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Research Article

Approximate Controllability of Semilinear Control System Using Tikhonov Regularization

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For an approximately controllable semilinear system, the problem of computing control for a given target state is converted into an equivalent problem of solving operator equation which is ill-posed. We exhibit a sequence of regularized controls which steers the semilinear control system from an arbitrary initial state x^0 to an ϵ neighbourhood of the target state x_τ at time $\tau > 0$ under the assumption that the nonlinear function f is Lipschitz continuous. The convergence of the sequences of regularized controls and the corresponding mild solutions are shown under some assumptions on the system operators. It is also proved that the target state corresponding to the regularized control is close to the actual state to be attained.

1. Introduction

Controllability is one of the qualitative properties of a control system that occupies an important place in control theory. Controllable systems have many applications in different branches of science and engineering (see [1–12] for an extensive review on controllability literature).

Let V and U be Hilbert spaces called state and control spaces, respectively. Let $Y = L_2(J, U)$ and $X = L_2(J, V)$ be the function spaces. The inner product and the corresponding norm on a Hilbert space are denoted by $\langle\cdot,\cdot\rangle$, $\|\cdot\|$, respectively.

Consider the semilinear control system

$$\begin{aligned} \frac{dx}{dt} &= Ax(t) + Bu(t) + f(t, x(t)), \\ x(t_0) &= x^0, \end{aligned} \tag{1}$$

where $A : D(A) \subseteq V \to V$ is a densely defined closed linear operator which generates a C_0 semigroup $T(t)$, $t \geq 0$. $B : U \to V$ is a bounded linear operator and $f : J \times V \to V$ is a nonlinear function where $J = [t_0, \tau] \subseteq [0, \infty)$. If $f \equiv 0$, then the resultant system is called the corresponding linear system which is denoted by $(1)^*$.

For $u \in Y$, the mild solution (see [13]) of (1) is given by

$$\begin{aligned} x(t) = T(t - t_0)x^0 &+ \int_{t_0}^{t} T(t-s)Bu(s)\,ds \\ &+ \int_{t_0}^{t} T(t-s) f(s, x(s))\,ds. \end{aligned} \tag{2}$$

The control system (1) is said to be exactly controllable if, for every x^0 and $x_\tau \in V$, there exists $u \in Y$ such that the mild solution $x \in X$ verifies the condition $x(\tau) = x_\tau$.

The control system (1) is said to be approximately controllable if, for every $\epsilon > 0$ and for every x^0 and $x_\tau \in V$, there exists $u \in Y$ such that the corresponding mild solution $x \in X$ satisfies

$$\left\| x(\tau) - x_\tau \right\| \leq \epsilon. \tag{3}$$

In [3], Naito proved the approximate controllability of semilinear system (1) under some assumptions which are given below.

Theorem 1 (see [3]). *The semilinear control system (1) is approximately controllable under the following conditions:*

(i) *The C_0 semigroup $T(t)$ is compact $\forall t > 0$.*

(ii) *The nonlinear function $f(t,x)$ is Lipschitz continuous with respect to x; that is, $\|f(t,x_1)-f(t,x_2)\| \le \|x_1 - x_2\| \forall x_1, x_2 \in X$, $t \in J$, where $c > 0$ is Lipschitz constant.*

(iii) $\|f(t,x)\| \le M_0$, *where M_0 is a positive constant.*

(iv) *For every $p \in X$, there exists a $q \in \overline{R(B)}$ such that $\mathscr{L}_1 p = \mathscr{L}_1 q$, where $R(B)$ is the range of the bounded linear operator B and $\mathscr{L}_1 : X \to V$ is a bounded linear operator defined as*

$$\mathscr{L}_1 p = \int_{t_0}^{\tau} T(\tau - s)\, p(s)\, ds. \tag{4}$$

Condition (iv) of Theorem 1 implies that the corresponding linear system $(1)^*$ is approximately controllable; for more details one can see the proof in [3].

