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Research Article

Linearization of Fifth-Order Ordinary Differential Equations by Generalized Sundman Transformations

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In this article, the linearization problem of fifth-order ordinary differential equation is presented by using the generalized Sundman transformation. The necessary and sufficient conditions which allow the nonlinear fifth-order ordinary differential equation to be transformed to the simplest linear equation are found. There is only one case in the part of sufficient conditions which is surprisingly less than the number of cases in the same part for order 2, 3, and 4. Moreover, the derivations of the explicit forms for the linearizing transformation are exhibited. Examples for the main results are included.

1. Introduction

Nonlinear problems are of interest to engineers, physicists, mathematicians and many other scientists since most equations are inherently nonlinear in nature. Although linear ordinary differential equations can be solved by a large number of methods but this situation does not hold for nonlinear equations. One common method to solve nonlinear ordinary differential equations is to change their unknowns by suitable variables so as to get linear ordinary differential equations.

The main tools used to solve the linearization problem are transformations such as point, contact, tangent, and generalized Sundman transformations.

It was recognized that Lie [1] is the first person who solved linearization problem for ordinary differential equations in 1883. He discovered the linearization of second-order ordinary differential equations by point transformations. Later, Liouville [2] and Tresse [3] attacked the equivalence problems for second-order ordinary differential equations via group of point transformations. Moreover, Cartan [4] approached the second-order ordinary differential equations by geometric structure of a certain form.

Mahomed and Leach [5] indicated that the nth-order ($n > 3$) linear ordinary differential equation has exactly one of $n+1$, $n+2$, or $n+4$ point symmetries. They suggested that the necessary and sufficient conditions for the nth-order ($n \geq 3$) to be linearizable by a point transformation must admit the n dimensional Abelian algebra.

The linearization problem of third-order ordinary differential equations under point transformations was solved by Bocharov et al. [6], Grebot [7], and Ibragimov and Meleshko [8]. Fourth-order ordinary differential equation was studied by Ibragimov et al. [9]. They found the necessary and sufficient conditions for a complete linearization problem. The linearization problem of a fifth-order ordinary differential equation with respect to fiber preserving transformations was considered by Suksern and Pinyo [10].

In the series of articles [8, 11–14] the linearization problem of a third-order ordinary differential equation via the contact transformations was solved. For a fourth-order ordinary differential equation, this problem was studied in [15, 16]. The criteria of the linearization problem of fifth-order ordinary differential equations were discovered by Suksern [17].

The linearization problems of third-order and fourth-order ordinary differential equations by the tangent transformations are examined in [18, 19]. These are the first application of tangent (essentially) transformations to the linearization problems of third-order and fourth-order ordinary

differential equations. Necessary and sufficient conditions for third-order and fourth-order ordinary differential equations to be linearizable are obtained there.

Sundman introduced the generalized Sundman transformations in 1992. Later on Duarte et al. [20] applied this method to transform second-order ordinary differential equations into free particle equations. In addition, Muriel and Romero [21] characterized the equations that can be linearized by means of generalized Sundman transformations in terms of first integral. A new characterization of linearizable equations in terms of the coefficients of ordinary differential equation and one auxiliary function was given by Mustafa et al. [22]. Moreover, Nakpim and Meleshko [23] pointed out that the solution given by Duarte et al. using the Laguerre form is not complete.

For the third-order ordinary differential equations, the linearization by the generalized Sundman transformation was investigated by [24] for the form $X'''(T) = 0$ and [25] for the Laguerre form. Some applications of the generalized Sundman transformation to ordinary differential equations can be found in [26]. More information of the generalized Sundman transformation are collected in the book [27].

The linearization problem of a fourth-order ordinary differential equation with respect to generalized Sundman transformations was studied in [28]. They found the necessary and sufficient conditions which allow the fourth-order ordinary differential equation to be transformed to the simplest linear equation.

In this article, we intend to use the generalized Sundman transformations to linearize the fifth-order ordinary differential equations in some particular cases. We use computer algebra system Reduce to compute the necessary and sufficient conditions of the linearization. We provide some examples to illustrate the conditions that we have found and also obtain the linearizing transformations.

2. Necessary Conditions

We now concentrate on finding the fifth-order ordinary differential equations

$$x^{(5)} = f\left(t, x, x', x'', x''', x^{(4)}\right), \tag{1}$$

which can be transformed to the linear equation

$$X^{(5)}(T) = 0, \tag{2}$$

under the generalized Sundman transformation

$$\begin{aligned} X &= F(t, x), \\ dT &= G(t, x)\, dt. \end{aligned} \tag{3}$$

It turns out that those equations must be in the form of the following theorem.

Theorem 1. *Any linearizable fifth-order ordinary differential equations that can be transformed by a generalized Sundman transformation has to be in the form*

$$\begin{aligned} x^{(5)} &+ \left(A_1 x' + A_0\right) x^{(4)} \\ &+ \left(B_3 x'' + B_2 x'^2 + B_1 x' + B_0\right) x''' \\ &+ \left(C_1 x' + C_0\right) x''^2 \\ &+ \left(D_3 x'^3 + D_2 x'^2 + D_1 x' + D_0\right) x'' + H_5 x'^5 \\ &+ H_4 x'^4 + H_3 x'^3 + H_2 x'^2 + H_1 x' + H_0 = 0. \end{aligned} \tag{4}$$

Here $A_i = A_i(t, x)$, $B_i = B_i(t, x)$, $C_i = C_i(t, x)$, $D_i = D_i(t, x)$, and $H_i = H_i(t, x)$ are some functions of t and x. Expressions of these coefficients are presented in the appendix.

Proof. By a generalized Sundman transformation (3), we have

$$\begin{aligned}
X'(T) &= \frac{D_t F(t, x)}{D_t \int G(t, x)\, dt} = \frac{F_t + x' F_x}{G} = P\left(t, x, x'\right), \\
X''(T) &= \frac{D_t P\left(t, x, x'\right)}{D_t \int G(t, x)\, dt} = \frac{P_t + x' P_x + x'' P_{x'}}{G} \\
&= \frac{1}{G^3}\left[2F_{tx} G x' + F_{tt} G - F_t G_t - F_t G_x x' + F_{xx} G x'^2\right. \\
&\quad \left. - F_x G_t x' - F_x G_x x'^2 + F_x G x''\right] = Q\left(t, x, x', x''\right), \\
X'''(T) &= \frac{D_t Q}{D_t \int G(t, x)\, dt} \\
&= \frac{Q_t + x' Q_x + x'' Q_{x'} + x''' Q_{x''}}{G} = \frac{1}{G^5}\left[\left(F_x G^2\right) x''' \right. \\
&\quad + G\left(3F_{xx} G - 4F_x G_x\right) x' x'' \\
&\quad \left. + G\left(3F_{tx} G - F_t G_x - 3F_x G_t\right) x'' + \cdots\right] \\
&= R\left(t, x, x', x'', x'''\right), \\
X^{(4)}(T) &= \frac{D_t R}{D_t \int G(t, x)\, dt} \\
&= \frac{R_t + x' R_x + x'' R_{x'} + x''' R_{x''} + x^{(4)} R_{x'''}}{G} \\
&= \frac{1}{G^7}\left[\left(F_x G^3\right) x^{(4)} + G^2\left(4F_{xx} G - 7F_x G_x\right) x' x''' \right. \\
&\quad \left. + G^2\left(4F_{tx} G - F_t G_x - 6F_x G_t\right) x''' + \cdots\right] \\
&= S\left(t, x, x', x'', x''', x^{(4)}\right), \\
X^{(5)}(T) &= \frac{D_t S}{D_t \int G(t, x)\, dt} \\
&= \frac{S_t + x' S_x + x'' S_{x'} + x''' S_{x''} + x^{(4)} S_{x'''} G + x^{(5)} S_{x^{(4)}}}{G} \\
&= \frac{1}{G^9}\left[\left(F_x G^4\right) x^{(5)} + G^3\left(5F_{xx} G - 11F_x G_x\right) x' x^{(4)}\right.
\end{aligned}$$

$$+ G^3 \left(5F_{tx}G - F_tF_x - 10F_xG_t\right) x^{(4)} + \cdots \Big]$$
$$= V\left(t, x, x', x'', x''', x^{(4)}, x^{(5)}\right), \tag{5}$$

where $D_t = \partial/\partial t + x'(\partial/\partial x) + x''(\partial/\partial x') + x'''(\partial/\partial x'') + x^{(4)}(\partial/\partial x''') + x^{(5)}(\partial/\partial x^{(4)}) + \cdots$ is a total derivative. Replacing $X^{(5)}(T)$ in (2), we get that

$$\begin{aligned}
&x^{(5)} + \left(\left(\frac{5F_{xx}G - 11F_xG_x}{F_xG}\right)x' + \left(\frac{5F_{tx}G - F_tG_x - 10F_xG_t}{F_xG}\right)\right)x^{(4)} + \left(\left(\frac{5\left(2F_{xx}G - 3F_xG_x\right)}{F_xG}\right)x''\right.\\
&\quad + \left(\frac{10F_{xxx}G^2 - 45F_{xx}G_xG - 14F_xG_{xx}G + \cdots}{F_xG^2}\right)x'^2\\
&\quad + \left(\frac{20F_{txx}G^2 - 50F_{tx}G_xG - 4F_tG_{xx}G + 15F_tG_x^2 - 40F_{xx}G_tG + \cdots}{F_xG^2}\right)x' + \cdots\Big)x'''\\
&\quad + \left(\left(\frac{15F_{xxx}G^2 - 60F_{xx}G_xG - 18F_xG_{xx}G + 70F_xG_x^2}{F_xG^2}\right)x'\right.\\
&\quad + \left.\left(\frac{15F_{txx}G^2 - 30F_{tx}G_xG - 3F_tG_{xx}G + 10F_tG_x^2 - 30F_{xx}G_tG + \cdots}{F_xG^2}\right)\right)x''^2\\
&\quad + \left(\left(\frac{10F_{xxxx}G^3 - 70F_{xxx}G_xG^2 - 45F_{xx}G_{xx}G^2 + 195F_{xx}G_x^2G + \cdots}{F_xG^3}\right)x'^3 + \cdots\right)x''\\
&\quad + \left(\frac{F_{xxxxx}G^4 - 10F_{xxxx}G_xG^3 - 10F_{xxx}G_{xx}G^3 + 45F_{xxx}G_x^2G^2 + \cdots}{F_xG^4}\right)x'^5 + \cdots = 0.
\end{aligned} \tag{6}$$

