\frac{|p(t)|_{\infty}}{|b_0| - (\gamma + \varepsilon)}\right]^{1/k}. \tag{3.49}$$

Taking $[|p(t)|_{\infty}/(|b_0| - (\gamma + \varepsilon))]^{1/k} = B$, one has $|d| \leq B$, which implies Ω_2 is bounded. Let Ω be a nonempty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2}$. We can easily see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument, we have

(i) $Lx \neq \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$,

(ii) $QNx \neq 0$, for all $x \in \partial\Omega \cap \operatorname{Ker} L$.

At last we will prove that condition (iii) of Lemma 2.4 is satisfied. We take

$$\begin{gathered} H : (\Omega \cap \operatorname{Ker} L) \times [0,1] \longrightarrow \operatorname{Ker} L, \\ H(d,\mu) = \mu d + \frac{1-\mu}{T} \int_0^T \left[-b_0 d^k - f(t,d) + p(t) \right] dt. \end{gathered} \tag{3.50}$$

From assumptions (H_1) and (H_2), we can easily obtain $H(d,\mu) \neq 0$, for all $(d,\mu) \in \partial\Omega \cap \operatorname{Ker} L \times [0,1]$, which results in

$$\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0\} = \deg\{H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0\} \neq 0. \tag{3.51}$$

Hence, by using Lemma 2.2, we know that (1.1) has at least one T-periodic solution.

Theorem 3.2. *Suppose $n = 4m+1$, $m > 0$ an integer and conditions (H_1), (H_2) hold. If*

(H_5) *there is a positive integer $0 < s \leq m$ such that*

$$b_{4s-3} \neq 0, \qquad b_{4s-3+i} = 0, \quad i = 1,2,\ldots,4m-4s+3, \tag{3.52}$$

(H_6)

$$\begin{gathered} A_2(4s-3,k) + \theta_1 T^{(4s-4)k} + \frac{(\gamma+\theta_2)\left(A_1(4s-3,k) + \theta_1 T^{(4s-4)k}\right)}{|b_0| - \gamma - \theta_2} \\ +k|b_0| T^{4s-3} \left[\frac{A_1(4s-3,k) + \theta_1 T^{4s-4}}{|b_0| - \gamma - \theta_2} \right]^{(k-1)/k} < |b_{4s-3}|, \quad \text{if } 1 < s \leq m, \\ \theta_1 + \frac{(\gamma+\theta_2)(A_1(1,k) + \theta_1)}{|b_0| - \gamma - \theta_2} < |b_1|, \quad \text{if } s = 1, \end{gathered} \tag{3.53}$$

then (1.1) *has at least one T-periodic solution.*

Proof. From the proof of Theorem 3.1, one has

$$\left(\int_0^T |x(t)|^{k+1} dt \right)^{k/(k+1)} \leq \frac{A_1(4s-3,k) + \theta_1 T^{(4s-4)k}}{|b_0| - (\gamma+\varepsilon) - \theta_2} \left(\int_0^T \left| x^{(4s-3)}(t) \right|^{k+1} dt \right)^{k/(k+1)} + u_9, \tag{3.54}$$

where u_9 is a positive constant. Multiplying both sides of (3.8) by $x^{(4s-3)}(t)$, and integrating on $[0,T]$, one has

$$\begin{aligned}\int_0^T x^{(n)}(t)x^{(4s-3)}(t)dt = &-\lambda\sum_{i=0}^{4s-3} b_i\int_0^T \left[x^{(i)}(t)\right]^k x^{(4s-3)}(t)dt\\ &-\lambda\int_0^T f(t,x(t-\tau))x^{(4s-3)}(t)dt+\lambda\int_0^T p(t)x^{(4s-3)}(t)dt.\end{aligned} \tag{3.55}$$

Since

$$\int_0^T x^{(4m+1)}(t)x^{(4s-3)}(t)dt = (-1)^{2m-2s+2}\int_0^T \left[x^{(2m+2s-1)}(t)\right]^2 dt, \tag{3.56}$$

then it follows from (3.55) and (3.56) that

$$\begin{aligned}b_{4s-3}\int_0^T \left|x^{(4s-3)}(t)\right|^{k+1}dt \le &-\sum_{i=0}^{4s-4} b_i\int_0^T \left[x^{(i)}(t)\right]^k x^{(4s-3)}(t)dt\\ &-\int_0^T f(t,x(t-\tau))x^{(4s-3)}(t)dt+\int_0^T p(t)x^{(4s-3)}(t)dt.\end{aligned} \tag{3.57}$$

□

By using the same way as in the proof of Theorem 3.1, the following theorems can be proved in case $1 < s \le m$ or $s = 1$.