In this paper, we study the problem of computing control for an approximately controllable semilinear system for a given target state by converting it into an equivalent linear operator equation which is ill-posed. We find sequence of regularized controls $\{u_{n,\lambda} : n \in \mathbb{N}, \lambda > 0\}$ using Tikhonov regularization and the mild solutions $\{x_{n,\lambda} : n \in \mathbb{N}, \lambda > 0\}$ corresponding to $\{u_{n,\lambda} : n \in \mathbb{N}, \lambda > 0\}$. Under some assumptions we prove the convergence of $\{u_{n,\lambda}\}$ and $\{x_{n,\lambda}\}$.

The outline of the paper is as follows. In Section 2, regularized control, its corresponding mild solutions, their convergence, and limitations due to the presence of nonlinearity are discussed. Section 3 is devoted to illustrating our theory through an example. Conclusions are made in Section 4.

2. Regularized Control

Definition 2 (well-posed problem). Let $\mathscr{U}$ and $\mathscr{V}$ be normed linear spaces and $\mathscr{L} : \mathscr{U} \to \mathscr{V}$ be a linear operator. The equation

$$\mathscr{L}u = v \tag{5}$$

is said to be well-posed if the following holds:

(i) For every $v \in \mathscr{V}$, there exists a unique $u \in \mathscr{U}$ such that $\mathscr{L}u = v$.

(ii) $\mathscr{L}^{-1}$ is a bounded operator. Equivalently, for every $v \in \mathscr{V}$ and for every $\epsilon > 0$, there exists a $\delta > 0$ with the following properties: If $\tilde{v} \in \mathscr{V}$ with $\|\tilde{v} - v\| \le \delta$ and if $u, \tilde{u} \in \mathscr{U}$ are such that $\mathscr{L}u = v$ and $\mathscr{L}\tilde{u} = \tilde{v}$, then $\|u - \tilde{u}\| \le \epsilon$.

Definition 3 (ill-posed problem). Equation (5) is said to be *ill-posed* if $\mathscr{L}$ violates one of the conditions for well-posedness.

Theorem 4 (Tikhonov regularization, see [14]). *Let $\mathscr{U}$ and $\mathscr{V}$ be Hilbert spaces and $\mathscr{L} : \mathscr{U} \to \mathscr{V}$ be a bounded linear operator. Then for each $v \in \mathscr{V}$ and $\lambda > 0$, there exists a unique $u_\lambda(v) \in \mathscr{U}$ which minimizes the map*

$$u \longmapsto \|\mathscr{L}u - v\|^2 + \lambda \|u\|^2, \quad u \in \mathscr{U}. \tag{6}$$

Moreover, for each $\lambda > 0$, the map

$$R_\lambda : v \longmapsto u_\lambda(v), \quad v \in \mathscr{V}, \tag{7}$$

is a bounded linear operator from $\mathscr{V}$ to $\mathscr{U}$ and $R_\lambda v := (\mathscr{L}^*\mathscr{L} + \lambda I)^{-1}\mathscr{L}^* v$, where $\mathscr{L}^*$ is the unique adjoint of the bounded linear operator $\mathscr{L}$.

Theorem 5 (see [14]). *For $\lambda > 0$, the solution u_λ of the operator equation*

$$(\mathscr{L}^*\mathscr{L} + \lambda I)\, u_\lambda = \mathscr{L}^* v \tag{8}$$

minimizes the function $u \to \|\mathscr{L}u - v\|^2 + \lambda\|u\|^2$, $u \in \mathscr{U}$, and $\|\mathscr{L}u_\lambda - v\| \to 0$ as $\lambda \to 0$.

Definition 6. For $v \in \mathscr{V}$ and $\lambda > 0$, the element $u_\lambda \in \mathscr{U}$ as in Theorems 4 and 5 is called the Tikhonov regularized solution of $\mathscr{L}u = v$.

Lemma 7. *Let $\mathscr{U}, \mathscr{V}$ be Hilbert spaces and $\mathscr{L} \in \mathscr{B}(\mathscr{U}, \mathscr{V})$. Then for $\lambda > 0$,*

$$\begin{gathered} \left\|(\mathscr{L}^*\mathscr{L} + \lambda I)^{-1}\right\| = \left\|(\mathscr{L}\mathscr{L}^* + \lambda I)^{-1}\right\| \le \frac{1}{\lambda}, \\ \left\|(\mathscr{L}^*\mathscr{L} + \lambda I)^{-1}\mathscr{L}^*\right\| = \left\|\mathscr{L}^*\left(\mathscr{L}\mathscr{L}^* + \lambda I\right)^{-1}\right\| \le \frac{1}{2\sqrt{\lambda}}. \end{gathered} \tag{9}$$

For more details on ill-posed problems and regularization methods one can refer to [14–20].