Denoting A_i, B_i, C_i, D_i, and H_i as (A.1)–(A.18), we obtain the necessary form (4). This proves the theorem. □

3. Sufficient Conditions and Linearizing Transformation

To get the sufficient conditions, we consider (A.1)–(A.18) appearing in the previous section. After using the compatibility theory to those equations, we derive the following results.

Theorem 2. *Equation (4) can be linearizable by the generalized Sundman transformation if its coefficients satisfy the following equations:*

$$S_{4x} = \frac{\left(7A_1S_4 + 49S_1 + 23S_4^2\right)}{280}, \tag{7}$$

$$S_{8t} = \left(27720S_2S_8 - 8520S_5S_8 + 62S_6S_8 + 4928S_7 - 115S_8^2\right)\left(277200S_4\right)^{-1}, \tag{8}$$

$$S_{8x} = \left(21A_1S_4S_8 + 147S_1S_8 + 73920S_2S_4^2 - 22720S_4^2S_5 - 40S_4^2S_6 + 69S_4^2S_8\right)\left(840S_4\right)^{-1}, \tag{9}$$

$$\begin{aligned}
B_{2xx} = {}&\Big(-2195200B_{2x}A_1 + 823200B_{2x}S_4 - 548800S_{1xx} - 644840S_{1x}A_1 + 82320S_{1x}S_4 + 10633A_1^3S_4 - 10633A_1^2S_1\\
&+ 1519A_1^2S_4^2 + 89180A_1B_2S_4 + 72716A_1S_1S_4 + 2604A_1S_4^3 - 548800B_2^2 - 260680B_2S_1 + 11760B_2S_4^2 + 54880000H_5\\
&+ 43218S_1^2 + 1176S_1S_4^2 + 558S_4^4\Big)\left(5488000\right)^{-1},
\end{aligned} \tag{10}$$

$$\begin{aligned}
S_{1x} = {}&\left(-78400B_{2x} - 637A_1^2S_4 - 15680A_1B_2 - 5635A_1S_1 + 105A_1S_4^2 + 2940B_2S_4 + 78400D_3 + 294S_1S_4 + 30S_4^3\right)\\
&\cdot\left(5880\right)^{-1},
\end{aligned} \tag{11}$$

$$C_{0x} = \left(-5488560A_{1t}S_4^2 - 38419920S_{1t}S_4 + 25347840S_{5x}S_4 + 75600S_{6x}S_4 - 18627840A_1C_0S_4 - 6137208A_1S_2S_4 + 6337320A_1S_4S_5 + 16929A_1S_4S_6 + 6573A_1S_4S_8 + 23950080C_0S_4^2 + 46569600D_2S_4 + 2794176S_1S_2 - 5188848S_1S_5 - 12096S_1S_6 - 6237S_1S_8 - 5207664S_2S_4^2 + 2080288S_4^2S_5 + 3682S_4^2S_6 + 5925S_4^2S_8\right)\left(93139200S_4\right)^{-1}, \tag{12}$$

$$A_{1t} = \left(-133056A_1S_2S_4 + 168240A_1S_4S_5 + 576A_1S_4S_6 - 231A_1S_4S_8 + 443520C_0S_4^2 + 25872S_1S_5 + 231S_1S_8 - 133056S_2S_4^2 + 21424S_4^2S_5 + 144S_4^2S_6 - 215S_4^2S_8\right)\left(1330560S_4^2\right)^{-1}, \tag{13}$$

$$S_{1t} = \left(13587840S_{5x}S_4 + 40320S_{6x}S_4 - 1862784A_1S_2S_4 + 2662464A_1S_4S_5 + 7056A_1S_4S_6 + 2541A_1S_4S_8 + 6209280C_0S_4^2 + 1862784S_1S_2 - 3235008S_1S_5 - 8064S_1S_6 - 2541S_1S_8 - 1862784S_2S_4^2 + 483392S_4^2S_5 + 504S_4^2S_6 + 2165S_4^2S_8\right) \cdot \left(18627840S_4\right)^{-1}, \tag{14}$$

$$S_{3x} = \left(-\left(\left(246400\left(144\left(3\left(308B_0 + 491S_3\right)S_4 + 5S_{6t}\right)S_4^2 - \left(337S_{5x} + S_{6x}\right)\left(112S_5 + S_8\right)\right) - \left(7020925440S_5^2 + 33795200S_5S_6 - 2623280S_5S_8 + 61440S_6^2 - 41624S_6S_8 + 2776064S_7 + 52785S_8^2\right)S_4\right)S_4 - 77\left(149022720S_3S_4^2 - 23206400S_5^2 - 62720S_5S_6 - 225680S_5S_8 - 224S_6S_8 + 21504S_7 - 765S_8^2\right)S_1 + 443520\left(32S_6 + 23S_8 + 27344S_5\right)S_2S_4^2 + 77\left(949760S_5^2 + 8960S_5S_6 - 10000S_5S_8 + 416S_6S_8 + 21504S_7 - 765S_8^2 + 149022720S_3S_4^2\right) \cdot A_1S_4\right)\right)\left(65569996800S_4^3\right)^{-1}, \tag{15}$$

$$S_{5x} = \left(-155232A_1S_4S_5 - 1617A_1S_4S_8 + 362208S_1S_5 + 1617S_1S_8 - 680064S_2S_4^2 - 69856S_4^2S_5 - 136S_4^2S_6 - 1449S_4^2S_8\right) \cdot \left(1034880S_4\right)^{-1}, \tag{16}$$

$$S_{7x} = \left(77616000S_{6t}S_4^3 - 64680S_{6x}S_4S_8 + 1617A_1S_4S_6S_8 + 206976A_1S_4S_7 + 20490624000B_0S_4^4 + 11319S_1S_6S_8 + 1448832S_1S_7 + 4719052800S_2S_4^2S_5 + 2069760S_2S_4^2S_6 + 17001600S_2S_4^2S_8 + 18255283200S_3S_4^4 - 3786137600S_4^2S_5^2 - 18085760S_4^2S_5S_6 - 5225600S_4^2S_5S_8 - 30520S_4^2S_6^2 - 3887S_4^2S_6S_8 - 1022336S_4^2S_7\right)\left(4139520S_4\right)^{-1}, \tag{17}$$

$$C_1 = \frac{\left(9A_1S_4 + 60B_2 + 3S_1 + S_4^2\right)}{40}, \tag{18}$$

and (A.19), (A.20), (A.21), (A.22), (A.23), (A.24), (A.25) (moved to the appendix in order to avoid the huge expressions), where

$$S_1 = -10A_{1x} - 2A_1^2 + 5B_2,$$
$$S_2 = -20A_{0x} - 4A_0A_1 + 5B_1,$$
$$S_3 = 10A_{0t} + 2A_0^2 - 5B_0,$$
$$S_4 = -2A_1 + B_3,$$
$$S_5 = -80S_{4t} + 2A_0S_4 + 7S_2,$$
$$S_6 = -462A_0S_4 + 231S_2 - 337S_5,$$
$$S_7 = -13860S_{5t}S_4 + 1386S_2S_5 + 8316S_3S_4^2 - 202S_5^2 + S_5S_6,$$
$$S_8 = -41580B_1 + 55440C_0 - 8316S_2 + 9176S_5 + 27S_6. \tag{19}$$