Theorem 3.3. *Suppose $n = 4m+1$, $m > 0$ for a positive integer and conditions (H_1), (H_2) hold. If*

(H_7) *there is a positive integer $0 < s \le m$ such that*

$$b_{4s-1} \ne 0, \qquad b_{4s-1+i} = 0, \quad i = 1,2,\dots,4m-4s+1, \tag{3.58}$$

(H_8)

$$\begin{aligned}&A_2(4s-1,k)+\theta_1T^{(4s-2)k}+\frac{(\gamma+\theta_2)\left(A_1(4s-1,k)+\theta_1T^{(4s-2)k}\right)}{|b_0|-\gamma-\theta_2}\\ &\quad+k|b_0|T^{4s-1}\left[\frac{A_1(4s-1,k)+\theta_1T^{(4s-2)k}}{|b_0|-\gamma-\theta_2}\right]^{(k-1)/k} < |b_{4s-1}|,\end{aligned} \tag{3.59}$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.4. *Suppose $n = 4m+3$, $m \ge 0$ an integer and conditions (H_1), (H_2) hold. If*

(H_9) *there is a positive integer $0 \le s \le m$ such that*

$$b_{4s+1} \ne 0, \qquad b_{4s+1+i} = 0, \quad i = 1,2,\dots,4m-4s+1, \tag{3.60}$$

(H_{10})

$$
\begin{aligned}
&A_2(4s+1,k)+\theta_1 T^{4sk}+\frac{(\gamma+\theta_2)\left(A_1(4s+1,k)+\theta_1 T^{4sk}\right)}{|b_0|-\gamma-\theta_2}\\
&\quad+k|b_0|T^{4s+1}\left[\frac{A_1(4s+1,k)+\theta_1 T^{4sk}}{|b_0|-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{4s+1}|,\quad \text{if } 0<s\leq m,\\
&\theta_1+\frac{(\gamma+\theta_2)(A_1(1,k)+\theta_1)}{|b_0|-\gamma-\theta_2}<|b_1|,\quad \text{if } s=0,
\end{aligned}
\tag{3.61}
$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.5. *Suppose $n=4m+3, m>0$ an integer and conditions $(H_1),(H_2)$ hold. If*

(H_{11}) *there is a positive integer $0<s\leq m$ such that*

$$
b_{4s-1}\neq 0,\qquad b_{4s-1+i}=0,\quad i=1,2,\ldots,4m-4s+3,
\tag{3.62}
$$

(H_{12})

$$
\begin{aligned}
&A_2(4-1,k)+\theta_1 T^{(4s-2)k}+\frac{(\gamma+\theta_2)\left(A_1(4s-1,k)+\theta_1 T^{(4s-2)k}\right)}{|b_0|-\gamma-\theta_2}\\
&\quad+k|b_0|T^{4s-1}\left[\frac{A_1(4s-1,k)+\theta_1 T^{(4s-2)k}}{|b_0|-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{4s-1}|,
\end{aligned}
\tag{3.63}
$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.6. *Suppose $n=4m$, $m>0$ an integer and conditions (H_1) hold. If*

(H_{13})

$$
b_0>\gamma+\theta_2,
\tag{3.64}
$$

(H_{14}) *there is a positive integer $0<s\leq 2m$ such that*

$$
\begin{aligned}
&b_{2s-1}\neq 0,\quad \text{if } s=2m,\\
&b_{2s-1}\neq 0,\quad b_{2s-1+i}=0,\quad i=1,2,\ldots,4m-2s,\ \text{if } 0<s<2m,
\end{aligned}
\tag{3.65}
$$

