

Research Article

On Certain Subclasses of Analytic Multivalent Functions Using Generalized Salagean Operator

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We introduce and study two subclasses of multivalent functions denoted by $\mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1; \lambda_2)$ and $\mathcal{N}_{p,n}^{m,\alpha,\beta,\sigma}(\mu, \delta; \gamma)$. Further, by using the method of differential subordination, certain inclusion relations between the two subclasses aforementioned are given. Moreover, several consequences of the main results are also discussed.

1. Introduction

Let $\mathcal{A}_{(p,n)}$ denote the class of the functions f of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (n, p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let denote $\mathcal{A} := \mathcal{A}_{(1,1)}$.

A function $f \in \mathcal{A}_{(p,n)}$ is said to be multivalent starlike functions of order α in \mathbb{U} , if it satisfies the following inequality:

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad z \in \mathbb{U}, \quad (0 \leq \alpha < p, \quad p \in \mathbb{N}), \quad (2)$$

and we denote this class by $S_{p,n}^*(\alpha)$.

A function $f \in \mathcal{A}_{(p,n)}$ is said to be multivalent convex functions of order α in \mathbb{U} , if it satisfies the following inequality:

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad z \in \mathbb{U}, \quad (0 \leq \alpha < p, \quad p \in \mathbb{N}), \quad (3)$$

and we denote this class by $C_{p,n}(\alpha)$.

For a function $f \in \mathcal{A}_{(p,n)}$, Goyal et al. [1] introduced the following generalized Salagean differential operator:

$$D_{\sigma}^0 f(z) = f(z), \quad (4)$$

$$D_{\sigma}^1 f(z) = D_{\sigma} f(z) = (1 - \sigma) f(z) + \frac{\sigma}{p} z f'(z), \quad (5)$$

$(\sigma \geq 0),$

$$D_{\sigma}^m f(z) = D_{\sigma} (D_{\sigma}^{m-1} f(z)), \quad (m \in \mathbb{N}). \quad (6)$$

If f is given by (1), then from (5) and (6) we have

$$D_{\sigma}^m f(z) = z^p + \sum_{k=p+n}^{\infty} \left[1 + \left(\frac{k}{p} - 1 \right) \sigma \right]^m a_k z^k. \quad (7)$$

Remark 1. For $\sigma = p = 1$, the differential operator $D_{\sigma}^m f(z)$ reduces to Salagean differential operator $D^m f(z)$ [2].

Definition 2. Let $\mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1; \lambda_2)$ be the class of functions $f \in \mathcal{A}_{(p,n)}$ that satisfy the condition

$$\Re \left\{ (1 - \lambda_1) \frac{z (D_{\sigma}^{\alpha, \beta, m} f(z))'}{D_{\sigma}^{\alpha, \beta, m} f(z)} + \lambda_1 \left(1 + \frac{z (D_{\sigma}^{\alpha, \beta, m} f(z))''}{(D_{\sigma}^{\alpha, \beta, m} f(z))'} \right) \right\} > \lambda_2, \quad z \in \mathbb{U}, \tag{8}$$

$$\frac{(D_{\sigma}^{\alpha, \beta, m} f(z))(D_{\sigma}^{\alpha, \beta, m} f(z))'}{z^{2p-1}} \neq 0, \quad z \in \mathbb{U},$$

$$\Re \left\{ \left(\frac{D_{\sigma}^{\alpha, \beta, m} f(z)}{z^p} \right)^{\mu} \left(\frac{(D_{\sigma}^{\alpha, \beta, m} f(z))'}{z^{p-1}} \right)^{\delta} \right\} > \gamma, \quad z \in \mathbb{U}, \tag{9}$$

$$(0 \leq \sigma, \delta, \mu \in \mathbb{R}, 0 \leq \alpha < \beta \leq 1, 0 \leq \gamma < p^{\delta} (\alpha + \beta p)^{\delta + \mu}, m, p \in \mathbb{N}),$$

where

$$D_{\sigma}^{\alpha, \beta, m} f(z) = \alpha D_{\sigma}^m f(z) + \beta z (D_{\sigma}^m f(z))'. \tag{10}$$

Remark 3. By specifying different values, we have some well-known subclasses of the classes $A_{(p,n)}$ and $A_{(n)} = A_{(1,n)}$ appearing from the families of the classes $\mathcal{M}_{p,n}^{m, \alpha, \beta, \sigma}(\lambda_1; \lambda_2)$ and $\mathcal{N}_{p,n}^{m, \alpha, \beta, \sigma}(\mu, \delta; \gamma)$.

