

Research Article

Computational Optimization of Residual Power Series Algorithm for Certain Classes of Fuzzy Fractional Differential Equations

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This paper aims to present a novel optimization technique, the residual power series (RPS), for handling certain classes of fuzzy fractional differential equations of order $1 < \gamma \leq 2$ under strongly generalized differentiability. The proposed technique relies on generalized Taylor formula under Caputo sense aiming at extracting a supportive analytical solution in convergent series form. The RPS algorithm is significant and straightforward tool for creating a fractional power series solution without linearization, limitation on the problem's nature, sort of classification, or perturbation. Some illustrative examples are provided to demonstrate the feasibility of the RPS scheme. The results obtained show that the scheme is simple and reliable and there is good agreement with exact solution.

1. Introduction

Fuzzy fractional differential equation is hot and important branch of mathematics. It has attracted much attention recently due to potential applications in artificial intelligence, industrial engineering, physics, chemistry, and other fields of science. Parameters and variables in many of the nature studies and technological processes that were designed utilizing the fractional differential equation (FDE) are specific and completely defined. Indeed, such information may be vague and uncertain because of experimentation and measurement errors that then lead to uncertain models, which cannot handle these studies. The process of analyzing the relative influence of uncertainty in inputs information to outputs led us to study solutions to the qualitative behavior of equations. Therefore, it is necessary to obtain some mathematical tools to understand the complex structure of uncertainty models [1–5]. On the other hand, the theory of fractional calculus, which is a generalization of classical calculus, deals with the discussion of the integrals and derivatives of noninteger

order, has a long history, and dates back to the seventeenth century [6–10]. Different forms of fractional operators are introduced to study FDEs such as Riemann–Liouville, Grunwald–Letnikov, and Caputo. Out of these forms, the Caputo concept is an appropriate tool for modeling practical situations due to its countless benefits as it allows the process to be performed based on initial and boundary conditions as is traditional and its derivative is zero for constant [11–17].

The residual power series (RPS) method developed in [18] is considered as an effective optimization technique to determine and define the power series solution's values of coefficients of first- and second-order fuzzy differential equations [19–22]. Furthermore, the RPS is characterized as an applicable and easy technique to create power series solutions for strongly linear and nonlinear equations without being linearized, discretized, or exposed to perturbation [23–27]. Unlike the classical power series method, the RPS neither requires comparing the corresponding coefficients nor is a recursion relation needed as well. Besides that, it calculates the power series coefficients through chain of equations of

one or more variables and offers convergence of a series solution whose terms approach quickly, especially when the exact solution is polynomial.

The remainder of this paper is organized as follows. In Section 2, essential facts and results related to the fuzzy fractional calculus will be shown. In Section 3, the concept of Caputo's H-differentiability will be presented together with some closely related results. In Section 4, basic idea of the RPS method will be presented to solve the fuzzy FDEs of order $1 < \gamma \leq 2$. In Section 5, numerical application will be performed to show capability, potentiality, and simplicity of the method. Conclusions will be given in Section 6.

2. Preliminaries

In this section, necessary definitions and results relating to fuzzy fractional calculus are presented. For the fuzzy derivative concept, the strongly generalized differentiability will be adopted, which is considered H-differentiability modification.

A fuzzy set v in a nonempty set U is described by its membership function $v : U \rightarrow [0, 1]$. So, for each $\eta \in U$ the degree of membership of η in v is defined by $v(\eta)$.

Definition 1 ([28]). Suppose that v is a fuzzy subset of \mathbb{R} . Then, v is called a fuzzy number such that v is upper semicontinuous membership function of bounded support, normal, and convex.

If v is a fuzzy number, then $[v]^\sigma = [v_1(\sigma), v_2(\sigma)]$, where $v_1(\sigma) = \min\{\eta \mid \eta \in [v]^\sigma\}$ and $v_2(\sigma) = \max\{\eta \mid \eta \in [v]^\sigma\}$ for each $\sigma \in [0, 1]$. The symbol $[v]^\sigma$ is called the σ -level representation or the parametric form of a fuzzy number v .

Theorem 2 ([29]). Suppose that $v_1, v_2 : [0, 1] \rightarrow \mathbb{R}$ satisfy the following conditions:

- (1) v_1 is a bounded nondecreasing function.
- (2) v_2 is a bounded nonincreasing function.
- (3) $v_1(1) \leq v_2(1)$.
- (4) for each $k \in (0, 1)$, $\lim_{\sigma \rightarrow k^-} v_1(\sigma) = v_1(k)$ and $\lim_{\sigma \rightarrow k^-} v_2(\sigma) = v_2(k)$.
- (5) $\lim_{\sigma \rightarrow 0^+} v_1(\sigma) = v_1(0)$ and $\lim_{\sigma \rightarrow 0^+} v_2(\sigma) = v_2(0)$.

