Research Article

Asymptotic Behavior of Global Entropy Solutions for Nonstrictly Hyperbolic Systems with Linear Damping

Richard Alexander De la Cruz Guerrero,¹
Juan Carlos Juajibioy Otero,² and Leonardo Rendon²

¹ School of Mathematics and Statistics, Pedagogical and Technological University of Colombia, Tunja, Colombia

² Department of Mathematics, National University of Colombia, Bogotá, Colombia

Correspondence should be addressed to Juan Carlos Juajibioy Otero; jcuajibioy@unal.edu.co

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We investigate the large time behavior of the global weak entropy solutions to the symmetric Keyfitz-Kranzer system with linear damping. It is proved that as $t \rightarrow \infty$ the entropy solutions tend to zero in the L^p norm.

1. Introduction

In this paper, we consider the Cauchy problem to the symmetric system of Keyfitz-Kranzer type with linear damping

$$\begin{aligned} u_t + (u\phi(r))_x + au &= 0, \\ v_t + (v\phi(r))_x + bv &= 0, \end{aligned} \quad (1)$$

with initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x). \quad (2)$$

This system models the propagation models of propagation of forward longitudinal and transverse waves of elastic string which moves in a plane; see [1, 2]. General source term for the system (1) was considered in [3]. The damping term in the system (1) represents external forces proportional to velocity, and this term can produce loss of total energy of system. Consider the scalar case; for example,

$$u_t + au_x + bu = 0, \quad u(x, 0) = u_0(x). \quad (3)$$

From the integral representation of (3), it is easy to find the following solution:

$$u(x, t) = u_0(x - at) e^{-bt}. \quad (4)$$

In this case, the solution of (3) tends to zero when $t \rightarrow \infty$. In Figure 1, we show the graph of solution for the advection equation with initial data

$$u(x, 0) = \begin{cases} (1 - x^2), & \text{if } x \in (-1, 1), \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

For more general case, the behavior of solutions and its computation can be more complicated; for example, we consider Burger's equation with a particular initial data and linear damping; this equation models the component of the velocity in one-dimensional flow

$$u_t + \left(\frac{1}{2}u^2\right)_x + u = 0 \quad (6)$$

with initial data

$$u(x, 0) = \begin{cases} A(1 - x^2), & \text{if } x \in (-1, 1), \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

where A is a constant. By an application to the characteristics method, we have that solutions emanating from x_0 are given by

$$\begin{aligned} X(t, x_0) &= x_0 + A(1 - x_0^2)(1 - e^{-t}), \\ u(X, t) &= u_0(x) e^{-t}. \end{aligned} \quad (8)$$

FIGURE 1: Graph of solution $u_t + u_x + u = 0$, $A = 1$. FIGURE 3: Linear damping $A = 1/2$. FIGURE 2: Characteristics for $u = 0$, $A = 1$. FIGURE 4: Linear damping $A = 2$.

In general, in systems with source term, the characteristics are nonlinear functions and they could have asymptotic behavior; for example, the characteristics solutions for (6), (7) are asymptotic to the lines

$$\lim_{t \rightarrow \infty} X(t, x_0) = x_0 + A(1 - x_0^2). \quad (9)$$

It is easy to see that the shock occurs when $x_0 + A(1 - x_0^2) > 1$. In Figures 2, 3, and 4, we show the characteristics solutions for several values of A .

We are looking for conditions under which the terms a, b have a dissipative effect in the solutions of (1).

Let $r(x, t) = \sqrt{u(x, t)^2 + v(x, t)^2}$, and we are going to show the following main theorem.

Theorem 1. *If the initial data $(u_0(x), v_0(x)) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, then the Cauchy problems (1) and (2) have a weak entropy solutions satisfying*

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} < M. \quad (10)$$

Moreover, $r(u, v)$ converges to zero in L^p with exponential time decay; that is,

$$\|r(x, t)\|_{L^p(\mathbb{R})} \leq K e^{-Mt} \|r(x, 0)\|_{L^p(\mathbb{R})}. \quad (11)$$

2. Preliminaries

We start with some preliminaries about the general systems of conservation laws; see [4] chapter 5. Let $f: \Omega \rightarrow \mathbb{R}^n$ be a smooth vector field. Consider Cauchy problem for the system

$$\begin{aligned} u_t + f(u)_x &= g(u), \\ u(x, 0) &= u_0(x). \end{aligned} \quad (12)$$

When $g(u) = 0$, the system (12) is called homogeneous system of conservation laws, if $g(u) \neq 0$, the system (12) is called inhomogeneous system or balance system of

conservation laws. We will work also with the parabolic perturbation to the system (12); namely,

$$\begin{aligned} u_t + f(u)_x &= \epsilon u_{xx} + g(u), \\ u(x, 0) &= u_0(x). \end{aligned} \quad (13)$$

Denote by $A(u) = Df(u)$ the Jacobian matrix of partial derivatives of f .