- (i) $\mathcal{M}_{p,n}^{0,1,0,\sigma}(0; \lambda_1) = \mathcal{N}_{p,n}^{0,1,0,\sigma}(-1, 1, \lambda_1) = S_{p,n}^*$, ($0 \leq \lambda_1 < p$) is the class of multivalent starlike functions of order λ_1 .
- (ii) $\mathcal{M}_{1,n}^{0,1,0,\sigma}(0; \lambda_1) = \mathcal{N}_{1,n}^{0,1,0,\sigma}(-1, 1, \lambda_1) = S_{1,n}^* = S_n^*$, ($0 \leq \lambda_1 < 1$) is the class of starlike functions of order λ_1 .
- (iii) $\mathcal{M}_{p,n}^{1,0,1,1}(0; \lambda_1) = \mathcal{C}_{p,n}(\lambda_1)$, ($0 \leq \lambda_1 < p$) is the class of multivalent convex functions of order λ_1 .
- (iv) $\mathcal{M}_{1,n}^{1,0,1,1}(0; \lambda_1) = \mathcal{C}_{1,n}(\lambda_1) = \mathcal{C}_n(\lambda_1)$, ($0 \leq \lambda_1 < 1$) is the class of convex functions of order γ .
- (v) $\mathcal{N}_{1,n}^{1,1,0,1}(1, \delta; \lambda_1) = \mathcal{B}_n(\delta; \lambda_1)$, ($\delta \geq -1, 0 \leq \lambda_1 < 1$) is the subclass of Bazilevič functions.

Let $\mathcal{H}[a, n]$ be denoted by the class

$$\mathcal{H}[a, n] = \{h \in \mathcal{H}(\mathbb{U}) : h(z) = a + a_n z^n + \dots, z \in \mathbb{U}\}. \tag{11}$$

In this investigation, we focus on certain inequalities consisting of the following differential operator $\mathcal{F}_{p,n}^{m, \alpha, \beta, \sigma}(\mu, \delta) : \mathcal{A}_{(p,n)} \rightarrow \mathcal{H}[(\mu + \delta), n + p]$:

$$\mathcal{F}_{p,n}^{m, \alpha, \beta, \sigma}(\mu, \delta) f(z) = \mu \frac{z (D_{\sigma}^{\alpha, \beta, m} f(z))'}{D_{\sigma}^{\alpha, \beta, m} f(z)} + \delta \left(1 + \frac{z (D_{\sigma}^{\alpha, \beta, m} f(z))''}{(D_{\sigma}^{\alpha, \beta, m} f(z))'} \right) \tag{12}$$

where ($0 \leq \sigma, 0 \leq \alpha < \beta \leq 1, \lambda_1 \in \mathbb{R}, 0 \leq \lambda_2 < p, m, p \in \mathbb{N}$), and let $\mathcal{N}_{p,n}^{m, \alpha, \beta, \sigma}(\mu, \delta; \gamma)$ be the class of functions $f \in \mathcal{A}_{(p,n)}$ that satisfy the conditions

that generalizes the expression used in the definition of class $\mathcal{M}_{p,n}^{m, \alpha, \beta, \sigma}(\lambda_1; \lambda_2)$ and we receive several properties of the expression

$$\left(\frac{D_{\sigma}^{\alpha, \beta, m} f(z)}{z^p} \right)^{\mu} \left(\frac{(D_{\sigma}^{\alpha, \beta, m} f(z))'}{z^{p-1}} \right)^{\delta}, \quad (z \in \mathbb{U}), \tag{13}$$

including relations between classes $\mathcal{M}_{p,n}^{m, \alpha, \beta, \sigma}(\lambda_1; \lambda_2)$ and $\mathcal{N}_{p,n}^{m, \alpha, \beta, \sigma}(\mu, \delta; \gamma)$.

In order to prove our main results, we will need the following lemmas due to Miller and Mocanu [3].