Then $v : \mathbb{R} \rightarrow [0, 1]$ given by $v(x) = \sup\{\sigma \mid v_1(\sigma) \leq x \leq v_2(\sigma)\}$ is a fuzzy number with parameterization $[v_1(\sigma), v_2(\sigma)]$.

Definition 3 ([29]). Let $v, w \in \mathbb{R}_{\mathcal{F}}$. If there exists an element $\mathcal{P} \in \mathbb{R}_{\mathcal{F}}$ such that $v = w + \mathcal{P}$, then we say that \mathcal{P} is the Hukuhara difference (H-difference) of v and w , denoted by $v \ominus w$.

The sign \ominus stands always for Hukuhara difference. Thus, it should be noted that $v \ominus w \neq v + (-1)w$. Normally, $v + (-1)w$ is denoted by $v - w$. If the H-difference $v \ominus w$ exists, then $[v \ominus w]^\sigma = [v_1(\sigma) - w_1(\sigma), v_2(\sigma) - w_2(\sigma)]$.

Definition 4 ([30]). The complete metric structure on $\mathbb{R}_{\mathcal{F}}$ is given by the Hausdorff distance mapping $D_H : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}^+ \cup \{0\}$ such that

$$D_H(v, w) = \sup_{0 \leq \sigma \leq 1} \max\{|v_{1\sigma} - w_{1\sigma}|, |v_{2\sigma} - w_{2\sigma}|\}, \quad (1)$$

for arbitrary fuzzy numbers $v = (v_1, v_2)$ and $w = (w_1, w_2)$.

Definition 5 ([30]). Let $\varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$. Then the function φ is continuous at $x_0 \in [a, b]$ if for every $\epsilon > 0$, $\exists \delta = \delta(x_0, \epsilon) > 0$ such that $D_H(\varphi(x), \varphi(x_0)) < \epsilon$, for each $x \in [a, b]$, whenever $|x - x_0| < \delta$.

Remark 6. If the function $\varphi(x)$ is continuous for each $x \in [a, b]$, where the continuity is one-sided at endpoints of $[a, b]$, then $\varphi(x)$ is continuous function on $[a, b]$. This means that $\varphi(x)$ is continuous on $[a, b]$ if and only if $\varphi_{1\sigma}$ and $\varphi_{2\sigma}$ are continuous on $[a, b]$.

Definition 7 ([28]). For fixed $x_0 \in [a, b]$ and $\varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$, the function φ is called a strongly generalized differentiable at x_0 , if there is an element $\varphi'(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that either

- (i) the H-differences $\varphi(x_0 + \xi) \ominus \varphi(x_0)$, $\varphi(x_0) \ominus \varphi(x_0 - \xi)$ exist, for each $\xi > 0$ sufficiently tends to 0 and $\lim_{\xi \rightarrow 0^+} ((\varphi(x_0 + \xi) \ominus \varphi(x_0))/\xi) = \varphi'(x_0) = \lim_{\xi \rightarrow 0^+} ((\varphi(x_0) \ominus \varphi(x_0 - \xi))/\xi)$, or
- (ii) the H-differences $\varphi(x_0) \ominus \varphi(x_0 + \xi)$, $\varphi(x_0 - \xi) \ominus \varphi(x_0)$ exist, for each $\xi > 0$ sufficiently tends to 0 and $\lim_{\xi \rightarrow 0^+} ((\varphi(x_0) \ominus \varphi(x_0 + \xi))/-\xi) = \varphi'(x_0) = \lim_{\xi \rightarrow 0^+} ((\varphi(x_0 - \xi) \ominus \varphi(x_0))/-\xi)$,

where the limit here is taken in the complete metric space $(\mathbb{R}_{\mathcal{F}}, D_H)$.

Theorem 8 ([31]). Suppose that $\varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$, where $[\varphi(x)]^\sigma = [\varphi_{1\sigma}(x), \varphi_{2\sigma}(x)]$, $\forall \sigma \in [0, 1]$, then

- (1) the functions $\varphi_{1\sigma}$ and $\varphi_{2\sigma}$ are two differentiable functions and $[D_1^1 \varphi(x)]^\sigma = [\varphi'_{1\sigma}(x), \varphi'_{2\sigma}(x)]$, when φ is (1)-differentiable;
- (2) the functions $\varphi_{1\sigma}$ and $\varphi_{2\sigma}$ are two differentiable functions and $[D_2^1 \varphi(x)]^\sigma = [\varphi'_{2\sigma}(x), \varphi'_{1\sigma}(x)]$, when φ is (2)-differentiable.