Let $L : Y \to V$ be a linear operator defined as

$$Lu = \int_{t_0}^{\tau} T(\tau - s)\, Bu(s)\, ds. \tag{10}$$

Assumption 8. (i) System (1) is approximately controllable.

(ii) $P = [M^2 bc(\tau - t_0)^{3/2} + 2Mc\sqrt{\lambda}(\tau - t_0)]/2\sqrt{\lambda} < 1$, where $\lambda > 0$ is a regularization parameter (to be chosen appropriately) and M, b are given by

$$\begin{gathered} \|T(t)\| \le M, \\ \|B\| \le b. \end{gathered} \tag{11}$$

In our analysis, we assume that the control system (1) satisfies Assumption 8. We obtain a sequence of controls and corresponding mild solutions for semilinear system (1) iteratively and also prove that this sequence of controls steers the semilinear control system from an initial state x^0 to an ϵ neighbourhood of the final state x_τ at time $\tau > 0$.

Consider

$$\begin{aligned} x_n(t) = T(t - t_0)\, x^0 &+ \int_{t_0}^{t} T(t - s)\, Bu_n(s)\, ds \\ &+ \int_{t_0}^{t} T(t - s)\, f\left(s, x_{n-1}(s)\right) ds, \end{aligned} \tag{12}$$

where $x_n(\tau) = x^1$, for all $n = 0, 1, 2, \ldots$, and $u_n(t)$ is a control function such that $x_n(\tau) = x^1$. We start with an initial (guess) mild solution $x_0(t)$. To find $u_n(t)$ such that $x_n(\tau) = x^1$, we need to solve

$$Lu_n = v_n, \tag{13}$$

where

$$\begin{aligned} Lu_n &= \int_{t_0}^{\tau} T(\tau - s) Bu_n(s)\, ds, \\ v_n &= x^1 - T(\tau - t_0) x^0 \\ &\quad - \int_{t_0}^{\tau} T(\tau - s) f(s, x_{n-1}(s))\, ds, \\ &\qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{14}$$

Since (13) is ill-posed in the sense of Hadamard [21], any small perturbations in v_n can lead to large deviations in the solution. Hence, in practice it is not advisable to solve (13) directly to obtain u_n; one has to look for stable approximations $u_{n,\lambda}$, $\lambda > 0$, such that $\|Lu_{n,\lambda} - v_n\| \to 0$ as $\lambda \to 0$. For this we shall use the Tikhonov regularization for obtaining the control function $u_{n,\lambda}$ which is given below:

$$u_{n,\lambda} = L^* (LL^* + \lambda I)^{-1} v_n, \quad n = 0, 1, 2, \ldots . \tag{15}$$

Convergence of $\{u_{n,\lambda}\}$ *and* $\{x_{n,\lambda}\}$. We have the sequence of regularized controls $\{u_{n,\lambda}\}$ and the sequence of corresponding mild solutions $\{x_{n,\lambda}\}$ for each $n \in \mathbb{N}$, $\lambda > 0$. The inner product and the corresponding norm on the function space $L_2(J,U)$ are given below.

For $u, w \in L_2(J,U)$,

$$\begin{aligned} \langle u, w\rangle_{L_2(J,U)} &= \int_J \langle u(s), w(s)\rangle_U\, ds, \\ \|w\|_{L_2(J,U)} &= \sqrt{\int_J \|w(s)\|_U^2\, ds}. \end{aligned} \tag{16}$$

Theorem 9. *Under Assumption 8 and for fixed $\lambda > c$, the sequences $\{u_{n,\lambda}\}$, $\{x_{n,\lambda}\}$ are convergent with respect to n in $L_2(J,U)$, $L_2(J,V)$, respectively.*

Proof. As $L_2(J,U)$, $L_2(J,V)$ are complete spaces, it is sufficient to prove $\{u_{n,\lambda}\}$ and $\{x_{n,\lambda}\}$ are Cauchy sequences in $L_2(J,U)$, $L_2(J,V)$, respectively.