Proof. We start with the coefficients A_i, B_i, C_i, D_i, and H_i in Theorem 1 through the unknown functions F and G. From (A.1) and (A.2), we have the derivatives

$$F_{xx} = \frac{\left(F_x\left(11G_x + A_1G\right)\right)}{(5G)}, \tag{20}$$

$$F_{tx} = \frac{\left(F_tG_x + 10F_xG_t + F_xA_0G\right)}{(5G)}. \tag{21}$$

From (A.4), one obtains the derivative

$$G_{xx} = \frac{\left(63G_x^2 + G_xA_1G + G^2S_1\right)}{(40G)}, \tag{22}$$

where

$$S_1 = -10A_{1x} - 2A_1^2 + 5B_2. \tag{23}$$

From (A.5), one gets the derivative

$$G_{tx} = \frac{\left(-9F_tG_x^2 + 135F_xG_tG_x + F_xG\left(2G_xA_0 + GS_2\right)\right)}{\left(80F_xG\right)}, \tag{24}$$

where

$$S_2 = -20A_{0x} - 4A_0A_1 + 5B_1. \tag{25}$$

From (A.6), one finds the derivative

$$F_{tt} = \frac{\left(9F_t^2G_x^2 + 225F_tF_xG_tG_x + F_tF_xG\left(14G_xA_0 - GS_2\right) + 400F_x^2G_{tt}G - 600F_x^2G_t^2 + 8F_x^2G^2S_3\right)}{\left(120F_xG_xG\right)}, \tag{26}$$

where

$$S_3 = 10A_{0t} + 2A_0^2 - 5B_0. \tag{27}$$

From (A.3), one obtains the derivative

$$G_x = \frac{\left(GS_4\right)}{7}, \tag{28}$$

where

$$S_4 = -2A_1 + B_3. \tag{29}$$

We note that, for the case $G_x = 0$, the generalized Sundman transformations are indeed the point transformations. We then suppose $G_x \neq 0$, which also implies $S_4 \neq 0$.

The relations $(G_x)_x = G_{xx}$ and $(G_x)_t = G_{tx}$ provide condition (7) and the derivative

$$F_t = \frac{\left(385F_xG_tS_4 + 7F_xGS_5\right)}{\left(9GS_4^2\right)}, \tag{30}$$

where

$$S_5 = -80S_{4t} + 2A_0S_4 + 7S_2. \tag{31}$$

The relation $(F_t)_t = F_{tt}$ gives the derivative

$$G_{tt} = \frac{\left(-2156000G_t^2S_4^2 + 385G_tGS_4S_6 + 4G^2S_7\right)}{\left(1386000GS_4^2\right)}, \tag{32}$$

where

$$\begin{aligned} S_6 &= -462A_0S_4 + 231S_2 - 337S_5, \\ S_7 &= -13860S_{5t}S_4 + 1386S_2S_5 + 8316S_3S_4^2 - 202S_5^2 \\ &\quad + S_5S_6. \end{aligned} \tag{33}$$

Substituting A_0 into A_{0x} and A_{0t}, one obtains the conditions

$$\begin{aligned} S_{2x} &= \left(94360S_{5x}S_4 + 280S_{6x}S_4 - 11319A_1S_2S_4 \right. \\ &\quad + 16513A_1S_4S_5 + 49A_1S_4S_6 + 32340B_1S_4^2 \\ &\quad + 11319S_1S_2 - 16513S_1S_5 - 49S_1S_6 - 1155S_2S_4^2 \\ &\quad \left. - 7751S_4^2S_5 - 23S_4^2S_6\right)\left(64680S_4\right)^{-1}, \\ S_{2t} &= \left(889680S_{5t}S_4 + 2640S_{6t}S_4 + 609840B_0S_4^2 \right. \\ &\quad + 70224S_2S_5 + 231S_2S_6 + 121968S_3S_4^2 - 102448S_5^2 \\ &\quad \left. - 641S_5S_6 - S_6^2\right)\left(609840S_4\right)^{-1}. \end{aligned} \tag{34}$$

From (A.8), we have

$$G_t = \frac{\left(GS_8\right)}{\left(6160S_4\right)}, \tag{35}$$

where

$$\begin{aligned} S_8 &= -41580B_1 + 55440C_0 - 8316S_2 + 9176S_5 \\ &\quad + 27S_6. \end{aligned} \tag{36}$$

The relations $(G_t)_t = G_{tt}$ and $(G_t)_x = G_{tx}$ provide conditions (8) and (9). From (A.18), (A.15), (A.17), (A.13), and (A.11), we obtain conditions (A.19)–(A.21), (10), (A.22). Substituting the relation C_{0t} into C_{0tt}, one obtains condition (A.23). Equations (A.9), (A.10), and (A.12) provide conditions (11), (12), (A.24). Comparing the mixed derivatives $(F_{xx})_t = (F_{tx})_x$, $(G_{xx})_t = (G_{tx})_x$, $(F_{tt})_x = (F_{tx})_t$, $(F_t)_x = F_{tx}$, we obtain conditions (13)–(16). Substituting the relation S_{5x} into S_{5xx}, one obtains condition (A.25). Comparing the mixed derivative $(G_{tt})_x = (G_{tx})_t$, one arrives at condition (17). From (A.7), one obtains condition (18). This proves the theorem. □

Corollary 3. *Under the sufficient conditions in Theorem 2, the transformation (3) mapping equation (4) to a linear equation (2) can be solved by the compatible system of (20), (28), (30), and (35).*

Remark 4. In the part of sufficient conditions for second-order, there are 2 cases in [20] and 3 cases in [23]. For the third-order, there are 3 cases in [24] and 4 cases in [25]. For the fourth-order, there are 2 cases in [28]. But for the fifth-order there is only one case.

4. Examples

Example 1. For the fifth-order ordinary differential equation

$$x^{(5)}x^4 - 11x'x^{(4)}x^3 - 15x''x'''x^3 + 60x'^2x'''x^2 + 70x'x''^2x^2 - 210x'^3x''x + 105x'^5 = 0, \tag{37}$$

we can verify that this equation cannot be linearized by a point transformation [10] or contact transformation [17]. However, (37) is in fact the form (4) in Theorem 1 with the coefficients

$$\begin{aligned}
A_1 &= \frac{-11}{x}, \\
A_0 &= 0, \\
B_3 &= \frac{-15}{x}, \\
B_2 &= \frac{60}{x^2}, \\
B_1 &= 0, \\
B_0 &= 0, \\
C_1 &= \frac{70}{x^2}, \\
C_0 &= 0, \\
D_3 &= \frac{-210}{x^3}, \\
D_2 &= 0, \\
D_1 &= 0, \\
D_0 &= 0, \\
H_5 &= \frac{105}{x^4}, \\
H_4 &= 0, \\
H_3 &= 0, \\
H_2 &= 0, \\
H_1 &= 0, \\
H_0 &= 0, \\
S_1 &= \frac{-52}{x^2}, \\
S_2 &= 0, \\
S_3 &= 0, \\
S_4 &= \frac{7}{x}, \\
S_5 &= 0, \\
S_6 &= 0, \\
S_7 &= 0, \\
S_8 &= 0.
\end{aligned} \tag{38}$$

Moreover, these coefficients also satisfy the conditions in Theorem 2. We now conclude that (37) is linearizable by a generalized Sundman transformation. Corollary 3 yields the linearizing transformation by solving the following equations:

$$F_{xx} = 0, \tag{39}$$

$$G_x = \frac{G}{x}, \tag{40}$$

$$G_t = 0, \tag{41}$$

$$F_t = 0. \tag{42}$$

Considering (40), one arrives at

$$G = xK_1(t). \tag{43}$$

Considering (41), one obtains

$$G = K_2(x). \tag{44}$$

From (43) and (44), one can choose $K_1(t) = 1$ and $K_2(x) = x$; then we have

$$G = x. \tag{45}$$

Considering (39), one gets

$$F = K_3(t)x + K_4(t). \tag{46}$$

Considering (42), one arrives at

$$F = K_5(x). \tag{47}$$

From (46) and (47), one can choose $K_3(t) = 1$, $K_4(t) = 0$, and $K_5(x) = x$; then we obtain

$$F = x. \tag{48}$$

So the linearizing transformation is

$$\begin{aligned}
X &= x, \\
dT &= xdt.
\end{aligned} \tag{49}$$

Hence, by (49), (37) becomes

$$X^{(5)} = 0. \tag{50}$$

The general solution of (50) is

$$X = \frac{c_1}{24}T^4 + \frac{c_2}{6}T^3 + \frac{c_3}{2}T^2 + c_4T + c_5, \tag{51}$$

where c_1, c_2, c_3, c_4, and c_5 are arbitrary constants. Substituting (49) into (51), the general solution of (37) is

$$x(t) = \frac{c_1}{24}\phi(t)^4 + \frac{c_2}{6}\phi(t)^3 + \frac{c_3}{2}\phi(t)^2 + c_4\phi(t) + c_5, \quad (52)$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \frac{c_1}{24}T^4 + \frac{c_2}{6}T^3 + \frac{c_3}{2}T^2 + c_4T + c_5. \quad (53)$$