Lemma 4. Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ satisfies $\psi(Ke^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $K \geq Mn, \theta \in \mathbb{R}$, and $z \in \mathbb{U}$. If $h(z) = a + h_n z^n + \dots$ is analytic in \mathbb{U} and $\psi(h(z), zh'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $|h(z)| < M, z \in \mathbb{U}$.

Lemma 5. Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for all $x \in \mathbb{R}, y \leq -n(1 + x^2)/2$, and $z \in \mathbb{U}$. If $h(z) = a + h_n z^n + \dots$ is analytic in \mathbb{U} and $\psi(h(z), zh'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re\{h(z)\} > 0, z \in \mathbb{U}$.

2. Main Results

Following the same techniques and procedure given by Goswami et al. [4], we have the following results.

Theorem 6. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_{\sigma}^{\alpha, \beta, m} f(z))(D_{\sigma}^{\alpha, \beta, m} f(z))' / z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where $D_{\sigma}^{\alpha, \beta, m}$ is given by (10), and also let $\mu, \delta \in \mathbb{R}$. If

$$\Re \left\{ \mathcal{F}_{p,n}^{m, \alpha, \beta, \sigma}(\mu, \delta) f(z) \right\} < p(\delta + \mu) + \frac{nM}{M + p^{\delta} (\alpha + \beta p)^{\delta + \mu}}, \quad (z \in \mathbb{U}), \tag{14}$$

where $p^\delta(\alpha + \beta p)^{\delta+\mu} \leq M$, then

$$\left| \left(\frac{D_\sigma^{\alpha,\beta,m} f(z)}{z^p} \right)^\mu \left(\frac{(D_\sigma^{\alpha,\beta,m} f(z))'}{z^{p-1}} \right)^\delta - p^\delta(\alpha + \beta p)^{\delta+\mu} \right| < M, \quad (z \in \mathbb{U}), \tag{15}$$

where the powers are the principal ones.

Proof. Let the function $h(z)$ be defined by

$$h(z) = \left(\frac{D_\sigma^{\alpha,\beta,m} f(z)}{z^p} \right)^\mu \left(\frac{(D_\sigma^{\alpha,\beta,m} f(z))'}{z^{p-1}} \right)^\delta - p^\delta(\alpha + \beta p)^{\delta+\mu}. \tag{16}$$

From the assumptions $f \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^{\alpha,\beta,m} f(z))(D_\sigma^{\alpha,\beta,m} f(z))'/z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, we have that $h \in \mathcal{H}[0, n]$. By a simple manipulation, we have

$$\begin{aligned} \mathcal{F}_{p,n}^{m,\alpha,\beta,\sigma}(\mu, \delta) f(z) &= p(\delta + \mu) \\ &+ \frac{zh'(z)}{h(z) + p^\delta(\alpha + \beta p)^{\delta+\mu}}. \end{aligned} \tag{17}$$

Now letting

$$\begin{aligned} \psi(r, s, z) &= p(\delta + \mu) + \frac{s}{r + p^\delta(\alpha + \beta p)^{\delta+\mu}}, \\ \Omega &= \left\{ w \in \mathbb{C} : \Re(w) < p(\delta + \mu) \right. \\ &\left. + \frac{nM}{M + p^\delta(\alpha + \beta p)^{\delta+\mu}} \right\}, \end{aligned} \tag{18}$$

then $f \in \mathcal{N}_{p,n}^{m,\sigma,\alpha,\beta}(\mu, \delta; \gamma)$.

Proof. Suppose that

$$h(z) = \frac{1}{p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma} \left[\left(\frac{D_\sigma^{\alpha,\beta,m} f(z)}{z^p} \right)^\mu \right.$$

we have from (17) and (14) that

$$\begin{aligned} \psi(h(z), zh'(z); z) &= \mathcal{F}_{p,n}^{m,\alpha,\beta,\sigma}(\mu, \delta) f(z) \in \Omega \\ &\forall z \in \mathbb{U}. \end{aligned} \tag{19}$$