We have $u_{n,\lambda} = L^*(LL^* + \lambda I)^{-1} v_n$,

$$\begin{aligned} &\|u_{n+1,\lambda} - u_{n,\lambda}\|_{L_2(J,U)} = \left\|L^* (LL^* + \lambda I)^{-1} (v_{n+1} - v_n)\right\|_{L_2(J,U)} \\ &\le \frac{1}{2\sqrt{\lambda}} \|v_{n+1} - v_n\|_V \le \frac{1}{2\sqrt{\lambda}} \left\|\int_{t_0}^{\tau} T(\tau - s) \right. \\ &\quad \cdot \left. [f(s, x_{n,\lambda}(s)) - f(s, x_{n-1,\lambda}(s))]\, ds\right\|_V \le \frac{1}{2\sqrt{\lambda}} \\ &\quad \cdot Mc \int_{t_0}^{\tau} \|x_{n,\lambda}(s) - x_{n-1,\lambda}(s)\|_V\, ds \le \frac{1}{2\sqrt{\lambda}} \\ &\quad \cdot Mc \sqrt{(\tau - t_0)}\, \|x_{n,\lambda} - x_{n-1,\lambda}\|_{L_2(J,V)} \le \frac{1}{2\sqrt{\lambda}} \\ &\quad \cdot Mc \sqrt{(\tau - t_0)} P \|x_{n-1,\lambda} - x_{n-2,\lambda}\|_{L_2(J,V)} \le \frac{1}{2\sqrt{\lambda}} \\ &\quad \cdot Mc \sqrt{(\tau - t_0)} P^2 \|x_{n-2,\lambda} - x_{n-3,\lambda}\|_{L_2(J,V)} \\ &\qquad \vdots \\ &\le \frac{1}{2\sqrt{\lambda}} Mc \sqrt{(\tau - t_0)} P^{n-1} \|x_{1,\lambda} - x_0\|_{L_2(J,V)} = \frac{1}{2\sqrt{\lambda}} \\ &\quad \cdot Mc \sqrt{(\tau - t_0)} P^{n-1} \|u_{1,\lambda} - u_{0,\lambda}\|_{L_2(J,U)}, \end{aligned} \tag{17}$$

where $P = [M^2 bc(\tau - t_0)^{3/2} + 2Mc\sqrt{\lambda}(\tau - t_0)]/2\sqrt{\lambda}$.

By Assumption 8 of (ii), $P < 1$; hence for large value of n, the sequence $\{u_{n,\lambda}\}$ is Cauchy. Therefore $\{u_{n,\lambda}\}$ converges.

Similarly, we have

$$\begin{aligned} x_{n,\lambda}(t) &= T(t - t_0) x^0 + \int_{t_0}^{t} T(t - s) Bu_{n,\lambda}(s)\, ds \\ &\quad + \int_{t_0}^{t} T(t - s) f(s, x_{n-1,\lambda}(s))\, ds, \\ \|x_{n+1,\lambda}(t) - x_{n,\lambda}(t)\|_V &= \left\| \int_{t_0}^{t} T(t - s) \right. \\ &\quad \cdot (Bu_{n+1,\lambda}(s) - Bu_{n,\lambda}(s))\, ds + \int_{t_0}^{t} T(t - s) \\ &\quad \cdot \left. (f(s, x_{n,\lambda}(s)) - f(s, x_{n-1,\lambda}(s)))\, ds \right\|_V \\ &\le M \int_{t_0}^{t} \|Bu_{n+1,\lambda}(s) - Bu_{n,\lambda}(s)\|_V\, ds \\ &\quad + M \int_{t_0}^{t} \|f(s, x_{n,\lambda}(s)) - f(s, x_{n-1,\lambda}(s))\|_V\, ds \\ &\le Mb \int_{t_0}^{t} \|u_{n+1,\lambda}(s) - u_{n,\lambda}(s)\|_U\, ds \\ &\quad + Mc \int_{t_0}^{t} \|x_{n,\lambda}(s) - x_{n-1,\lambda}(s)\|_V\, ds \\ &\le Mb \int_{t_0}^{\tau} \|u_{n+1,\lambda}(s) - u_{n,\lambda}(s)\|_U\, ds \\ &\quad + Mc \int_{t_0}^{\tau} \|x_{n,\lambda}(s) - x_{n-1,\lambda}(s)\|_V\, ds \\ &\le Mb \sqrt{(\tau - t_0)}\, \|u_{n+1,\lambda} - u_{n,\lambda}\|_{L_2(J,U)} \\ &\quad + Mc \sqrt{(\tau - t_0)}\, \|x_{n,\lambda} - x_{n-1,\lambda}\|_{L_2(J,V)} \\ &\le \frac{MbMc(\tau - t_0)}{2\sqrt{\lambda}} \|x_{n,\lambda} - x_{n-1,\lambda}\|_{L_2(J,V)} \\ &\quad + Mc \sqrt{(\tau - t_0)}\, \|x_{n,\lambda} - x_{n-1,\lambda}\|_{L_2(J,V)} \end{aligned}$$