Definition 9 ([31]). Suppose that $\varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$. One can say that φ is (n, m) -differentiable at $x_0 \in (a, b)$, if $D_n^1 \varphi$ exists on a neighborhood of x_0 as a fuzzy function and it is (m) -differentiable at x_0 . The second-order derivatives of φ at x are indicated by $\varphi''(x) = D_{n,m}^2 \varphi(x)$ for $n, m = \{1, 2\}$.

Theorem 10 ([32]). Let $D_1^1 \varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ and $D_2^1 \varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$, where $[\varphi(x)]^\sigma = [\varphi_{1\sigma}(x), \varphi_{2\sigma}(x)]$ for each $\sigma \in [0, 1]$:

- (1) If $D_1^1 \varphi$ is (1)-differentiable, then $\varphi'_{1\sigma}$ and $\varphi'_{2\sigma}$ are differentiable functions and $[D_{1,1}^2 \varphi(x)]^\sigma = [\varphi''_{1\sigma}(x), \varphi''_{2\sigma}(x)]$,
- (2) If $D_1^1 \varphi$ is (2)-differentiable, then $\varphi'_{1\sigma}$ and $\varphi'_{2\sigma}$ are differentiable functions and $[D_{1,2}^2 \varphi(x)]^\sigma = [\varphi''_{2\sigma}(x), \varphi''_{1\sigma}(x)]$,

- (3) If $D_{2,1}^1 \varphi$ is (1)-differentiable, then $\varphi'_{1\sigma}$ and $\varphi'_{2\sigma}$ are differentiable functions and $[D_{2,1}^2 \varphi(x)]^\sigma = [\varphi''_{2\sigma}(x), \varphi''_{1\sigma}(x)]$,
- (4) If $D_{2,2}^1 \varphi$ is (2)-differentiable, then $\varphi'_{1\sigma}$ and $\varphi'_{2\sigma}$ are differentiable functions and $[D_{2,2}^2 \varphi(x)]^\sigma = [\varphi''_{1\sigma}(x), \varphi''_{2\sigma}(x)]$.

Definition 11 ([32]). Let $\varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ and $\varphi \in C^{\mathcal{F}}[a, b] \cap L^{\mathcal{F}}[a, b]$. One can say that φ is Caputo fuzzy H -differentiable at x when $({}^C D_{a+}^{\gamma} \varphi)(x) = (1/\Gamma(1-\gamma)) \int_a^x \varphi'(\tau)/(x-\tau)^{\gamma} d\tau$ exists, where $0 < \gamma \leq 1$. Also, we say that φ is Caputo $[(1)-\gamma]$ -differentiable if φ is (1)-differentiable and φ is Caputo $[(2)-\gamma]$ differentiable if φ is (2)-differentiable, where $C^{\mathcal{F}}[a, b]$ and $L^{\mathcal{F}}[a, b]$ stand for the space of all continuous and Lebesgue integrable fuzzy-valued functions on $[a, b]$, respectively.

Theorem 12 ([33]). Let $0 < \gamma \leq 1$ and $\varphi \in C^{\mathcal{F}}[a, b]$. Then, for each $\sigma \in [0, 1]$, the Caputo fuzzy fractional derivative exists on (a, b) such that

$$\begin{aligned} [({}^C D_{a+}^{\gamma} \varphi)(x)]^{\sigma} &= \left[\frac{1}{\Gamma(1-\gamma)} \right. \\ &\quad \cdot \left. \int_a^x \frac{\varphi'_{1\sigma}(\tau)}{(x-\tau)^{\gamma}} d\tau, \frac{1}{\Gamma(1-\gamma)} \int_a^x \frac{\varphi'_{2\sigma}(\tau)}{(x-\tau)^{\gamma}} d\tau \right] \end{aligned} \quad (2)$$

for (1)-differentiable and

$$\begin{aligned} [({}^C D_{a+}^{\gamma} \varphi)(x)]^{\sigma} &= \left[\frac{1}{\Gamma(1-\gamma)} \right. \\ &\quad \cdot \left. \int_a^x \frac{\varphi'_{2\sigma}(\tau)}{(x-\tau)^{\gamma}} d\tau, \frac{1}{\Gamma(1-\gamma)} \int_a^x \frac{\varphi'_{1\sigma}(\tau)}{(x-\tau)^{\gamma}} d\tau \right] \end{aligned} \quad (3)$$

for (2)-differentiable.

The next characterization theorem shows a way to convert the FFDEs into a system of ordinary fractional differential equations (OFDEs), ignoring the fuzzy setting approach.

Theorem 13 ([34]). Consider the below fuzzy fractional IVPs

$$({}^C D_{t_0+}^{\gamma} \varphi)(t) = f(t, \varphi(t)), \quad t > t_0, \quad (4)$$

subject to

$$\varphi(t_0) = \varphi_0, \quad (5)$$

where $f : [a, b] \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ such that

(i) $[f(t, \varphi(t))]^{\sigma} = [f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t))]$.