Further, for any $\theta \in \mathbb{R}$, $K \geq nM$, and $z \in \mathbb{U}$, since $M \geq p^\delta(\alpha + \beta p)^{\delta+\mu}$, we also have

$$\begin{aligned} \Re\{\psi(Me^{i\theta}, Ke^{i\theta}; z)\} &= p(\delta + \mu) + K \Re\left(\frac{1}{M + e^{-i\theta} p^\delta(\alpha + \beta p)^{\delta+\mu}}\right) \\ &\geq p(\delta + \mu) + \frac{nM}{M + p^\delta(\alpha + \beta p)^{\delta+\mu}}, \quad (z \in \mathbb{U}), \end{aligned} \tag{20}$$

which shows that $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $\theta \in \mathbb{R}$, $K \geq nM$, and $z \in \mathbb{U}$. Therefore, according to Lemma 4, we obtain $|h(z)| < M$ ($z \in \mathbb{U}$). Hence, (15) is proven. \square

Theorem 7. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^{\alpha,\beta,m} f(z))(D_\sigma^{\alpha,\beta,m} f(z))'/z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where $D_\sigma^{\alpha,\beta,m}$ is given by (10), and also let $\mu, \delta \in \mathbb{R}$. If

$$\begin{aligned} \Re\{\mathcal{F}_{p,n}^{m,\alpha,\beta,\sigma}(\mu, \delta) f(z)\} &> k(\mu, \delta, \alpha, \beta; \gamma), \\ &(z \in \mathbb{U}), \end{aligned} \tag{21}$$

where $\gamma \in [0, p^\delta(\alpha + \beta p)^{\delta+\mu})$ and

$$k(\mu, \delta, \alpha, \beta; \gamma) = \begin{cases} p(\delta + \mu) - \frac{n\gamma}{2[p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma]}, & \text{if } \gamma \in \left[0, \frac{p^\delta(\alpha + \beta p)^{\delta+\mu}}{2}\right] \\ p(\delta + \mu) - \frac{n[p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma]}{2\gamma}, & \text{if } \gamma \in \left[\frac{p^\delta(\alpha + \beta p)^{\delta+\mu}}{2}, p^\delta(\alpha + \beta p)^{\delta+\mu}\right), \end{cases} \tag{22}$$

$$\cdot \left[\left(\frac{D_\sigma^{\alpha,\beta,m} f(z)}{z^p} \right)^\mu - \gamma \right]. \tag{23}$$

Then, $h(z) = 1 + h_n z^n + \dots$ is analytic in \mathbb{U} . It is easily seen from (23) that

$$\begin{aligned} & \mathcal{F}_{p,n}^{m,\alpha,\beta,\sigma}(\mu, \delta) f(z) \\ &= p(\delta + \mu) + \frac{(p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma)zh'(z)}{(p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma)h(z) + \gamma}. \end{aligned} \tag{24}$$

Further, since

$$\begin{aligned} \psi(r, s; z) &= p(\delta + \mu) + \frac{(p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma)s}{(p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma)r + \gamma}, \\ \Omega &= \{w \in \mathbb{C} : \Re(w) > k(\mu, \delta, \alpha, \beta; \gamma)\}, \end{aligned} \tag{25}$$

it leads to

$$\begin{aligned} \psi(h(z), zh'(z); z) &= \mathcal{F}_{p,n}^{m,\alpha,\beta,\sigma}(\mu, \delta) f(z) \in \Omega \\ & \forall z \in \mathbb{U}. \end{aligned} \tag{26}$$

Also, for any $x \in \mathbb{R}, y \leq -n(1 + x^2)/2$ and $z \in \mathbb{U}$, we have

$$\begin{aligned} \Re\{\psi(ix, y; z)\} &= p(\delta + \mu) + \frac{\gamma(p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma)y}{[p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma]^2 x^2 + \gamma^2} \\ &\leq p(\delta + \mu) \\ &\quad - \frac{n\gamma[p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma]}{2} \frac{1 + x^2}{[p^\delta(\alpha + \beta p)^{\delta+\mu} - \gamma]^2 x^2 + \gamma^2} \\ &\equiv q(z) \leq k(\mu, \delta, \alpha, \beta; \gamma) \\ &= \begin{cases} \lim_{x \rightarrow \infty} q(z), & \text{if } \gamma \in \left[0, \frac{p^\delta(\alpha + \beta p)^{\delta+\mu}}{2}\right] \\ q(0), & \text{if } \gamma \in \left[\frac{p^\delta(\alpha + \beta p)^{\delta+\mu}}{2}, p^\delta(\alpha + \beta p)^{\delta+\mu}\right); \end{cases} \end{aligned} \tag{27}$$

that is, $\psi(ix, y; z) \notin \Omega$. Finally, by Lemma 5, we obtain that $\text{Re}(h(z)) > 0$. The proof of Theorem 7 is complete. \square

3. Corollaries and Consequences

We will discuss some interesting consequences of the main theorems that extend some previous results obtained in ([4, 5]).