$$\leq \frac{\left[M^2bc(\tau-t_0)^{3/2}+2Mc\sqrt{\lambda}(\tau-t_0)\right]}{2\sqrt{\lambda}}\|x_{n,\lambda} - x_{n-1,\lambda}\|_{L_2(J,V)}. \quad (18)$$

Thus

$$\begin{aligned}\|x_{n+1,\lambda}(t)-x_{n,\lambda}(t)\|_V &\leq \frac{\left[M^2bc(\tau-t_0)^{3/2}+2Mc\sqrt{\lambda}(\tau-t_0)\right]}{2\sqrt{\lambda}}\|x_{n,\lambda} - x_{n-1,\lambda}\|_{L_2(J,V)}.\end{aligned} \quad (19)$$

We have

$$\|x_{n+1,\lambda}-x_{n,\lambda}\|^2_{L_2(J,V)} = \int_J \|x_{n+1,\lambda}(t)-x_{n,\lambda}(t)\|^2_V\,dt. \quad (20)$$

From (19) and (20), we get

$$\begin{aligned}\|x_{n+1,\lambda}-x_{n,\lambda}\|_{L_2(J,V)} &\leq P\|x_{n,\lambda}-x_{n-1,\lambda}\|_{L_2(J,V)}\\ &\leq P^2\|x_{n-1,\lambda}-x_{n-2,\lambda}\|_{L_2(J,V)}\\ &\leq P^3\|x_{n-2,\lambda}-x_{n-3,\lambda}\|_{L_2(J,V)}\\ &\vdots\\ &\leq P^n\|x_{1,\lambda}-x_0\|_{L_2(J,V)}.\end{aligned} \quad (21)$$

Since $P < 1$, for large value of n, the sequence $\{x_{n,\lambda}\}$ is also Cauchy; hence it converges.

This completes the proof. □

Remark 10. In practice, to obtain better approximation to the sequence of controls, λ (regularization parameter) can be chosen such that $P < 1$; that is,

$$\begin{aligned}&\frac{M^2bc(\tau-t_0)^{3/2}+2Mc\sqrt{\lambda}(\tau-t_0)}{2\sqrt{\lambda}} < 1 \Longrightarrow\\ &M^2bc(\tau-t_0)^{3/2}+2Mc\sqrt{\lambda}(\tau-t_0) < 2\sqrt{\lambda} \Longrightarrow\\ &2\sqrt{\lambda}(Mc(\tau-t_0)-1) < -M^2bc(\tau-t_0)^{3/2} \Longrightarrow\\ &\lambda > \frac{M^4b^2c^2(\tau-t_0)^3}{(2-2Mc(\tau-t_0))^2} = Q \quad (\text{say}).\end{aligned} \quad (22)$$

If $Mc(\tau-t_0) \ll 1$ then Q is very small. Then we get better approximation.