(ii) for any $\epsilon > 0$ there exist $\delta > 0$ such that $|f_{1\sigma}(t, s, u) - f_{1\sigma}(t_1, s_1, u_1)| < \epsilon$ and $|f_{2\sigma}(t, s, u) - f_{2\sigma}(t_1, s_1, u_1)| < \epsilon$, $\forall \sigma \in [0, 1]$, whenever (t, s, u) and $(t_1, s_1, u_1) \in [a, b] \times \mathbb{R}^2$, $\|(t, s, u) - (t_1, s_1, u_1)\|_{\mathbb{R}^3} < \delta$ and $f_{1\sigma}, f_{2\sigma}$ are uniformly bounded on any bounded set.

(iii) there is a constant (say) $\ell > 0$ such that

$$\begin{aligned} &|f_{1\sigma}(t_2, s_2, u_2) - f_{1\sigma}(t_1, s_1, u_1)| \\ &\leq \ell \cdot \max\{|s_2 - s_1|, |u_2 - u_1|\}, \quad \forall \sigma \in [0, 1] \end{aligned} \quad (6)$$

and

$$\begin{aligned} &|f_{2\sigma}(t_2, s_2, u_2) - f_{2\sigma}(t_1, s_1, u_1)| \\ &\leq \ell \cdot \max\{|s_2 - s_1|, |u_2 - u_1|\}, \quad \forall \sigma \in [0, 1]. \end{aligned} \quad (7)$$

Therefore, there are two systems of OFDEs that are equivalent to FFDEs (4) and (5) as follows:

Case 1. When $\varphi(t)$ is Caputo $[(1)-\gamma]$ -differentiable

$$\begin{aligned} ({}^C D_{t_0+}^{\gamma} \varphi_{1\sigma})(t) &= f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \\ ({}^C D_{t_0+}^{\gamma} \varphi_{2\sigma})(t) &= f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \end{aligned} \quad (8)$$

with $\varphi_{1\sigma}(t_0) = \varphi_{01\sigma}$, $\varphi_{2\sigma}(t_0) = \varphi_{02\sigma}$.

Case 2. When $\varphi(t)$ is Caputo $[(2)-\gamma]$ -differentiable

$$\begin{aligned} ({}^C D_{t_0+}^{\gamma} \varphi_{1\sigma})(t) &= f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \\ ({}^C D_{t_0+}^{\gamma} \varphi_{2\sigma})(t) &= f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \end{aligned} \quad (9)$$

with $\varphi_{1\sigma}(t_0) = \varphi_{01\sigma}$, $\varphi_{2\sigma}(t_0) = \varphi_{02\sigma}$.

3. Formulation of Fuzzy Fractional IVPs of Order $1 < \gamma \leq 2$

Consider the below fuzzy fractional differential equation

$$\begin{aligned} ({}^C D_{a+}^{\gamma} \varphi)(t) &= g(t) \varphi'(t) + f(t, \varphi(t)), \\ a \leq t \leq b, \quad 1 < \gamma \leq 2, \end{aligned} \quad (10)$$

subject to fuzzy initial conditions

$$\begin{aligned} \varphi(a) &= \alpha, \\ \varphi'(a) &= \beta. \end{aligned} \quad (11)$$

where $\alpha, \beta \in \mathbb{R}_{\mathcal{F}}$, $f : [a, b] \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ is a linear or nonlinear continuous fuzzy-valued function, $g(t)$ is a continuous real valued function with nonnegative values on $[a, b]$, and $\varphi(t)$ is unknown analytical fuzzy function to be determined. We assume that the fuzzy fractional IVPs (10) and (11) have unique smooth solution on the domain of interest.

Next, some theorems and definitions which are used later in this paper are presented.

Definition 14. Let $\varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy function such that $\varphi, \varphi' \in C^{\mathcal{F}}[a, b] \cap L^{\mathcal{F}}[a, b]$. Then, for $1 < \gamma \leq 2$, Caputo's H -derivative of φ at $x \in (a, b)$ is defined as

$$({}^C D_{a+}^{\gamma} \varphi)(x) = \frac{1}{\Gamma(2-\gamma)} \int_a^x \varphi''(\tau) (x-\tau)^{1-\gamma} d\tau. \quad (12)$$

Also, we say that φ is Caputo $[(n, m) - \gamma]$ -differentiable for $n, m \in \{1, 2\}$, when $({}^C D_{a+}^{\gamma} \varphi)(x)$ exists, and φ is (n, m) -differentiable.