Example 2. For the fifth-order ordinary differential equation

$$\begin{aligned} &x^{(5)}tx^4 - 22x'x^{(4)}tx^3 + 3x^{(4)}x^4 - 30x''x'''tx^3 \\ &\quad + 212x'^2x'''tx^2 - 48x'x'''x^3 + 244x'x''^2tx^2 \\ &\quad - 26x''^2x^3 - 1180x'^3x''tx + 320x'^2x''x^2 \\ &\quad + 880x'^5t - 320x'^4x = 0 \end{aligned} \quad (54)$$

we can verify that this equation cannot be linearized by a point transformation [10] or contact transformation [17]. However, (54) is in fact the form (4) in Theorem 1 with the coefficients

$$\begin{aligned} A_1 &= \frac{-22}{x}, \\ A_0 &= \frac{3}{t}, \\ B_3 &= \frac{-30}{x}, \\ B_2 &= \frac{212}{x^2}, \\ B_1 &= \frac{-48}{tx}, \\ B_0 &= 0, \\ C_1 &= \frac{244}{x^2}, \\ C_0 &= \frac{-26}{tx}, \\ D_3 &= \frac{-1180}{x^3}, \\ D_2 &= \frac{320}{tx^2}, \\ D_1 &= 0, \\ D_0 &= 0, \\ H_5 &= \frac{880}{x^4}, \\ H_4 &= \frac{-320}{tx^3}, \\ H_3 &= 0, \\ H_2 &= 0, \\ H_1 &= 0, \\ H_0 &= 0, \\ S_1 &= \frac{-128}{x^2}, \\ S_2 &= \frac{24}{tx}, \\ S_3 &= \frac{-12}{t^2}, \\ S_4 &= \frac{14}{x}, \\ S_5 &= \frac{252}{tx}, \\ S_6 &= \frac{-98784}{tx}, \\ S_7 &= 0, \\ S_8 &= 0. \end{aligned} \quad (55)$$

Moreover, these coefficients also satisfy the conditions in Theorem 2. We now conclude that (54) is linearizable by a generalized Sundman transformation. Corollary 3 yields the linearizing transformation by solving the following equations:

$$F_{xx} = 0, \quad (56)$$

$$G_x = \frac{(2G)}{x}, \quad (57)$$

$$G_t = 0, \quad (58)$$

$$F_t = \frac{(F_x x)}{t}. \quad (59)$$

Considering (57), one arrives at

$$G = K_1(t)x^2. \quad (60)$$

Considering (58), one obtains

$$G = K_2(x). \quad (61)$$

From (60) and (61), one can choose $K_1(t) = 1$ and $K_2(x) = x^2$; then we obtain

$$G = x^2. \quad (62)$$

Equation (59) becomes

$$tF_t - xF_x = 0, \quad (63)$$

and by Cauchy method, one arrives at

$$F = tx. \quad (64)$$

This solution satisfies (56), so the linearizing transformation is

$$X = tx, \qquad dT = x^2 dt. \tag{65}$$

Hence, by (65), (54) becomes

$$X^{(5)} = 0. \tag{66}$$

The general solution of (66) is

$$X = \frac{c_1}{24}T^4 + \frac{c_2}{6}T^3 + \frac{c_3}{2}T^2 + c_4 T + c_5, \tag{67}$$

where c_1, c_2, c_3, c_4, and c_5 are arbitrary constants. Substituting (65) into (67), the general solution of (54) is

$$x(t) = \frac{\left((c_1/24)\phi(t)^4 + (c_2/6)\phi(t)^3 + (c_3/2)\phi(t)^2 + c_4\phi(t) + c_5\right)}{t}, \tag{68}$$

where the function $T = \phi(t)$ is a solution of the equation

$$\frac{dT}{dt} = \left(\frac{(c_1/24)T^4 + (c_2/6)T^3 + (c_3/2)T^2 + c_4 T + c_5}{t}\right)^2. \tag{69}$$

Appendix

Equations for Theorem 1 in Section 2

$$A_1 = \frac{(5F_{xx}G - 11F_xG_x)}{(F_xG)}, \tag{A.1}$$

$$A_0 = \frac{(5F_{tx}G - F_tG_x - 10F_xG_t)}{(F_xG)}, \tag{A.2}$$

$$B_3 = \frac{5(2F_{xx}G - 3F_xG_x)}{(F_xG)}, \tag{A.3}$$

$$B_2 = \frac{\left(10F_{xxx}G^2 - 45F_{xx}G_xG - 14F_xG_{xx}G + 60F_xG_x^2\right)}{(F_xG^2)}, \tag{A.4}$$

$$\begin{aligned} B_1 = {} & \left(20F_{txx}G^2 - 50F_{tx}G_xG - 4F_tG_{xx}G + 15F_tG_x^2 \right. \\ & \left. - 40F_{xx}G_tG - 24F_xG_{tx}G + 105F_xG_tG_x\right)\left(F_xG^2\right)^{-1}, \end{aligned} \tag{A.5}$$

$$\begin{aligned} B_0 = {} & \left(-40F_{tx}G_tG + 10F_{ttx}G^2 - 5F_{tt}G_xG - 4F_tG_{tx}G \right. \\ & \left. + 15F_tG_tG_x - 10F_xG_{tt}G + 45F_xG_t^2\right)\left(F_xG^2\right)^{-1}, \end{aligned} \tag{A.6}$$

$$C_1 = \frac{\left(15F_{xxx}G^2 - 60F_{xx}G_xG - 18F_xG_{xx}G + 70F_xG_x^2\right)}{(F_xG^2)}, \tag{A.7}$$

$$\begin{aligned} C_0 = {} & \left(15F_{txx}G^2 - 30F_{tx}G_xG - 3F_tG_{xx}G + 10F_tG_x^2 \right. \\ & \left. - 30F_{xx}G_tG - 15F_xG_{tx}G + 60F_xG_tG_x\right)\left(F_xG^2\right)^{-1}, \end{aligned} \tag{A.8}$$

$$\begin{aligned} D_3 = {} & \left(10F_{xxxx}G^3 - 70F_{xxx}G_xG^2 - 45F_{xx}G_{xx}G^2 \right. \\ & + 195F_{xx}G_x^2G - 11F_xG_{xxx}G^2 + 125F_xG_{xx}G_xG \\ & \left. - 210F_xG_x^3\right)\left(F_xG^3\right)^{-1}, \end{aligned} \tag{A.9}$$

$$\begin{aligned} D_2 = {} & \left(30F_{txxx}G^3 - 150F_{txx}G_xG^2 - 60F_{tx}G_{xx}G^2 \right. \\ & + 255F_{tx}G_x^2G - 6F_tG_{xxx}G^2 + 65F_tG_{xx}G_xG \\ & - 105F_tG_x^3 - 60F_{xxx}G_tG^2 - 75F_{xx}G_{tx}G^2 \\ & + 330F_{xx}G_tG_xG - 27F_xG_{txx}G^2 + 205F_xG_{tx}G_xG \\ & \left. + 105F_xG_tG_{xx}G - 525F_xG_tG_x^2\right)\left(F_xG^3\right)^{-1}, \end{aligned} \tag{A.10}$$

$$\begin{aligned} D_1 = {} & \left(-120F_{txx}G_tG^2 - 90F_{tx}G_{tx}G^2 + 390F_{tx}G_tG_xG \right. \\ & + 30F_{ttxx}G^3 - 90F_{ttx}G_xG^2 - 15F_{tt}G_{xx}G^2 \\ & + 60F_{tt}G_x^2G - 12F_tG_{txx}G^2 + 85F_tG_{tx}G_xG \\ & + 45F_tG_tG_{xx}G - 210F_tG_tG_x^2 - 30F_{xx}G_{tt}G^2 \\ & + 135F_{xx}G_t^2G + 165F_xG_{tx}G_tG - 21F_xG_{ttx}G^2 \\ & \left. + 80F_xG_{tt}G_xG - 420F_xG_t^2G_x\right)\left(F_xG^3\right)^{-1}, \end{aligned} \tag{A.11}$$

$$\begin{aligned} D_0 = {} & \left(-30F_{tx}G_{tt}G^2 + 135F_{tx}G_t^2G + 10F_{tttx}G^3 \right. \\ & - 10F_{ttt}G_xG^2 - 60F_{ttx}G_tG^2 - 15F_{tt}G_{tx}G^2 \\ & + 60F_{tt}G_tG_xG + 45F_tG_{tx}G_tG - 6F_tG_{ttx}G^2 \\ & + 20F_tG_{tt}G_xG - 105F_tG_t^2G_x - 5F_xG_{ttt}G^2 \\ & \left. + 60F_xG_{tt}G_tG - 105F_xG_t^3\right)\left(F_xG^3\right)^{-1}, \end{aligned} \tag{A.12}$$

$$\begin{aligned} H_5 = {} & \left(F_{xxxxx}G^4 - 10F_{xxxx}G_xG^3 - 10F_{xxx}G_{xx}G^3 \right. \\ & + 45F_{xxx}G_x^2G^2 - 5F_{xx}G_{xxx}G^3 + 60F_{xx}G_{xx}G_xG^2 \\ & - 105F_{xx}G_x^3G - F_xG_{xxxx}G^3 + 15F_xG_{xxx}G_xG^2 \\ & \left. + 10F_xG_{xx}^2G^2 - 105F_xG_{xx}G_x^2G + 105F_xG_x^4\right) \\ & \cdot\left(F_xG^4\right)^{-1}, \end{aligned} \tag{A.13}$$

$$\begin{aligned} H_4 = {} & \left(5F_{txxxx}G^4 - 40F_{txxx}G_xG^3 - 30F_{txx}G_{xx}G^3 \right. \\ & + 135F_{txx}G_x^2G^2 - 10F_{tx}G_{xxx}G^3 + 120F_{tx}G_{xx}G_xG^2 \\ & - 210F_{tx}G_x^3G - F_tG_{xxxx}G^3 + 15F_tG_{xxx}G_xG^2 \end{aligned}$$