Putting $\alpha = 1, \beta = 0$ in Theorems 6 and 7, we get the following corollaries.

Corollary 8. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\mu, \delta \in \mathbb{R}$. If

$$\begin{aligned} & \Re \left\{ \mu \frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \delta \left(1 + \frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'} \right) \right\} \\ & < p(\delta + \mu) + \frac{nM}{M + p^\delta}, \quad (z \in \mathbb{U}), \end{aligned} \tag{28}$$

where $p^\delta \leq M$, then

$$\begin{aligned} & \left| \left(\frac{D_\sigma^m f(z)}{z^p} \right)^\mu \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^\delta - p^\delta \right| < M, \\ & (z \in \mathbb{U}), \end{aligned} \tag{29}$$

where the powers are the principal ones.

Corollary 9. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\mu, \delta \in \mathbb{R}$. If

$$\begin{aligned} & \Re \left\{ \mu \frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \delta \left(1 + \frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'} \right) \right\} \\ & > \varphi(\mu, \delta; \gamma), \quad (z \in \mathbb{U}), \end{aligned} \tag{30}$$

where $\gamma \in [0, p^\delta)$ and

$$\begin{aligned} & \varphi(\mu, \delta; \gamma) = k(\mu, \delta, 1, 0; \gamma) \\ &= \begin{cases} p(\delta + \mu) - \frac{n\gamma}{2[p^\delta - \gamma]}, & \text{if } \gamma \in \left[0, \frac{p^\delta}{2}\right] \\ p(\delta + \mu) - \frac{n[p^\delta - \gamma]}{2\gamma}, & \text{if } \gamma \in \left[\frac{p^\delta}{2}, p^\delta\right), \end{cases} \end{aligned} \tag{31}$$

then

$$\Re \left\{ \left(\frac{D_\sigma^m f(z)}{z^p} \right)^\mu \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^\delta \right\} > \gamma, \quad (z \in \mathbb{U}), \tag{32}$$

where the powers are the principal ones.

Taking $\mu = 1 - \lambda_1$ and $\delta = \lambda_1$ in Corollaries 8 and 9, respectively, we obtain the following special cases.

Corollary 10. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\lambda_1 \in \mathbb{R}$. If

$$\begin{aligned} & \Re \left\{ (1 - \lambda_1) \frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} \right. \\ & \left. + \lambda_1 \left(1 + \frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'} \right) \right\} < p + \frac{nM}{M + p^{\lambda_1}}, \\ & (z \in \mathbb{U}), \end{aligned} \tag{33}$$

where $p^{\lambda_1} \leq M$, then

$$\begin{aligned} & \left| \left(\frac{D_\sigma^m f(z)}{z^p} \right)^{1-\lambda_1} \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^{\lambda_1} - p^{\lambda_1} \right| < M, \\ & (z \in \mathbb{U}), \end{aligned} \tag{34}$$

where the powers are the principal ones.

Corollary 11. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))(D_\sigma^m f(z))'/z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\lambda_1 \in \mathbb{R}$. If

$$\Re \left\{ (1 - \lambda_1) \frac{z(D_\sigma^m f(z))'}{D_\sigma^m f(z)} + \lambda_1 \left(1 + \frac{z(D_\sigma^m f(z))''}{(D_\sigma^m f(z))'} \right) \right\} > \chi(\lambda_1; \gamma), \tag{35}$$

$(z \in \mathbb{U}),$

where $\gamma \in [0, p^{\lambda_1})$ and

$$\chi(\lambda_1; \gamma) = k(1 - \lambda_1, \lambda_1, 1, 0; \gamma) = \begin{cases} p - \frac{n\gamma}{2[p^{\lambda_1} - \gamma]}, & \text{if } \gamma \in \left[0, \frac{p^{\lambda_1}}{2}\right] \\ p - \frac{n[p^{\lambda_1} - \gamma]}{2\gamma}, & \text{if } \gamma \in \left[\frac{p^{\lambda_1}}{2}, p^{\lambda_1}\right), \end{cases} \tag{36}$$

then

$$\Re \left\{ \left(\frac{D_\sigma^m f(z)}{z^p} \right)^{1-\lambda_1} \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^{\lambda_1} \right\} > \gamma, \tag{37}$$

$(z \in \mathbb{U}),$

where the powers are the principal ones.