Definition 2. The system (12) is strictly hyperbolic if, for every $u \in \Omega$, the matrix $A(u)$ has n real distinct eigenvalues $\lambda_1(u) < \dots < \lambda_n(u)$.

Let $r_i(u)$ be the corresponding eigenvector to $\lambda_i(u)$. Then, one can see the following.

Definition 3. One says that the i th characteristic field is genuinely nonlinear if

$$\nabla \lambda_i(u) \cdot r_i(0) \neq 0. \quad (14)$$

If instead

$$\nabla \lambda_i(u) \cdot r_i(0) = 0, \quad (15)$$

we say that the i th characteristic field is linearly degenerate.

For the following definitions, see [5, 6].

Definition 4. A k -Riemann invariant is a smooth function $w_k : \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$\nabla w_k(u) \cdot r_k(u) = 0. \quad (16)$$

Definition 5. A pair of function $\eta, q : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a entropy-entropy flux pair if it satisfies

$$\nabla \eta(u) A(u) = \nabla q(u), \quad (17)$$

if $\eta(u)$ is a convex function, then the pair (η, q) is called convex entropy-entropy flux pair.

Definition 6. A bounded measurable function $u(x, t)$ is an entropy (or admissible) solution for the Cauchy problem (12) if it satisfies the following inequality:

$$\eta(u)_t + q(u)_x + \nabla \eta(u) g(u) \leq 0, \quad (18)$$

in the distributional sense, where (η, q) is any convex entropy-entropy flux pair.

We consider the general system of Keyfitz-Kranzer type

$$\begin{aligned} u_t + (u\phi(u, v))_x &= 0, \\ v_t + (v\phi(u, v))_x &= 0, \end{aligned} \quad (19)$$

to get some general observations about this type of system. Let $F(u, v) = (u\phi(u, v), v\phi(u, v))$ in (19), and we have that

the eigenvalues and eigenvector of the Jacobian matrix Df are given by

$$\lambda_1(u, v) = \phi(u, v), \quad r_1 = \left(1, -\frac{\phi_u}{\phi_v}\right), \quad (20)$$

$$\lambda_2(u, v) = \phi(u, v) + (u, v) \cdot \nabla \phi(u, v), \quad r_2 = \left(1, \frac{v}{u}\right). \quad (21)$$

From (20) and (21), we have that $\nabla \phi \cdot r_1 = 0$ and $\nabla Z(u, v) \cdot r_2 = 0$, where $Z(u, v) = u/v$, and then the Riemann invariants are given by

$$\begin{aligned} W(u, v) &= \phi(r), \\ Z(u, v) &= \frac{u}{v}. \end{aligned} \quad (22)$$

Lemma 7. The system (1) is always linear degenerate in the first characteristic field. If

$$(u, v) \nabla \phi(u, v) \neq 0, \quad (23)$$

then the system (1) is strictly hyperbolic and nonlinear degenerate in the second characteristic field. Moreover,

$$\nabla \lambda_2(u, v) \cdot r_2 = \frac{2(u, v) \nabla \phi(u, v) + (u, v) H(\phi)(u, v)^T}{u}, \quad (24)$$

where H represents the Hessian matrix.

Lemma 8. Let $\eta(u, v) \in C^1(\mathbb{R}_+)$ be a Lipschitz function in a neighborhood of the origin, and let $q(u, v) = \psi(u, v) + \eta(u, v)\phi(u, v)$ be a function, such that ψ satisfies

$$\nabla \psi(u, v) = ((u, v) \cdot \nabla \eta(u, v) - \eta(u, v)) \nabla \phi(u, v). \quad (25)$$

Then, the pair

$$(n(u, v), q(u, v)) \quad (26)$$

is a entropy-entropy flux pair for the system (1). Moreover, if $\eta(u, v)$ is a convex function, then the pair (26) is a convex entropy-entropy flux pair.