$$
\begin{aligned}
&+ 10F_tG_{xx}^2G^2 - 105F_tG_{xx}G_x^2G + 105F_tG_x^4 \\
&- 10F_{xxxx}G_tG^3 - 20F_{xxx}G_{tx}G^3 + 90F_{xxx}G_tG_xG^2 \\
&- 15F_{xx}G_{txx}G^3 + 120F_{xx}G_{tx}G_xG^2 + 60F_{xx}G_tG_{xx}G^2 \\
&- 315F_{xx}G_tG_x^2G - 4F_xG_{txxx}G^3 + 45F_xG_{txx}G_xG^2 \\
&+ 40F_xG_{tx}G_{xx}G^2 - 210F_xG_{tx}G_x^2G + 15F_xG_tG_{xxx}G^2 \\
&- 210F_xG_tG_{xx}G_xG + 420F_xG_tG_x^3\big)\left(F_xG^4\right)^{-1},
\end{aligned} \tag{A.14}
$$

$$
\begin{aligned}
H_3 = \big(&-40F_{txxx}G_tG^3 - 60F_{txx}G_{tx}G^3 + 270F_{txx}G_tG_xG^2 \\
&- 30F_{tx}G_{txx}G^3 + 240F_{tx}G_{tx}G_xG^2 + 120F_{tx}G_tG_{xx}G^2 \\
&- 630F_{tx}G_tG_x^2G + 10F_{ttxxx}G^4 - 60F_{ttxx}G_xG^3 \\
&- 30F_{ttx}G_{xx}G^3 + 135F_{ttx}G_x^2G^2 - 5F_{tt}G_{xxx}G^3 \\
&+ 60F_{tt}G_{xx}G_xG^2 - 105F_{tt}G_x^3G - 4F_tG_{txxx}G^3 \\
&+ 45F_tG_{txx}G_xG^2 + 40F_tG_{tx}G_{xx}G^2 - 210F_tG_{tx}G_x^2G \\
&+ 15F_tG_tG_{xxx}G^2 - 210F_tG_tG_{xx}G_xG + 420F_tG_tG_x^3 \\
&- 10F_{xxx}G_{tt}G^3 + 45F_{xxx}G_t^2G^2 + 120F_{xx}G_{tx}G_tG^2 \\
&- 15F_{xx}G_{ttx}G^3 + 60F_{xx}G_{tt}G_xG^2 - 315F_{xx}G_t^2G_xG \\
&+ 45F_xG_{txx}G_tG^2 + 40F_xG_{tx}^2G^2 - 420F_xG_{tx}G_tG_xG \\
&- 6F_xG_{ttxx}G^3 + 45F_xG_{ttx}G_xG^2 + 20F_xG_{tt}G_{xx}G^2 \\
&- 105F_xG_{tt}G_x^2G - 105F_xG_t^2G_{xx}G + 630F_xG_t^2G_x^2\big) \\
&\cdot\left(F_xG^4\right)^{-1},
\end{aligned} \tag{A.15}
$$

$$
\begin{aligned}
H_2 = \big(&-30F_{txx}G_{tt}G^3 + 135F_{txx}G_t^2G^2 + 240F_{tx}G_{tx}G_tG^2 \\
&- 30F_{tx}G_{ttx}G^3 + 120F_{tx}G_{tt}G_xG^2 - 630F_{tx}G_t^2G_xG \\
&+ 10F_{tttxx}G^4 - 40F_{tttx}G_xG^3 - 10F_{ttt}G_{xx}G^3 \\
&+ 45F_{ttt}G_x^2G^2 - 60F_{ttxx}G_tG^3 - 60F_{ttx}G_{tx}G^3 \\
&+ 270F_{ttx}G_tG_xG^2 - 15F_{tt}G_{txx}G^3 + 120F_{tt}G_{tx}G_xG^2 \\
&+ 60F_{tt}G_tG_{xx}G^2 - 315F_{tt}G_tG_x^2G + 45F_tG_{txx}G_tG^2 \\
&+ 40F_tG_{tx}^2G^2 - 420F_tG_{tx}G_tG_xG - 6F_tG_{ttxx}G^3 \\
&+ 45F_tG_{ttx}G_xG^2 + 20F_tG_{tt}G_{xx}G^2 - 105F_tG_{tt}G_x^2G \\
&- 105F_tG_t^2G_{xx}G + 630F_tG_t^2G_x^2 - 5F_{xx}G_{ttt}G^3 \\
&+ 60F_{xx}G_{tt}G_tG^2 - 105F_{xx}G_t^3G + 40F_xG_{tx}G_{tt}G^2 \\
&- 210F_xG_{tx}G_t^2G - 4F_xG_{tttx}G^3 + 15F_xG_{ttt}G_xG^2 \\
&+ 45F_xG_{ttx}G_tG^2 - 210F_xG_{tt}G_tG_xG + 420F_xG_t^3G_x\big) \\
&\cdot\left(F_xG^4\right)^{-1},
\end{aligned} \tag{A.16}
$$

$$
\begin{aligned}
H_1 = \big(&-10F_{tx}G_{ttt}G^3 + 120F_{tx}G_{tt}G_tG^2 - 210F_{tx}G_t^3G \\
&+ 5F_{ttttx}G^4 - 10F_{tttt}G_xG^3 - 40F_{tttx}G_tG^3 \\
&- 20F_{ttt}G_{tx}G^3 + 90F_{ttt}G_tG_xG^2 - 30F_{ttx}G_{tt}G^3 \\
&+ 135F_{ttx}G_t^2G^2 + 120F_{tt}G_{tx}G_tG^2 - 15F_{tt}G_{ttx}G^3 \\
&+ 60F_{tt}G_{tt}G_xG^2 - 315F_{tt}G_t^2G_xG + 40F_tG_{tx}G_{tt}G^2 \\
&- 210F_tG_{tx}G_t^2G - 4F_tG_{tttx}G^3 + 15F_tG_{ttt}G_xG^2 \\
&+ 45F_tG_{ttx}G_tG^2 - 210F_tG_{tt}G_tG_xG + 420F_tG_t^3G_x \\
&- F_xG_{ttttt}G^3 + 15F_xG_{ttt}G_tG^2 + 10F_xG_{tt}^2G^2 \\
&- 105F_xG_{tt}G_t^2G + 105F_xG_t^4\big)\left(F_xG^4\right)^{-1},
\end{aligned} \tag{A.17}
$$

$$
\begin{aligned}
H_0 = \big(&F_{ttttt}G^4 - 10F_{tttt}G_tG^3 - 10F_{ttt}G_{tt}G^3 \\
&+ 45F_{ttt}G_t^2G^2 - 5F_{tt}G_{ttt}G^3 + 60F_{tt}G_{tt}G_tG^2 \\
&- 105F_{tt}G_t^3G - F_tG_{tttt}G^3 + 15F_tG_{ttt}G_tG^2 \\
&+ 10F_tG_{tt}^2G^2 - 105F_tG_{tt}G_t^2G + 105F_tG_t^4\big)\left(F_xG^4\right)^{-1}.
\end{aligned} \tag{A.18}
$$

Equations for Theorem 2 in Section 3

$$
\begin{aligned}
S_{3ttt} = \big(&23856004085760000000S_{6ttt}S_4^3S_5 \\
&+ 85200014592000000S_{6ttt}S_4^3S_8 \\
&- 45181825920000000S_{6t}^2S_4^2S_5 \\
&- 122636384640000S_{6t}^2S_4^2S_8 \\
&- 8179201400832000000S_{7ttt}S_4^3 \\
&+ 1214611408023552000000000H_0S_4^6 \\
&+ 1288224220631040000S_2^2S_3S_4^2S_6 \\
&+ 333433269227520000S_2^2S_5^2S_6 \\
&+ 202242458880000S_2^2S_5S_6^2 \\
&+ 1247633114112000S_2^2S_5S_6S_8 \\
&- 398786229338112000S_2^2S_5S_7 \\
&+ 2142294739200S_2^2S_6^2S_8 \\
&- 169957431705600S_2^2S_6S_7 \\
&- 1775000304000S_2^2S_6S_8^2 \\
&- 147532492800000S_2^2S_7S_8 \\
&+ 487653901701120000S_2S_3S_4^2S_5S_6 \\
&+ 2581818624000000S_2S_3S_4^2S_6^2 \\
&+ 3947600676096000S_2S_3S_4^2S_6S_8
\end{aligned}
$$