Theorem 15. Let $\varphi, \varphi' \in C^{\mathcal{F}}[a, b]$, such that $[\varphi(x)]^{\sigma} = [\varphi_{1\sigma}(x), \varphi_{2\sigma}(x)]$, $\forall \sigma \in [0, 1]$. Caputo's H -derivative of order $1 < \gamma \leq 2$ exists on (a, b) such that

$$(i) \text{ If } \varphi \text{ is } (1,1)\text{-differentiable, then } [(^C D_{a+}^{\gamma} \varphi)(x)]^{\sigma} = [M \int_a^x \varphi_{1\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau, M \int_a^x \varphi_{2\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau] = [(^C D_{a+}^{\gamma} \varphi_{1\sigma})(x), (^C D_{a+}^{\gamma} \varphi_{2\sigma})(x)].$$

$$(ii) \text{ If } \varphi \text{ is } (1,2)\text{-differentiable, then } [(^C D_{a+}^{\gamma} \varphi)(x)]^{\sigma} = [M \int_a^x \varphi_{2\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau, M \int_a^x \varphi_{1\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau] = [(^C D_{a+}^{\gamma} \varphi_{2\sigma})(x), (^C D_{a+}^{\gamma} \varphi_{1\sigma})(x)].$$

$$(iii) \text{ If } \varphi \text{ is } (2,1)\text{-differentiable, then } [(^C D_{a+}^{\gamma} \varphi)(x)]^{\sigma} = [M \int_a^x \varphi_{2\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau, M \int_a^x \varphi_{1\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau] = [(^C D_{a+}^{\gamma} \varphi_{2\sigma})(x), (^C D_{a+}^{\gamma} \varphi_{1\sigma})(x)].$$

$$(iv) \text{ If } \varphi \text{ is } (2,2)\text{-differentiable, then } [(^C D_{a+}^{\gamma} \varphi)(x)]^{\sigma} = [M \int_a^x \varphi_{1\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau, M \int_a^x \varphi_{2\sigma}''(\tau)(x-\tau)^{1-\gamma} d\tau] = [(^C D_{a+}^{\gamma} \varphi_{1\sigma})(x), (^C D_{a+}^{\gamma} \varphi_{2\sigma})(x)], \text{ where } M = 1/\Gamma(2-\gamma).$$

The (n, m) -solution of fuzzy fractional IVPs (10) and (11) is a function $\varphi : [a, b] \rightarrow \mathbb{R}_{\mathcal{F}}$ that has Caputo $[(n, m) - \gamma]$ -differentiable and satisfies the FFIVPs (10) and (11). To compute it, we firstly convert the fuzzy problem into equivalent system of second OFDEs, called correspondence (n, m) -system, based upon the type of derivative chosen. Then, by utilizing the σ -cut representation of $\varphi(t)$, $f(t, \varphi(t))$, and the initial data in (11) such that $[\varphi(t)]^{\sigma} = [\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[f(t, \varphi(t))]^{\sigma} = [f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t))]$, $[\varphi(a)]^{\sigma} = [\varphi_{1\sigma}(a), \varphi_{2\sigma}(a)] = [\alpha_{1\sigma}, \alpha_{2\sigma}]$, and $[\varphi'(a)]^{\sigma} = [\varphi'_{1\sigma}(a), \varphi'_{2\sigma}(a)] = [\beta_{1\sigma}, \beta_{2\sigma}]$, the following corresponding (n, m) -systems will be hold:

(i) the (1,1)-system such that

$$\begin{aligned} (^C D_{a+}^{\gamma} \varphi_{1\sigma})(t) &= g(t) \varphi'_{1\sigma}(t) \\ &\quad + f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \\ (^C D_{a+}^{\gamma} \varphi_{2\sigma})(t) &= g(t) \varphi'_{2\sigma}(t) \\ &\quad + f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \end{aligned} \quad (13)$$

(ii) the (1,2)-system such that

$$\begin{aligned} (^C D_{a+}^{\gamma} \varphi_{2\sigma})(t) &= g(t) \varphi'_{1\sigma}(t) \\ &\quad + f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \\ (^C D_{a+}^{\gamma} \varphi_{1\sigma})(t) &= g(t) \varphi'_{2\sigma}(t) \\ &\quad + f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \end{aligned} \quad (14)$$

(iii) the (2,1)-system such that

$$\begin{aligned} (^C D_{a+}^{\gamma} \varphi_{2\sigma})(t) &= g(t) \varphi'_{2\sigma}(t) \\ &\quad + f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \\ (^C D_{a+}^{\gamma} \varphi_{1\sigma})(t) &= g(t) \varphi'_{1\sigma}(t) \\ &\quad + f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \end{aligned} \quad (15)$$