Next, upon taking $\alpha = 0, \beta = 1$ in Theorems 6 and 7, we obtain the following results.

Corollary 12. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\mu, \delta \in \mathbb{R}$. If

$$\Re \left\{ \mathcal{F}_{p,n}^{m,0,1,\sigma}(\mu, \delta) f(z) \right\} < p(\delta + \mu) + \frac{nM}{M + p^{2\delta+\mu}}, \tag{38}$$

$(z \in \mathbb{U}),$

where $p^{2\delta+\mu} \leq M$, then

$$\left| \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^\mu \left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}} \right)^\delta - p^{2\delta+\mu} \right| < M, \quad (z \in \mathbb{U}), \tag{39}$$

where the powers are the principal ones.

Corollary 13. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\mu, \delta \in \mathbb{R}$. If

$$\Re \left\{ \mathcal{F}_{p,n}^{m,0,1,\sigma}(\mu, \delta) f(z) \right\} > \phi(\mu, \delta; \gamma), \quad (z \in \mathbb{U}), \tag{40}$$

where $\gamma \in [0, p^{2\delta+\mu})$ and

$$\phi(\mu, \delta; \gamma) = k(\mu, \delta, 0, 1; \gamma) = \begin{cases} p(\delta + \mu) - \frac{n\gamma}{2[p^{2\delta+\mu} - \gamma]}, & \text{if } \gamma \in \left[0, \frac{p^{2\delta+\mu}}{2}\right] \\ p(\delta + \mu) - \frac{n[p^{2\delta+\mu} - \gamma]}{2\gamma}, & \text{if } \gamma \in \left[\frac{p^{2\delta+\mu}}{2}, p^{2\delta+\mu}\right), \end{cases} \tag{41}$$

then

$$\Re \left\{ \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^\mu \cdot \left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}} \right)^\delta \right\} > \gamma, \tag{42}$$

$(z \in \mathbb{U}),$

where the powers are the principal ones.

Taking $\mu = 1 - \lambda_1$ and $\delta = \lambda_1$ in Corollaries 12 and 13, respectively, we obtain the following special cases.

Corollary 14. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\lambda_1 \in \mathbb{R}$. If

$$\Re \left\{ \mathcal{F}_{p,n}^{m,0,1,\sigma}(1 - \lambda_1, \lambda_1) f(z) \right\} < p + \frac{nM}{M + p^{\lambda_1+1}}, \tag{43}$$

$(z \in \mathbb{U}),$

where $p^{\lambda_1+1} \leq M$, then

$$\left| \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^{1-\lambda_1} \cdot \left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}} \right)^{\lambda_1} - p^{\lambda_1+1} \right| < M, \tag{44}$$

$(z \in \mathbb{U}),$

where the powers are the principal ones.

Corollary 15. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^m f(z))'[(D_\sigma^m f(z))' + z(D_\sigma^m f(z))'']/z^{2(p-1)} \neq 0$ for all $z \in \mathbb{U}$, where D_σ^m is given by (7), and also let $\lambda_1 \in \mathbb{R}$. If

$$\Re \left\{ \mathcal{F}_{p,n}^{m,0,1,\sigma}(1 - \lambda_1, \lambda_1) f(z) \right\} > \psi(\lambda_1; \gamma), \tag{45}$$

$(z \in \mathbb{U}),$

where $\gamma \in [0, p^{\lambda_1+1})$ and

$$\begin{aligned} \psi(\lambda_1; \gamma) &= k(1 - \lambda_1, \lambda_1, 0, 1; \gamma) \\ &= \begin{cases} p(\delta + \mu) - \frac{n\gamma}{2[p^{\lambda_1+1} - \gamma]}, & \text{if } \gamma \in \left[0, \frac{p^{\lambda_1+1}}{2}\right] \\ p(\delta + \mu) - \frac{n[p^{2\delta+\mu} - \gamma]}{2\gamma}, & \text{if } \gamma \in \left[\frac{p^{\lambda_1+1}}{2}, p^{\lambda_1+1}\right), \end{cases} \end{aligned} \tag{46}$$

then

$$\Re \left\{ \left(\frac{(D_\sigma^m f(z))'}{z^{p-1}} \right)^{1-\lambda_1} \cdot \left(\frac{(D_\sigma^m f(z))' + z(D_\sigma^m f(z))''}{z^{p-1}} \right)^{\lambda_1} \right\} > \gamma, \tag{47}$$

$(z \in \mathbb{U}),$

where the powers are the principal ones.