$$
\begin{aligned}
&- 1004306790187008000S_2S_3S_4^2S_7 \\
&- 332554631270400000S_2S_5^3S_6 \\
&- 1429140303360000S_2S_5^2S_6^2 \\
&- 1233512146944000S_2S_5^2S_6S_8 \\
&+ 353437928718336000S_2S_5^2S_7 \\
&- 2831431680000S_2S_5S_6^3 \\
&- 6606314668800S_2S_5S_6^2S_8 \\
&+ 1176324891033600S_2S_5S_6S_7 \\
&+ 2489610816000S_2S_5S_6S_8^2 \\
&+ 149604618240000S_2S_5S_7S_8 \\
&- 10866683520S_2S_6^3S_8 \\
&+ 2607397309440S_2S_6^2S_7 \\
&+ 4354257600S_2S_6^2S_8^2 \\
&+ 130235212800S_2S_6S_7S_8 \\
&- 3590433000S_2S_6S_8^3 \\
&+ 176538275020800S_2S_7^2 \\
&- 512265600000S_2S_7S_8^2 \\
&+ 159828655472640000000S_3S_4^2S_5^3 \\
&+ 2184672479109120000S_3S_4^2S_5^2S_6 \\
&+ 672295138099200000S_3S_4^2S_5^2S_8 \\
&+ 107085863808000000S_3S_4^2S_5S_6^2 \\
&+ 1042969036032000S_3S_4^2S_5S_6S_8 \\
&+ 337584386506752000S_3S_4^2S_5S_7 \\
&+ 5249697868800000S_3S_4^2S_5S_8^2 \\
&+ 14883644160000S_3S_4^2S_6^3 \\
&+ 4409023449600S_3S_4^2S_6^2S_8 \\
&- 74094096384000S_3S_4^2S_6S_7 \\
&+ 4756386096000S_3S_4^2S_6S_8^2 \\
&+ 291637463040000S_3S_4^2S_7S_8 \\
&+ 21679812000000S_3S_4^2S_8^3 \\
&- 552494117683200000S_5^5 \\
&+ 682894773657600000S_5^4S_6 \\
&- 7047118848000000S_5^4S_8 \\
&+ 291110400000S_5^2S_6^3 \\
&+ 2269611955200S_5^2S_6^2S_8 \\
&- 402813609984000S_5^2S_6S_7 \\
&- 517887216000S_5^2S_6S_8^2 \\
&- 57156096000000S_5^2S_7S_8 \\
&- 2773848000000S_5^2S_8^3 \\
&- 1049440000S_5S_6^4 \\
&+ 6279324480S_5S_6^3S_8 \\
&- 650224460800S_5S_6^2S_7 \\
&- 9707266800S_5S_6^2S_8^2 \\
&- 637826227200S_5S_6S_7S_8 \\
&+ 11823801000S_5S_6S_8^3 \\
&- 17672824422400S_5S_7^2 \\
&+ 505169280000S_5S_7S_8^2 \\
&- 21954900000S_5S_8^4 \\
&+ 4937432S_6^4S_8 + 896097408S_6^3S_7 \\
&- 14788620S_6^3S_8^2 \\
&- 1103392000S_6^2S_7S_8 - 1631850S_6^2S_8^3 \\
&+ 151872430080S_6S_7^2 \\
&- 536659200S_6S_7S_8^2 \\
&+ 32170875S_6S_8^4 - 12909568000S_7^2S_8 \\
&+ 848760000S_7S_8^3 - 62184375S_8^5 \\
&+ 7683984000\big(155232000S_2S_5S_6 \\
&+ 554400S_2S_6S_8 - 170311680S_2S_7 \\
&- 838252800S_3S_4^2S_6 \\
&- 35795200S_5^2S_6 + 112000S_5S_6^2 \\
&- 164800S_5S_6S_8 + 48762880S_5S_7 \\
&- 272S_6^2S_8 - 96000S_6S_7 \\
&+ 1155S_6S_8^2 + 64000S_7S_8\big)B_0S_4^2 \\
&+ 17740800\big(276623424000B_0S_4^2 \\
&- 82987027200S_2^2 + 69144768000S_2S_5
\end{aligned}
$$

$$
\begin{aligned}
&- 59209920S_2S_6 + 55440000S_2S_8 \\
&+ 264608467200S_3S_4^2 \\
&- 49174841600S_5^2 - 232872800S_5S_6 \\
&- 1724000S_5S_8 \\
&- 514314S_6^2 - 1160S_6S_8 - 49754880S_7 \\
&+ 144375S_8^2\big) S_{7t}S_4 \\
&+ 25818186240000 \big(4158000S_{6t}S_4 \\
&- 415800S_2S_6 + 1096381440S_3S_4^2 \\
&- 129382400S_5^2 \\
&- 741880S_5S_6 - 268800S_5S_8 \\
&- 1280S_6^2 - 684S_6S_8 \\
&- 4992S_7 - 825S_8^2\big) S_{3t}S_4^3 \\
&+ 1022400175104000000 \big(77S_6 + 20S_8 \\
&+ 23968S_5\big) S_{3tt}S_4^4 \\
&- 619636469760000 \big(24213S_6 \\
&+ 4880S_8 + 4866176S_5\big) S_3^2S_4^4 \\
&+ 768000 \big(\big(417386935S_6 \\
&+ 422682482S_8\big) S_6 \\
&- 88 \big(1252844464S_7 + 2443875S_8^2\big)\big) S_5^3 \\
&- 511200087552000 \big(S_6S_8 \\
&- 384S_7 + 280S_5S_6\big) S_2^3 \\
&- 8520001459200000 \big(S_6S_8 \\
&- 192S_7 + 280S_5S_6\big) B_{0t}S_4^3 \\
&+ 19670999040000 \big(327S_6 - 125S_8 \\
&+ 26080S_5 + 249480S_2\big) S_{7tt}S_4^2 \\
&- 76839840000 \big(332640 \big(\big(280S_5 \\
&+ S_8\big) S_2 - 2520S_3S_4^2\big) \\
&+ 54790400S_5^2 + 380800S_5S_6 \\
&+ 158720S_5S_8 + 688S_6S_8 \\
&- 112896S_7 + 1155S_8^2\big) S_{6tt}S_4^2 \\
&- 277200 \big(6147187200 \big(3 \big(5B_0S_4^2 \\
&- S_2^2\big) \big(280S_5 + S_8\big) \\
&- 16S_{7t}S_4\big) - \big(5438522880000S_5^3 \\
&+ 30283456000S_5^2S_6 \\
&+ 18086822400S_5^2S_8 + 96768000S_5S_6^2 \\
&+ 99126880S_5S_6S_8 \\
&+ 6218168320S_5S_7 + 3603600S_5S_8^2 \\
&+ 211536S_6^2S_8 \\
&- 29151360S_6S_7 + 120120S_6S_8^2 \\
&+ 10810240S_7S_8 + 129525S_8^3\big) \\
&+ 93139200 \big(3160S_6 + 1397S_8 \\
&+ 879840S_5\big) S_3S_4^2 \\
&+ 55440 \big(81088000S_5^2 - 565600S_5S_6 \\
&+ 326560S_5S_8 - 844S_6S_8 + 467712S_7 \\
&- 1155S_8^2 + 838252800S_3S_4^2\big) S_2\big) S_{6t}S_4\big) \\
&\cdot \big(566818657077657600000000S_4^5\big)^{-1},
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
S_{6tx} = \big(\big(2 \big(40320 \big(2 \big(99 \big(259952S_2^2 \\
&- 436299S_3S_4^2 - 19404000H_3 \\
&+ 1724800C_0^2\big) S_4 \\
&+ 24736S_{7x} - 285959520S_{3x}S_4^2 \\
&+ 672348600S_{1tt}S_4 \\
&+ 1280664000C_{0tx}S_4 \\
&- 384199200B_{0x}S_4^2 \\
&+ 96049800A_{1tt}S_4^2\big) S_4 \\
&- 1334025 \big(112S_5 + S_8\big) S_{1tx}\big) \\
&- \big(48428561920S_5^2 \\
&+ 654609600S_5S_6 \\
&+ 600371360S_5S_8 \\
&+ 1929680S_6^2 + 9171338S_6S_8 \\
&+ 552802176S_7 \\
&- 10786905S_8^2\big) S_4^2
\end{aligned}
$$