In many practical semilinear control systems, the nonlinear part is a perturbation, in the sense that the Lipschitz constant is sufficiently small so that the system is approximately controllable. In particular, the regularization parameter $\lambda > c$, where c is very small. Then λ can also be chosen sufficiently small. Hence we get a regularized control close to the exact solution.

3. Application for an Approximately Controllable System

In this section, we illustrate the theory for an approximately controllable semilinear system. Let $u_\lambda := \lim_{n\to\infty} u_{n,\lambda}$ be the regularized control. Let x_λ be the mild solution corresponding to u_λ.

Then from Theorem 4 we see that

$$\|x_\lambda(\tau)-x(\tau)\| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0 \quad (23)$$

which shows that the target state corresponding to the regularized control $(x_\lambda(\tau))$ is close to the actual state $(x(\tau))$ to be attained.

Example 11. Consider the semilinear heat equation given by the partial differential equation

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t)+u(x,t)+f(t,z(x,t)), \quad x\in[0,\ell], \quad (24)$$

$$\begin{aligned}&z(0,t) = 0 = z(\ell,t),\\ &z(x,0) = g_0(x), \quad g_0\in L_2[0,\ell],\end{aligned} \quad (25)$$

where $z(x,t)$ represents the temperature at position x at time t, $g_0(x)$ is the initial temperature profile, and $u(x,t)$ is the heat input (control) along the rod and $f : J\times V \to V$ is a nonlinear function which is Lipschitz continuous.

We have

$$\begin{aligned}&J := [0,\tau],\\ &U = V = L_2[0,\ell],\\ &\mathscr{U} = L_2(J,U),\\ &\mathscr{V} = V.\end{aligned} \quad (26)$$

Define the operator A by

$$Ah = \frac{d^2h}{dx^2}, \quad h\in D(A), \quad (27)$$

where

$$D(A) := \left\{h\in L_2[0,\ell] : h''\in L_2[0,\ell],\ h(0)=0=h(\ell)\right\}. \quad (28)$$

Let $B = I$, the identity operator on $L_2[0,\ell]$. By using the notations $\tilde{z}(t) := z(\cdot,t)$, $\tilde{u}(t) := u(\cdot,t)$, $\tilde{f}(t,\tilde{z}(t)) := f(t,z(\cdot,t))$, (24) takes the form of a control system defined on $L_2[0,\ell]$ which is given below:

$$\frac{d\tilde{z}(t)}{dt} = A\tilde{z}(t)+\tilde{u}(t)+\tilde{f}(t,\tilde{z}(t)), \quad t\in[0,\tau], \quad (29)$$

$$\tilde{z}(0) = g_0. \quad (30)$$

The C_0 semigroup generated by the operator A [22] is

$$T(t)g = \sum_{j=1}^{\infty} e^{(-j^2\pi^2/\ell^2)t} \langle g, \phi_j \rangle \phi_j, \quad \text{where } \phi_j(x) = \sqrt{\frac{2}{\ell}} \sin\left(\frac{j\pi x}{\ell}\right). \tag{31}$$

For $\tilde{u} \in \mathscr{U}$, the mild solution of (29) is given by

$$\tilde{z}(t) = T(t) g_0 + \int_0^t T(t-s)\tilde{u}(s)\,ds + \int_0^t T(t-s)\tilde{f}(s, \tilde{z}(s))\,ds. \tag{32}$$

Let $L_0 : \mathscr{U} \to \mathscr{V}$ be the operator defined by

$$L_0 \tilde{u} = \int_0^{\tau} T(\tau - s)\tilde{u}(s)\,ds. \tag{33}$$

Then we have

$$\begin{aligned} (L_0^* v)(s) &:= \sum_{j=1}^{\infty} e^{(-j^2\pi^2/\ell^2)(\tau-s)} \langle v, \phi_j \rangle \phi_j, \\ L_0 L_0^* v &= \sum_{j=1}^{\infty} \sigma_j^2 \langle v, \phi_j \rangle \phi_j, \\ &\text{where } \sigma_j^2 := \left[\frac{1 - e^{((-2j^2\pi^2/\ell^2)\tau)}}{2j^2\pi^2}\right] \ell^2. \end{aligned} \tag{34}$$