(iv) the (2,2)-system such that

$$\begin{aligned} (^C D_{a+}^{\gamma} \varphi_{1\sigma})(t) &= g(t) \varphi'_{2\sigma}(t) \\ &\quad + f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \\ (^C D_{a+}^{\gamma} \varphi_{2\sigma})(t) &= g(t) \varphi'_{1\sigma}(t) \\ &\quad + f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \end{aligned} \quad (16)$$

subject to initial conditions

$$\begin{aligned} \varphi_{1\sigma}(a) &= \alpha_{1\sigma}, \\ \varphi'_{1\sigma}(a) &= \beta_{1\sigma}, \\ \varphi_{2\sigma}(a) &= \alpha_{2\sigma}, \\ \varphi'_{2\sigma}(a) &= \beta_{2\sigma}. \end{aligned} \quad (17)$$

Theorem 16 ([33]). Let $n, m \in \{1, 2\}$ and let $[\varphi(t)]^{\sigma} = [\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$ be an (n, m) -solution of FFIVPs (10) and (11) on $[a, b]$. Then, $\varphi_{1\sigma}(t)$ and $\varphi_{2\sigma}(t)$ will be a solution to the associated (n, m) -system.

Theorem 17 ([33]). Let $n, m \in \{1, 2\}$ and let $\varphi_{1\sigma}(t)$ and $\varphi_{2\sigma}(t)$ be the solution of (n, m) -system for each $\sigma \in [0, 1]$. If $[\varphi(t)]^{\sigma} = [\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$ has valid level sets and $\varphi(t)$ is Caputo $[(n, m) - \gamma]$ -differentiable, then $\varphi(t)$ is an (n, m) -solution of FFIVPs (10) and (11) on $[a, b]$.

The aim of the next algorithm is to perform a strategy to solve the FFIVPs (10) and (11) in terms of its σ -cut representation form. Indeed, there are four cases that depend on type of differentiability.

Algorithm 18. To determine the solutions of FFIVPs (10) and (11), do the following:

Case (I). If $\varphi(t)$ is Caputo $[(1,1)-\gamma]$ -differentiable and the FFIVPs (10) and (11) will be converted to crisp system described in (13) and (17), then do the following steps:

Step 1: Solve the required system.

Step 2: Ensure that $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{1\sigma}(t), \varphi'_{2\sigma}(t)]$ and $[\varphi''_{1\sigma}(t), \varphi''_{2\sigma}(t)]$ are valid level sets for each $\sigma \in [0, 1]$.

Step 3: Construct (1,1)-solution $\varphi(t)$ whose σ -cut representation is $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$.

Case (II). If $\varphi(t)$ is Caputo $[(1,2)-\gamma]$ -differentiable and the FFIVPs (10) and (11) will be converted to crisp system described in (14) and (17), then do the following steps:

Step 1: Solve the required system.

Step 2: Ensure that $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{1\sigma}(t), \varphi'_{2\sigma}(t)]$ and $[\varphi''_{2\sigma}(t), \varphi''_{1\sigma}(t)]$ are valid level sets for each $\sigma \in [0, 1]$.

Step 3: Construct (1,2)-solution $\varphi(t)$ whose σ -cut representation is $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$.

Case (III). If $\varphi(t)$ is Caputo $[(2,1)-\gamma]$ -differentiable and the FFIVPs (10) and (11) will be converted to crisp system described in (15) and (17), then do the following steps:

Step 1: Solve the required system.

Step 2: Ensure that $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{1\sigma}(t), \varphi'_{2\sigma}(t)]$ and $[\varphi''_{1\sigma}(t), \varphi''_{2\sigma}(t)]$ are valid level sets for each $\sigma \in [0, 1]$.

Step 3: Construct (2,1)-solution $\varphi(t)$ whose σ -cut representation is $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$.

Case (IV). If $\varphi(t)$ is Caputo $[(2,2)-\gamma]$ -differentiable and the FFIVPs (10) and (11) will be converted to crisp system described in (16) and (17), then do the following steps:

Step 1: Solve the required system.

Step 2: Ensure that $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{1\sigma}(t), \varphi'_{2\sigma}(t)]$ and $[\varphi''_{1\sigma}(t), \varphi''_{2\sigma}(t)]$ are valid level sets for each $\sigma \in [0, 1]$.

Step 3: Construct (2,2)-solution $\varphi(t)$ whose σ -cut representation is $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$.

4. Description of Fractional RPS Method

In this section, the RPS scheme is presented for constructing an analytical solution of FFIVPs (10) and (11) through substituting the expansion of fractional power series (FPS) among the truncated residual functions. In view of that, the resultant equation helps us to derive a recursion formula for the coefficients' computation, where the coefficients can be computed recursively through the recurrent fractional differentiating of the truncated residual function.

Definition 19 ([35]). A fractional power series (FPS) representation at t_0 has the following form:

$$\sum_{n=0}^{\infty} c_n (t - t_0)^{n\gamma} = c_0 + c_1 (t - t_0)^{\gamma} + c_2 (t - t_0)^{2\gamma} + \dots, \quad (18)$$

where $0 \leq m - 1 < \gamma \leq m$, $t \geq t_0$, and c_n 's are the coefficients of the series.