$$
\begin{aligned}
&- 1890\left(819624960S_2^2\right.\\
&- 2209320960S_2S_5\\
&- 7096320S_2S_6\\
&+ 1626240S_2S_8\\
&+ 204906240S_3S_4^2\\
&+ 1478702080S_5^2\\
&+ 9564160S_5S_6\\
&- 2372480S_5S_8\\
&+ 15360S_6^2 - 10274S_6S_8\\
&\left.- 206976S_7 + 5775S_8^2\right)B_2\Big)\\
&+ 189\left(13113999360S_2^2\right.\\
&- 35713413120S_2S_5\\
&- 113541120S_2S_6\\
&+ 22767360S_2S_8\\
&+ 1434343680S_3S_4^2\\
&+ 24190668800S_5^2\\
&+ 154603520S_5S_6\\
&- 33214720S_5S_8\\
&+ 245760S_6^2 - 95326S_6S_8\\
&\left.+ 206976S_7 - 5775S_8^2\right)A_1^2\\
&- 42\left(29506498560S_2^2\right.\\
&- 79535554560S_2S_5\\
&- 255467520S_2S_6\\
&+ 105114240S_2S_8\\
&- 101056032000S_3S_4^2\\
&+ 31943598080S_5^2\\
&+ 253961600S_5S_6\\
&- 60480320S_5S_8 + 552960S_6^2\\
&- 700362S_6S_8\\
&\left.- 15044736S_7 + 976015S_8^2\right)S_1\\
&- 663896217600\left(21S_1\right.\\
&\left.- 11S_4^2\right)B_0S_4^2\\
&+ 41309097984000\left(A_1\right.\\
&\left.- S_4\right)C_{0t}S_4^2\\
&+ 838252800\left(67A_1 + 6S_4\right)S_{6t}S_4^2\\
&- 34927200\left(8624A_1S_5 + 77A_1S_8\right.\\
&- 487872S_2S_4 + 265760S_4S_5\\
&\left.+ 2112S_4S_6 - 3298S_4S_8\right)S_{1t}\\
&- 73920\left(176576S_6 + 4499S_8\right.\\
&\left.+ 51980272S_5\right)S_2S_4^2\Big)S_4\\
&- 582120\left(18627840S_3S_4^2\right.\\
&+ 636160S_5^2 + 1120S_5S_6\\
&+ 5680S_5S_8 + 52S_6S_8\\
&+ 2688S_7 - 75S_8^2\\
&\left.- 18480\left(112S_5 + S_8\right)S_2\right)S_{1x}\\
&- 3725568000\left(48S_6 - 11S_8\right.\\
&\left.+ 14944S_5 - 11088S_2\right)C_{0x}S_4^2\\
&+ 698544000\left(192S_6 - 11S_8\right.\\
&\left.+ 63472S_5 - 44352S_2\right)B_{2t}S_4^2\\
&+ 4300800\left(17555S_6 - 2693S_8\right.\\
&\left.+ 4417608S_5 - 5580036S_2\right)S_{5x}S_4^2\\
&+ 13440\left(18360S_6 - 2657S_8\right.\\
&+ 3797800S_5\\
&\left.- 4241160S_2\right)S_{6x}S_4^2\\
&+ 8870400\left(1617\left(\left(112S_5\right.\right.\right.\\
&\left.\left.+ S_8\right)S_1 - 1728S_2S_4^2\right)\\
&+ \left(8064S_6 - 4841S_8\right.\\
&\left.+ 1852816S_5\right)S_4^2\Big)C_0S_4\\
&- 3\left(4851\left(18627840S_3S_4^2\right.\right.\\
&- 47676160S_5^2\\
&- 142240S_5S_6 - 425680S_5S_8\\
&- 1228S_6S_8 + 2688S_7 - 75S_8^2\\
&\left.+ 277200\left(112S_5 + S_8\right)S_2\right)S_1\\
&- \left(3098182348800B_0S_4^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ 1910545781760S_2^2 \\
&- 2436527681280S_2S_5 \\
&- 11120820480S_2S_6 \\
&+ 2970419760S_2S_8 \\
&+ 4109618177280S_3S_4^2 \\
&+ 394650233600S_5^2 \\
&+ 6026409760S_5S_6 \\
&- 4658100400S_5S_8 \\
&+ 12337920S_6^2 \\
&- 16712320S_6S_8 \\
&- 588303744S_7 \\
&+ 4150685S_8^2\big) S_4^2 \\
&- 186278400\left(64S_6 - 11S_8\right. \\
&+ \left. 20336S_5 - 14784S_2\right) C_0S_4^2\big) A_1 \\
&- 1663200\left(1617\left(192\left(80C_0 - 57S_2\right) S_4^2\right.\right. \\
&+ \left.\left(112S_5 + S_8\right) S_1\right) \\
&+ \left(49392S_6 - 9451S_8\right. \\
&+ \left. 15674288S_5\right) S_4^2 \\
&+ 42\left(1536S_6 - 77S_8 + 509008S_5\right. \\
&- \left. 354816S_2\right) A_1S_4\big) A_{1t}S_4\big) \\
&\cdot \left(167650560000S_4^3\right)^{-1},
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
S_{7tt} = {} & \big(852000145920000000B_{0tt}S_4^4 \\
&- 852000145920000000S_{3tt}S_4^4 \\
&+ 215151552000000S_{6tt}S_4^2S_5 \\
&+ 307359360000S_{6tt}S_4^2S_8 \\
&+ 85200014592000000B_0^2S_4^4 \\
&- 1704000291840000000H_1S_4^4 \\
&+ 5553778728960000S_2S_3S_4^2S_5 \\
&- 18646467840000S_2S_3S_4^2S_6 \\
&+ 3688312320000S_2S_3S_4^2S_8 \\
&- 176177971200000S_2S_5^3 \\
&- 3671754240000S_2S_5^2S_6 \\
&- 471905280000S_2S_5^2S_8 \\
&- 172480000S_2S_5S_6^2 \\
&- 11718537600S_2S_5S_6S_8 \\
&+ 4367800729600S_2S_5S_7 \\
&- 2550240000S_2S_5S_8^2 \\
&- 18098080S_2S_6^2S_8 \\
&+ 256650240S_2S_6S_7 \\
&+ 46985400S_2S_6S_8^2 \\
&+ 2848384000S_2S_7S_8 \\
&- 78540000S_2S_8^3 \\
&+ 399665522995200000S_3^2S_4^4 \\
&- 102956816793600000S_3S_4^2S_5^2 \\
&- 71642672640000S_3S_4^2S_5S_6 \\
&- 19314408960000S_3S_4^2S_5S_8 \\
&- 109438560000S_3S_4^2S_6^2 \\
&- 54992044800S_3S_4^2S_6S_8 \\
&+ 11248234905600S_3S_4^2S_7 \\
&- 48898080000S_3S_4^2S_8^2 \\
&+ 243674726400000S_5^4 \\
&+ 3398088960000S_5^3S_6 \\
&+ 717050880000S_5^3S_8 + 86176S_6^3S_8 \\
&- 29701056S_6^2S_7 - 56120S_6^2S_8^2 \\
&+ 4396160S_6S_7S_8 - 267075S_6S_8^3 \\
&- 1886760960S_7^2 - 15064000S_7S_8^2 \\
&+ 628125S_8^4 \\
&- 277200\big(86240000S_2S_5 \\
&+ 252560S_2S_8 - 1397088000S_3S_4^2 \\
&+ 72800000S_5^2 + 672000S_5S_6 \\
&- 56160S_5S_8 \\
&+ 876S_6S_8 - 276864S_7 - 455S_8^2\big) S_{6t}S_4
\end{aligned}
$$

$$
\begin{aligned}
&+ 6400\left(\left(2909900S_6 + 1186419S_8\right)S_6\right.\\
&\left.-\left(440601056S_7 - 1100025S_8^2\right)\right)S_5^2\\
&+ 170755200\\
&\cdot\left(41S_6S_8 - 22848S_7 + S_6 + 14000S_5S_6\right)S_2^2\\
&- 737662464000000\left(337S_5 - 231S_2\right)B_{0t}S_4^3\\
&+ 18441561600000\\
&\cdot\left(51S_6 + 4S_8 + 16312S_5 - 8008S_2\right)S_{3t}S_4^3\\
&+ 17740800\left(2373S_6 - 500S_8\right.\\
&\left.+ 152320S_5 + 1681680S_2\right)S_{7t}S_4\\
&+ 40\left(2\left(2\left(224875S_6 + 349638S_8\right)S_6^2\right.\right.\\
&\left.-75\left(520192S_7 - 29095S_8^2\right)S_8\right)\\
&-\left(304667776S_7 + 2689035S_8^2\right)S_6\Big)S_5\\
&- 30735936000\left(S_6S_8 - 672S_7\right.\\
&\left.+ 700S_5S_6 + 2217600S_3S_4^2\right)B_0S_4^2\Big)\\
&\cdot\left(1032727449600000S_4^2\right)^{-1},
\end{aligned}
\tag{A.21}
$$

$$
\begin{aligned}
C_{0t} = {} & \Big(-\Big(\Big(332640\Big(4\Big(45S_{6t}S_4\\
&- 27720B_0S_4^2 - 92400D_1S_4\\
&- 5544S_2^2 + 20110S_3S_4^2\Big)S_4\\
&- 385\left(112S_5 + S_8\right)S_{1t}\Big)\\
&-\Big(7120943360S_5^2\\
&+ 38913440S_5S_6\\
&- 10308880S_5S_8 + 60480S_6^2\\
&- 64408S_6S_8 - 769152S_7\\
&+ 57855S_8^2\Big)S_4\\
&- 9\Big(819624960S_2^2\\
&- 2209320960S_2S_5\\
&- 7096320S_2S_6\\
&+ 1626240S_2S_8\\
&+ 204906240S_3S_4^2\\
&+ 1478702080S_5^2\\
&+ 9564160S_5S_6\\
&- 2372480S_5S_8 + 15360S_6^2\\
&- 10274S_6S_8 - 206976S_7\\
&+ 5775S_8^2\Big)A_1\\
&+ 55440\left(828S_6 - 65S_8\right.\\
&\left.+ 330832S_5\right)S_2S_4\Big)S_4\\
&- 693\Big(18627840S_3S_4^2\\
&+ 636160S_5^2\\
&+ 1120S_5S_6 + 5680S_5S_8\\
&+ 52S_6S_8 + 2688S_7\\
&- 75S_8^2 - 18480\left(112S_5 + S_8\right)S_2\Big)S_1\\
&- 4435200\left(48S_6 - 11S_8\right.\\
&\left.+ 14944S_5 - 11088S_2\right)C_0S_4^2\\
&+ 1663200\left(192S_6 - 11S_8\right.\\
&\left.+ 63472S_5 - 44352S_2\right)A_{1t}S_4^2\Big)\Big)\\
&\cdot\left(245887488000S_4^3\right)^{-1},
\end{aligned}
\tag{A.22}
$$