Since the semigroup (31) is compact, L is a compact operator; consequently the control system (24) is approximately controllable. The control system (24) satisfies Assumption 8. Hence, the regularized control of system (24) for a given target state g_τ (desired temperature profile) is obtained as follows:

$$\tilde{z}_{n,\lambda}(t) = T(t) g_0 + \int_0^t T(t-s)\tilde{u}_{n,\lambda}(s)\,ds + \int_0^t T(t-s)\tilde{f}(s, \tilde{z}_{n-1,\lambda}(s))\,ds, \tag{35}$$

where $\tilde{z}_{n,\lambda}(\tau) = g_\tau$, for all $n = 0, 1, 2, \ldots$, and $\tilde{u}_{n,\lambda}(t)$ is a control function such that $\tilde{z}_{n,\lambda}(\tau) = g_\tau$.

$$\begin{aligned} L_0 L_0^* v_n &= \sum_{j=1}^{\infty} \sigma_j^2 \langle v_n, \phi_j \rangle \phi_j, \\ (L_0 L_0^* + \lambda I) v_n &= \sum_{j=1}^{\infty} \left(\sigma_j^2 + \lambda\right) \langle v_n, \phi_j \rangle \phi_j, \\ \left(L_0^* \phi_j\right)(s) &= e^{(-j^2\pi^2/\ell^2)(\tau-s)} \phi_j. \end{aligned} \tag{36}$$

We have

$$\tilde{u}_{n,\lambda} = L_0^* \left(L_0 L_0^* + \lambda I\right)^{-1} v_n. \tag{37}$$

Thus, using (36) in (37) we get

$$\begin{aligned} \tilde{u}_{n,\lambda}(s) &= \sum_{j=1}^{\infty} \frac{1}{\sigma_j^2 + \lambda} \langle v_n, \phi_j \rangle \left(L_0^* \phi_j\right)(s) \\ &= \sum_{j=1}^{\infty} \frac{e^{-j^2\pi^2(\tau-s)/\ell^2}}{\sigma_j^2 + \lambda} \langle v_n, \phi_j \rangle \phi_j, \quad n \in \mathbb{N}. \end{aligned} \tag{38}$$

Error involved in the regularization procedure is given by

$$v_n - L_0 \tilde{u}_{n,\lambda} = \sum_{j=1}^{\infty} \frac{\lambda}{\sigma_j^2 + \lambda} \langle v_n, \phi_j \rangle \phi_j, \tag{39}$$

$$\|v_n - L_0 \tilde{u}_{n,\lambda}\|^2 = \sum_{j=1}^{\infty} \frac{\lambda^2}{\left(\sigma_j^2 + \lambda\right)^2} \left|\langle v_n, \phi_j \rangle\right|^2, \quad n \in \mathbb{N}. \tag{40}$$

From (40), it is clear that $\|v_n - L_0 \tilde{u}_{n,\lambda}\| \to 0$ as $\lambda \to 0$.

Problem 12. Consider (24) and (25) with $f(t, z(x,t)) = c\int_0^1 \|z(x,t)\|^2 dx$, and $g_0(x) = \sin(\pi x)$, $t \in J := [0, 2]$, $\ell = 1$, $x \in [0, 1]$; that is,

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + u(x,t) + c\int_0^1 \|z(x,t)\|^2 dx, \tag{41}$$

$$\begin{aligned} z(0,t) &= 0 = z(1,t), \\ z(x,0) &= \sin(\pi x). \end{aligned} \tag{42}$$

Here we have the system constants: $M = 1$, $\tau = 2$, $t_0 = 0$, and c is the Lipschitz constant.

In order to obtain better approximation to the regularized control, the regularization parameter λ can be chosen in such a way that $\lambda > 8c^2/(2 - 4c)^2$, $c \neq 1/2$. Then the semilinear control system (41) satisfies Assumption 8. Hence, the convergence of the sequences of regularized controls $\{u_{n,\lambda}\}$ and the corresponding mild solutions $\{x_{n,\lambda}\}$ follows from Theorem 9.