(H_{15})

$$A_2(2s-1,k)+\theta_1 T^{(2s-2)k}+\frac{(\gamma+\theta_2)\left(A_1(2s-1,k)+\theta_1 T^{(2s-2)k}\right)}{b_0-\gamma-\theta_2}$$
$$+kb_0T^{2s-1}\left[\frac{A_1(2s-1,k)+\theta_1 T^{(2s-2)k}}{b_0-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{2s-1}|,\quad \text{if } 1<s\le 2m,$$
$$\theta_1+\frac{(\gamma+\theta_2)(A_1(1,k)+\theta_1)}{b_0-\gamma-\theta_2}<|b_1|,\quad \text{if } s=1, \tag{3.66}$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.7. *Suppose $n=4m+2$, $m>0$ an integer and conditions (H_1) hold. If*

(H_{16})

$$-b_0>\gamma+\theta_2, \tag{3.67}$$

(H_{17}) *there is a positive integer $0<s\le 2m+1$ such that*

$$b_{2s-1}\neq 0,\quad \text{if } s=2m+1,$$
$$b_{2s-1}\neq 0,\quad b_{2s-1+i}=0,\quad i=1,2,\ldots,\ 4m-2s,\ \text{if } 0<s<2m+1, \tag{3.68}$$

(H_{18})

$$A_2(2s-1,k)+\theta_1 T^{(2s-2)k}+\frac{(\gamma+\theta_2)\left(A_1(2s-1,k)+\theta_1 T^{(2s-2)k}\right)}{-b_0-\gamma-\theta_2}$$
$$-kb_0T^{2s-1}\left[\frac{A_1(2s-1,k)+\theta_1 T^{(2s-2)k}}{-b_0-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{2s-1}|,\quad \text{if } 1<s\le 2m+1,$$
$$\theta_1+\frac{(\gamma+\theta_2)(A_1(1,k)+\theta_1)}{-b_0-\gamma-\theta_2}<|b_1|,\quad \text{if } s=1, \tag{3.69}$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.8. *Suppose $n=4m$, $m>0$ is an integer, and conditions (H_1), (H_{13}) hold. If*

(H_{19}) *there is a positive integer $0<s\le m$ such that*

$$b_{4s-2}\neq 0,\qquad b_{4s-2+i}=0,\quad i=1,2,\ldots,4m-4s+1, \tag{3.70}$$

(H_{20})

$$A_2(4s-2,k)+\theta_1 T^{(4s-3)k}+\frac{(\gamma+\theta_2)\left(A_1(4s-2,k)+\theta_1 T^{(4s-3)k}\right)}{b_0-\gamma-\theta_2}$$
$$+kb_0T^{4s-2}\left[\frac{A_1(4s-2,k)+\theta_1 T^{(4s-3)k}}{b_0-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{4s-2}|,\quad \text{if } 1<s\le m, \tag{3.71}$$
$$\frac{(\gamma+\theta_2)\left(A_1(2,k)+\theta_1 T^{k}\right)}{b_0-\gamma-\theta_2}+kb_0T^{2}\left[\frac{A_1(2,k)+\theta_1 T^{k}}{b_0-\gamma-\theta_2}\right]^{(k-1)/k}<|b_2|,\quad \text{if } s=1,$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.9. *Suppose $n=4m$, $m>1$ an integer and conditions (H_1), (H_{13}) hold. If*

(H_{21}) *there is a positive integer $1<s\le m$ such that*

$$b_{4s-4}\neq 0,\qquad b_{4s-4+i}=0,\quad i=1,2,\ldots,4m-4s+3, \tag{3.72}$$

(H_{22})

$$\begin{aligned}&A_2(4s-4,k)+\theta_1 T^{(4s-5)k}+\frac{(\gamma+\theta_2)\left(A_1(4s-4,k)+\theta_1 T^{(4s-5)k}\right)}{b_0-\gamma-\theta_2}\\&+kb_0T^{4s-4}\left[\frac{A_1(4s-4,k)+\theta_1 T^{(4s-5)k}}{b_0-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{4s-4}|,\end{aligned} \tag{3.73}$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.10. *Suppose $n=4m+2$, $m\ge 1$ an integer and conditions (H_1), (H_{16}) hold. If*