$$
\begin{aligned}
S_{6tt} = {} & \Big(-\Big(\Big(36960\Big(15\Big(3049200\left(672\left(10\left(D_{1t}\right.\right.\right.\right.\\
&\left.\left.- 3H_2\right) - 3B_{0t}S_4\right)S_4^2\\
&-\left(A_{1tt}S_4 + 7S_{1tt}\right)\left(112S_5 + S_8\right)\Big)S_4\\
&+\Big(86240A_1S_4S_5 + 2387A_1S_4S_8\\
&+ 1509200S_1S_5\\
&+ 5390S_1S_8 + 3326400S_2S_4^2\\
&- 3437680S_4^2S_5 - 14400S_4^2S_6\\
&+ 7604S_4^2S_8\Big)S_{6t}\\
&+ 1848\left(72S_6 + 91S_8 + 104944S_5\right)S_2^2S_4\\
&- 24\left(330129S_6 - 20551S_8\right.\\
&\left.+ 94386224S_5\right)S_3S_4^3\\
&+ 231\Big(2383360S_5^2 + 7840S_5S_6\\
&+ 21280S_5S_8 + 28S_6S_8 - 2688S_7\\
&+ 75S_8^2 - 18627840S_3S_4^2\Big)S_{1t}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(3544441600S_5^2 + 15097240S_5S_6\right.\\
&+ 703120S_5S_8\\
&+ 17280S_6^2 + 3970S_6S_8\\
&+ 3094464S_7 + 13785S_8^2\\
&\left.- 34379009280S_3S_4^2\right) S_2S_4\\
&- 19958400\left(2695S_1 - 7099S_4^2\right.\\
&\left.+ 385A_1S_4\right) S_{3t}S_4^2\\
&- 2880\left(2695S_1 - 3183S_4^2 - 539A_1S_4\right) S_{7t}\\
&+ 5940\left(896S_7 - 25S_8^2\right.\\
&\left.+ 14S_6S_8 + 2069760S_3S_4^2\right) C_0S_4\\
&- 5544000\left(24S_6 - 11S_8\right.\\
&\left.+ 6856S_5 - 5544S_2\right) D_1S_4^2\Big)\\
&+ \Big(160\Big(\left(4626885S_6 - 4733737S_8\right) S_6\\
&+ 293259136S_7 + 5053935S_8^2\Big) S_5\\
&+ 13742136217600S_5^3\\
&+ 150598419200S_5^2S_6\\
&- 63989446400S_5^2S_8\\
&+ 1382400S_6^3 - 828024S_6^2S_8\\
&+ 190854144S_6S_7 - 2697315S_6S_8^2\\
&- 214158080S_7S_8 + 5466300S_8^3\Big) S_4\Big) S_4\\
&- 385\Big(776160\left(S_6S_8 - 192S_7\right.\\
&\left.+ 280S_5S_6\right) S_2\\
&- \Big(1341502525440S_3S_4^2S_5\\
&+ 3259872000S_3S_4^2S_6\\
&+ 3520661760S_3S_4^2S_8\\
&- 40040448000S_5^3 - 183456000S_5^2S_6\\
&- 371804160S_5^2S_8 - 431200S_5S_6^2\\
&- 274176S_5S_6S_8 + 19726336S_5S_7\\
&- 3265920S_5S_8^2 - 364S_6^2S_8\\
&+ 182784S_6S_7 - 441S_6S_8^2\\
&+ 103936S_7S_8 - 11220S_8^3\Big)\Big) S_1\\
&+ 15367968000\Big(77\Big(192\left(10C_0 - 3S_2\right) S_4^2\\
&+ \left(112S_5 + S_8\right) S_1\Big)\\
&+ \left(120S_6 + 17S_8 + 60272S_5\right) S_4^2\Big) B_0S_4^2\\
&- 91476000\Big(3696\Big(5040\left(4B_0 + S_3\right) S_4^2\\
&+ \left(112S_5 + S_8\right) S_2\Big)\\
&- \Big(603904S_5^2 + 1792S_5S_6\\
&+ 5392S_5S_8 - 26S_6S_8\\
&- 2688S_7 + 75S_8^2\Big)\Big) A_{1t}S_4^2\\
&+ 11\Big(11176704000\left(24S_6\right.\\
&\left.- 11S_8 + 6856S_5\right.\\
&\left.- 5544S_2\right) B_0S_4^2\\
&+ 21451820544000S_3S_4^2S_5\\
&+ 71624044800S_3S_4^2S_6\\
&- 20676902400S_3S_4^2S_8\\
&- 4004044800S_5^3\\
&+ 133593600S_5^2S_6\\
&- 107251200S_5^2S_8\\
&- 39200S_5S_6^2 + 49007280S_5S_6S_8\\
&+ 3140157440S_5S_7\\
&- 81867600S_5S_8^2 + 188188S_6^2S_8\\
&+ 12348672S_6S_7 - 476955S_6S_8^2\\
&- 10622080S_7S_8\\
&+ 249900S_8^3 - 55440\Big(847S_6S_8\\
&+ 59136S_7 - 1125S_8^2 + 7840S_5S_6\\
&+ 279417600S_3S_4^2\Big) S_2\Big) A_1S_4\Big)\Big)\\
&\cdot \left(5532468480000S_4^4\right)^{-1},
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
B_{0t} = {} & \Big(3688312320000S_{3t}S_4^3\\
& - 620928000S_{6t}S_4S_5\\
& + 277200S_{6t}S_4S_8\\
& + 372556800S_{7t}S_4
\end{aligned}
$$

$$
\begin{aligned}
&- 1106493696000B_0S_2S_4^2 \\
&+ 1614235392000B_0S_4^2S_5 \\
&+ 4790016000B_0S_4^2S_6 \\
&+ 11064936960000D_0S_4^3 \\
&+ 442597478400S_2S_3S_4^2 \\
&+ 62092800S_2S_5S_6 \\
&- 27720S_2S_6S_8 - 74511360S_2S_7 \\
&- 826011648000S_3S_4^2S_5 \\
&- 2567980800S_3S_4^2S_6 \\
&- 119750400S_3S_4^2S_8 \\
&+ 572006400S_5^3 \\
&- 19084800S_5^2S_6 \\
&+ 15321600S_5^2S_8 + 5600S_5S_6^2 \\
&+ 38760S_5S_6S_8 \\
&+ 36305920S_5S_7 + 82800S_5S_8^2 \\
&+ 512S_6^2S_8 + 49728S_6S_7 \\
&- 1935S_6S_8^2 - 34880S_7S_8 \\
&+2550S_8^3\big) \\
&\cdot \left(11064936960000S_4^3\right)^{-1},
\end{aligned}
\tag{A.24}
$$

$$
\begin{aligned}
S_{6xx} = {} & \big(7\big(33\big(240\big(108662400H_4 \\
&- 778447S_2S_4^2 \\
&+ 2199120D_2S_4 \\
&- 18110400D_{2x} \\
&- 1164240B_{2t}S_4\big)S_4^2 \\
&- 136171\left(112S_5 + S_8\right)S_1^2 \\
&- 53760\big(4851S_1 \\
&- 2188S_4^2\big)C_0S_4^2\big) \\
&+ 2\big(4303031040B_2S_2 \\
&- 2283635200B_2S_5 \\
&- 18627840B_2S_6 \\
&+ 20913200B_2S_8 \\
&- 1434343680C_0S_4^2 \\
&- 14343436800D_2S_4 \\
&+ 1290909312S_1S_2 \\
&- 1052792048S_1S_5 \\
&- 5588352S_1S_6 \\
&+ 6852307S_1S_8 \\
&+ 2189453112S_2S_4^2 \\
&- 3051480S_4^2S_5 \\
&+ 2469168S_4^2S_6 \\
&- 4097730S_4^2S_8\big)A_1S_4 \\
&- 388080\left(48A_1 - 79S_4\right)S_{6x}S_4^2\big) \\
&- \big(16666080S_6 + 97969487S_8 \\
&+ 9031282208S_5\big)S_4^4 \\
&+ 1764\big(7528052S_5 \\
&- 27160S_6 + 99515S_8 \\
&+ 15878940S_2\big)S_1S_4^2 \\
&+ 258720\big(297S_6 + 14S_8 \\
&+77780S_5 - 141372S_2\big)B_2S_4^2 \\
&- 3018400\big(432S_6 \\
&- 485S_8 + 52960S_5 \\
&- 99792S_2\big)\left(B_{2x} - D_3\right)S_4 \\
&- 3773\big(6912S_6 - 13183S_8 \\
&+ 819440S_5 - 1596672S_2\big)A_1^2S_4^2\big) \\
&\cdot \left(162993600S_4^2\right)^{-1}.
\end{aligned}
\tag{A.25}
$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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