3. Global Existence of Weak Entropy Solutions and Asymptotic Behavior

We consider the parabolic regularization of the system (1). Namely,

$$\begin{aligned} u_t + (u\phi(r))_x + au &= \epsilon u_{xx}, \\ v_t + (v\phi(r))_x + bv &= \epsilon v_{xx}, \end{aligned} \quad (27)$$

with initial data

$$u^\epsilon(x, 0) = u_0^\epsilon * j_\epsilon, \quad v^\epsilon(x, 0) = v_0^\epsilon * j_\epsilon, \quad (28)$$

where j_ε is a mollifier. In this case, $\phi(u, v) = \phi(r)$, with $r = \sqrt{u^2 + v^2}$. By (20), the eigenvectors and eigenvalues are given by

$$\begin{aligned}\lambda_1(u, v) &= \phi(r), & r_1 &= \left(1, -\frac{u}{v}\right), \\ \lambda_2(u, v) &= \phi(r) + r\phi'(r), & r_2 &= \left(1, \frac{v}{u}\right).\end{aligned}\quad (29)$$

The following conditions will be necessary in our next discussion:

$$\begin{aligned}(C_1) \quad & r\phi(r) \rightarrow 0, \text{ as } r \rightarrow 0, r\phi'(r) \neq 0; \\ (C_2) \quad & a > b.\end{aligned}$$

The condition (C_1) guarantees the strict hyperbolicity to the system (27) according to Lemma 7, while condition (C_2) ensures the existence of a positive invariant region. Now, we consider the following subset of $\mathbb{R}$:

$$\Sigma = \left\{ (u, v) : \phi(r) \leq C_0, \quad 0 < C_1 \leq \frac{u}{v} \leq C_2 \right\}. \quad (30)$$

We affirm that Σ is an invariant region. Let $h(u, v) = (au, bv)$, if $(\bar{u}, \bar{v}) \in \gamma_1$, where γ_1 is the level curve of $W = \phi(r)$. We have that

$$(\nabla W \cdot h)(\bar{u}, \bar{v}) = (a + b)r\phi'(r) > 0, \quad (31)$$

and if $\bar{u}, \bar{v} \in \gamma_2$, where γ_2 is the level curve of $Z = u/v$, we have that

$$(\nabla Z \cdot h)(\bar{u}, \bar{v}) = (a - b)\left(\frac{u}{v}\right) > 0. \quad (32)$$

By Theorem 14.7 of [5], Σ is an invariant region for the system (27). It is easy to verify that (au, bv) satisfies the condition $H_1 \cdots H_5$ in [3]; thus, we have the following Lemma.

Proposition 9. *If $(u_0, v_0) \in \Sigma$ and the C condition holds, then the Cauchy problems (27), (28) have a global weak entropy solution.*

Now, for the global behavior of solutions, using ideas of the author in [7], we construct the following entropy-entropy flux pairs as follows:

$$n(r) = r^m, \quad m \geq 2. \quad (33)$$

From (25), we have

$$q(r) = (m-1) \int_0^r s^m \phi'(s) ds + r^m \phi(r). \quad (34)$$

Integrating by parts, we have that

$$q(r) = m\phi(r) - m(m-1) \int_0^r s^{m-1} \phi(s) ds. \quad (35)$$

Let $M = \sup_{(u,v) \in [0, \|u\|_{L^\infty}] \times [0, \|v\|_{L^\infty}]} \{\phi(r)\}$. Then, we have that

$$|q(r)| \leq 2mMr^m. \quad (36)$$

Multiplying (1) by $\nabla \eta$, we have that

$$\eta(r)_t + q(r)_x \leq -3mMr^m. \quad (37)$$

Now, we choose $h(x) \in C^2(\mathbb{R})$ as a function, such as $|h'(x)| \leq 1$, $|h''(x)| \leq 1$, and $h(x) = |x|$ for $|x| \geq 1$ and set $k(x) = e^{-h(x)}$. Then, $k'(x) \leq k(x)$. Multiplying by $k(x)$ in (37) and integrating over x , we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} \eta(r) g(x) \leq \int_{-\infty}^{\infty} q(r) k'(x) + -3mM \int_{-\infty}^{\infty} r^m dx. \quad (38)$$

By the inequality (36), we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} \eta(r) k(x) dx \leq -mM \int_{-\infty}^{\infty} r^m k(x) dx. \quad (39)$$

If $\psi(t) = \int_{-\infty}^{\infty} \eta(r) k(x) dx$, we have

$$\frac{d}{dt} \psi(t) \leq -mM \psi(t). \quad (40)$$

By Gronwall's inequality, we have

$$\psi(t) \leq e^{-mMt} \psi(0). \quad (41)$$

Thus, we have

$$\left(\int_{-\infty}^{\infty} r^m(t) k(x) dx \right)^{1/m} \leq e^{-Mt} \left(\int_{-\infty}^{\infty} r^m(0) k(x) dx \right)^{1/m}. \quad (42)$$

Passing to limit $m \rightarrow \infty$, in (42), we have the inequality (10).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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