(H_{23}) *there is a positive integer $1\le s\le m$ such that*

$$b_{4s}\neq 0,\qquad b_{4s+i}=0,\quad i=1,2,\ldots,4m-4s+1, \tag{3.74}$$

(H_{24})

$$\begin{aligned}&A_2(4s,k)+\theta_1 T^{(4s-1)k}+\frac{(\gamma+\theta_2)\left(A_1(4s,k)+\theta_1 T^{(4s-1)k}\right)}{-b_0-\gamma-\theta_2}\\&-kb_0T^{4s}\left[\frac{A_1(4s,k)+\theta_1 T^{(4s-1)k}}{-b_0-\gamma-\theta_2}\right]^{(k-1)/k}<|b_{4s}|,\end{aligned} \tag{3.75}$$

then (1.1) *has at least one T-periodic solution.*

Theorem 3.11. *Suppose $n = 4m + 2$, $m \geq 1$ is an integer, and conditions (H_1), (H_{16}) hold. If*

(H_{25}) *there is a positive integer $1 \leq s \leq m$ such that*

$$b_{4s-2} \neq 0, \qquad b_{4s-2+i} = 0, \quad i = 1, 2, \ldots, 4m - 4s + 3, \tag{3.76}$$

(H_{26})

$$\begin{aligned} & A_2(4s-2,k) + \theta_1 T^{(4s-3)k} + \frac{(\gamma+\theta_2)\left(A_1(4s-2,k) + \theta_1 T^{(4s-3)k}\right)}{-b_0 - \gamma - \theta_2} \\ & \quad - k b_0 T^{4s-2}\left[\frac{A_1(4s-2,k) + \theta_1 T^{(4s-3)k}}{-b_0 - \gamma - \theta_2}\right]^{(k-1)/k} < |b_{4s-2}|, \quad \text{if } 1 < s \leq m, \\ & \theta_1 T^k + \frac{(\gamma+\theta_2)\left(A_1(2,k) + \theta_1 T^k\right)}{-b_0 - \gamma - \theta_2} - k b_0 T^2 \left[\frac{A_1(2,k) + \theta_1 T^k}{-b_0 - \gamma - \theta_2}\right]^{(k-1)/k} < |b_2|, \quad \text{if } s = 1, \end{aligned} \tag{3.77}$$

then (1.1) *has at least one T-periodic solution.*

The proofs of Theorem 3.3–3.11 are similar to that of Theorem 3.1.

Example 3.12. Consider the following equation:

$$x^{(5)}(t) + 300\left[x''(t)\right]^3 + \frac{1}{50}\left[x'(t)\right]^3 + \frac{1}{100}\left[x(t)\right]^3 + \frac{1}{300}(\sin t)\left[x\left(t - \frac{\pi}{10}\right)\right]^3 = \cos t, \tag{3.78}$$

where $n = 5$, $k = 3$, $b_4 = b_3 = 0$, $b_2 = 300$, $b_1 = 1/50$, $b_0 = 1/100$, $f(t,x) = 1/300(\sin t)x^3$, $p(t) = \cos t$, $\tau(t) = \pi/10$. Thus, $T = 2\pi$, $\gamma = 1/300$, $A_1(2,k) = |b_1|(2\pi)^3 + |b_2| = 1/50 \times (2\pi)^3 + 200$. Obviously assumptions (H_1)–(H_3) hold and

$$\theta_1 T^k + \frac{(\gamma+\theta_2)\left(A_1(2,k) + \theta_1 T^k\right)}{|b_0| - \gamma - \theta_2} + k|b_0|(2\pi)^2\left[\frac{A_1(2,k) + \theta_1 T^k}{|b_0| - \gamma - \theta_2}\right]^{(k-1)/k} < |b_2|. \tag{3.79}$$

By Theorem 3.1, we know that (3.78) has at least one 2π-periodic solution.

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