Theorem 20 ([35]). Suppose that f has the following FPS representation at t_0 :

$$f(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^{n\gamma}, \quad (19)$$

where $f(t) \in C[t_0, t_0 + R]$ and ${}^C D^{m\gamma} f(t) \in C(t_0, t_0 + R)$ for $n = 0, 1, 2, \dots$; then the coefficients c_n will be in the form $c_n = {}^C D^{m\gamma} f(t_0) / \Gamma(1 + n\gamma)$ such that ${}^C D^{m\gamma} = {}^C D^{\gamma} \cdot {}^C D^{\gamma} \cdot \dots \cdot {}^C D^{\gamma}$ (n -times).

Conveniently, for obtaining (n, m) -solution of FFIVPs (10) and (11) utilizing the solution of the corresponding (n, m) -system, we will explain the fashion to determine (1, 1)-solution equivalent to the solution for the system of OFDEs (13) and (17). Further, same manner can be applied to construct other type of (n, m) -solutions. To achieve our goal,

assume that the solution of OFDEs (13) and (17) at $t_0 = 0$ has the following form:

$$\begin{aligned} \varphi_{1\sigma}(t) &= \sum_{n=0}^{\infty} c_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}, \\ \varphi_{2\sigma}(t) &= \sum_{n=0}^{\infty} d_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}. \end{aligned} \quad (20)$$

Since $\varphi_{1\sigma}(t)$ and $\varphi_{2\sigma}(t)$ satisfy the initial conditions in (17), then the following polynomials $\varphi_{1\sigma}(t) = \alpha_{1\sigma} + \beta_{1\sigma}t$ and $\varphi_{2\sigma}(t) = \alpha_{2\sigma} + \beta_{2\sigma}t$ will be the initial guesses for the system and the solutions can also be represented by

$$\begin{aligned} \varphi_{1\sigma}(t) &= \alpha_{1\sigma} + \beta_{1\sigma}t + \sum_{n=1}^{\infty} c_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}, \\ \varphi_{2\sigma}(t) &= \alpha_{2\sigma} + \beta_{2\sigma}t + \sum_{n=1}^{\infty} d_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}. \end{aligned} \quad (21)$$

Consequently, the k th-truncated series solutions can be given by

$$\begin{aligned} \varphi_{k,1\sigma}(t) &= \alpha_{1\sigma} + \beta_{1\sigma}t + \sum_{n=1}^k c_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}, \\ \varphi_{k,2\sigma}(t) &= \alpha_{2\sigma} + \beta_{2\sigma}t + \sum_{n=1}^k d_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}. \end{aligned} \quad (22)$$

The residual functions $Res_{1\sigma}(t)$ and $Res_{2\sigma}(t)$ are defined as follows:

$$\begin{aligned} Res_{1\sigma}(t) &= ({}^C D_{0+}^{\gamma} \varphi_{1\sigma})(t) - g(t) \varphi'_{1\sigma}(t) \\ &\quad - f_{1\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \\ Res_{2\sigma}(t) &= ({}^C D_{0+}^{\gamma} \varphi_{2\sigma})(t) - g(t) \varphi'_{2\sigma}(t) \\ &\quad - f_{2\sigma}(t, \varphi_{1\sigma}(t), \varphi_{2\sigma}(t)), \end{aligned} \quad (23)$$

and the k th-residual functions $Res_{k,1\sigma}(t)$ and $Res_{k,2\sigma}(t)$ for $k = 1, 2, 3, \dots, n$ are defined as follows:

$$\begin{aligned} Res_{k,1\sigma}(t) &= ({}^C D_{0+}^{\gamma} \varphi_{k,1\sigma})(t) - g(t) \varphi'_{k,1\sigma}(t) \\ &\quad - f_{1\sigma}(t, \varphi_{k,1\sigma}(t), \varphi_{k,2\sigma}(t)), \\ Res_{k,2\sigma}(t) &= ({}^C D_{0+}^{\gamma} \varphi_{k,2\sigma})(t) - g(t) \varphi'_{k,2\sigma}(t) \\ &\quad - f_{2\sigma}(t, \varphi_{k,1\sigma}(t), \varphi_{k,2\sigma}(t)). \end{aligned} \quad (24)$$

From (23), we have $Res_{n\sigma}(t) = 0$ and $\lim_{k \rightarrow \infty} Res_{k,n\sigma}(t) = Res_{n\sigma} \equiv 0$ for $n = 1, 2$ and each $t \geq 0$, which leads to ${}^C D_t^{m\gamma} Res_{n\sigma}(t) = 0$. Also, the fractional derivatives ${}^C D_t^{m\gamma} Res_{n\sigma}(t)$ and ${}^C D_t^{m\gamma} Res_{k,n\sigma}(t)$ are equivalent at $t = 0$ for each $m = 0, 1, 2, \dots, k$, that is, ${}^C D_t^{m\gamma} Res_{n\sigma}(0) = {}^C D_t^{m\gamma} Res_{k,n\sigma}(0) = 0$. However, ${}^C D_t^{(k-1)\gamma} Res_{k,n\sigma}(0) = 0$ holds for $n = 1, 2$.