In the next result, we will find the relation between $\mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1; \gamma)$ and $\mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(1 - \lambda_1, \lambda_1; \gamma)$. For this purpose, taking $\mu = 1 - \lambda_1$ and $\delta = \lambda_1$ in Theorem 7, we obtain the following result.

Corollary 16. Let $f(z) \in \mathcal{A}_{(p,n)}$ with $(D_\sigma^{\alpha,\beta,m} f(z))(D_\sigma^{\alpha,\beta,m} f(z))' / z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where $D_\sigma^{\alpha,\beta,m}$ is given by (10), and also let $\lambda_1 \in \mathbb{R}$. If

$$f(z) \in \mathcal{M}_{p,n}^{m,\alpha,\beta,\sigma}(\lambda_1; \varrho(\lambda_1, \alpha, \beta; \gamma)), \tag{48}$$

where $\gamma \in [0, p^{\lambda_1}(\alpha + \beta p))$ and

$$\varrho(\lambda_1, \alpha, \beta; \gamma) = k(1 - \lambda_1, \lambda_1, \alpha, \beta; \gamma) = \begin{cases} p - \frac{n\gamma}{2[p^{\lambda_1}(\alpha + \beta p) - \gamma]}, & \text{if } \gamma \in \left[0, \frac{p^{\lambda_1}(\alpha + \beta p)}{2}\right] \\ p - \frac{n[p^{\lambda_1}(\alpha + \beta p) - \gamma]}{2\gamma}, & \text{if } \gamma \in \left[\frac{p^{\lambda_1}(\alpha + \beta p)}{2}, p^{\lambda_1}[(\alpha + \beta p)], \right) \end{cases} \tag{49}$$

then $f(z) \in \mathcal{N}_{p,n}^{m,\alpha,\beta,\sigma}(1 - \lambda_1, \lambda_1; \gamma)$.

Taking $\lambda_1 = 0$ and $n = 1$ in the above corollary, we get the next special result.

Corollary 17. Let $f(z) \in \mathcal{A}_{(p)}$ with $(D_\sigma^{\alpha,\beta,m} f(z))(D_\sigma^{\alpha,\beta,m} f(z))' / z^{2p-1} \neq 0$ for all $z \in \mathbb{U}$, where $D_\sigma^{\alpha,\beta,m}$ is given by (10), and also let $\lambda_1 \in \mathbb{R}$. If

$$f(z) \in \mathcal{M}_p^{m,\alpha,\beta,\sigma}(\varrho(\alpha, \beta; \gamma)), \tag{50}$$

where $\gamma \in [0, \alpha + \beta p)$ and

$$\begin{aligned} \varrho(\alpha, \beta; \gamma) &= k(1, 0, \alpha, \beta; \gamma) \\ &= \begin{cases} p - \frac{\gamma}{2[(\alpha + \beta p) - \gamma]}, & \text{if } \gamma \in \left[0, \frac{\alpha + \beta p}{2}\right] \\ p - \frac{[(\alpha + \beta p) - \gamma]}{2\gamma}, & \text{if } \gamma \in \left[\frac{\alpha + \beta p}{2}, (\alpha + \beta p)\right), \end{cases} \end{aligned} \tag{51}$$

then $f(z) \in \mathcal{N}_p^{m,\alpha,\beta,\sigma}(1, 0; \gamma)$.

Again, for the special cases of μ and δ , Theorems 6 and 7 reduce at once to some results obtained by [4, 5].

Remark 18. Taking $p = 1$ and $m = 0$ in (7) and $\alpha = 1$ and $\beta = 0$ in (10), we get a known result obtained by Irmak et al. [5].

Remark 19. Taking $m = 0$ in (7) and $\alpha = 1 - \beta$ in (10), we get a known result obtained by Goswami et al. [4].

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

Both authors agreed with the contents of the paper.

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