4. Conclusions

In the mathematical control theory literature, Tikhonov regularization is not given much attention to the problems related to approximately controllable system. We use the Tikhonov regularization method and exhibited a sequence of regularized controls and their corresponding mild solutions. The convergence of the sequences under some assumptions has also been established. The results are illustrated with an example. However, the case where $B \neq I$ should be considered for future work as the theory will change substantially.

Competing Interests

The authors declare that they have no competing interests.

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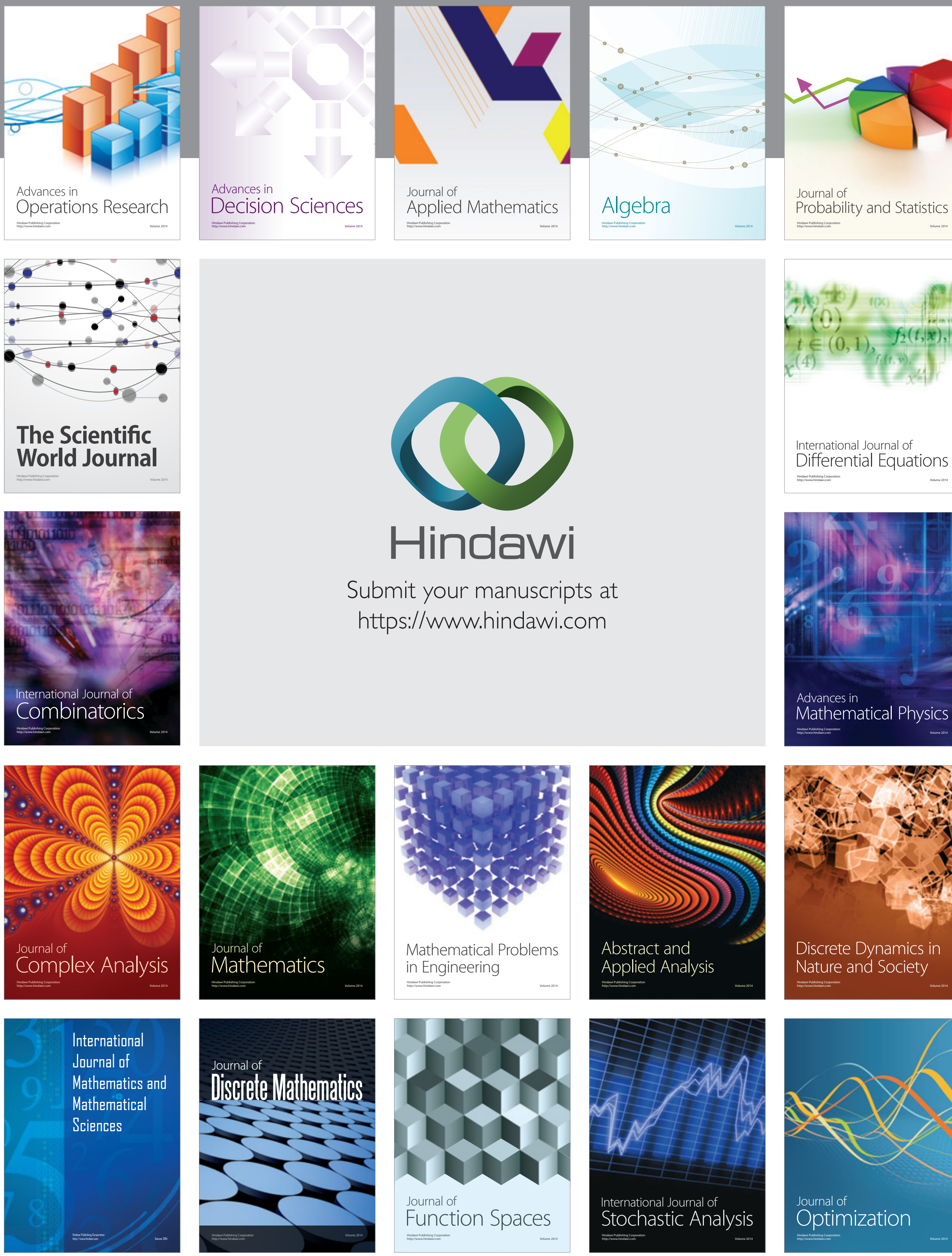
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