Regarding employing the RPS algorithm to obtain the 1st unknown coefficients, c_1 and d_1 , substitute the 1st approximations $\varphi_{1,1\sigma}(t) = \alpha_{1\sigma} + \beta_{1\sigma}t + c_1(t^\gamma/\Gamma(1+\gamma))$ and $\varphi_{1,2\sigma}(t) = \alpha_{2\sigma} + \beta_{2\sigma}t + d_1(t^\gamma/\Gamma(1+\gamma))$ into the 1st residual functions $Res_{1,1\sigma}(t)$ and $Res_{1,2\sigma}(t)$ of (24) such that

$$\begin{aligned} Res_{1,1\sigma}(t) &= \left({}^C D_{0^+}^\gamma \varphi_{1,1\sigma}\right)(t) - g(t) \varphi'_{1,1\sigma}(t) \\ &\quad - f_{1\sigma}(t, \varphi_{1,1\sigma}(t), \varphi_{1,2\sigma}(t)), \\ Res_{1,2\sigma}(t) &= \left({}^C D_{0^+}^\gamma \varphi_{1,2\sigma}\right)(t) - g(t) \varphi'_{1,2\sigma}(t) \\ &\quad - f_{2\sigma}(t, \varphi_{1,1\sigma}(t), \varphi_{1,2\sigma}(t)), \end{aligned} \quad (25)$$

and based upon the facts $Res_{1,1\sigma}(0) = Res_{1,2\sigma}(0) = 0$, we have $c_1 = g(0)\varphi'_{1,1\sigma}(0) - f_{1\sigma}(0, \alpha_{1\sigma}, \alpha_{2\sigma})$ and $d_1 = g(0)\varphi'_{1,2\sigma}(0) - f_{2\sigma}(0, \alpha_{1\sigma}, \alpha_{2\sigma})$. Therefore, the 1st RPS approximate solutions can be written as

$$\begin{aligned} \varphi_{1,1\sigma}(t) &= \alpha_{1\sigma} + \beta_{1\sigma}t \\ &\quad + \left(g(0)\varphi'_{1,1\sigma}(0) - f_{1\sigma}(0, \alpha_{1\sigma}, \alpha_{2\sigma})\right) \frac{t^\gamma}{\Gamma(1+\gamma)}, \end{aligned}$$

$$\begin{aligned} \varphi_{1,2\sigma}(t) &= \alpha_{2\sigma} + \beta_{2\sigma}t \\ &\quad + \left(g(0)\varphi'_{1,2\sigma}(0) - f_{2\sigma}(0, \alpha_{1\sigma}, \alpha_{2\sigma})\right) \frac{t^\gamma}{\Gamma(1+\gamma)}. \end{aligned} \quad (26)$$

Currently, for the 2nd unknown coefficients, c_2 and d_2 substitute $\varphi_{2,1\sigma}(t) = \alpha_{1\sigma} + \beta_{1\sigma}t + \sum_{n=1}^2 c_n(t^{n\gamma}/\Gamma(1+n\gamma))$ and $\varphi_{2,2\sigma}(t) = \alpha_{2\sigma} + \beta_{2\sigma}t + \sum_{n=1}^2 d_n(t^{n\gamma}/\Gamma(1+n\gamma))$ into the 2nd residual functions, $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$ of (24) such that

$$\begin{aligned} Res_{2,1\sigma}(t) &= \left({}^C D_{0^+}^\gamma \varphi_{2,1\sigma}\right)(t) - g(t) \varphi'_{2,1\sigma}(t) \\ &\quad - f_{1\sigma}(t, \varphi_{2,1\sigma}(t), \varphi_{2,2\sigma}(t)), \\ Res_{2,2\sigma}(t) &= \left({}^C D_{0^+}^\gamma \varphi_{2,2\sigma}\right)(t) - g(t) \varphi'_{2,2\sigma}(t) \\ &\quad - f_{2\sigma}(t, \varphi_{2,1\sigma}(t), \varphi_{2,2\sigma}(t)). \end{aligned} \quad (27)$$

Then, by applying the fractional derivative ${}^C D_t^\gamma$ on both sides of $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$, using the facts ${}^C D_t^\gamma Res_{2,1\sigma}(0) = {}^C D_t^\gamma Res_{2,2\sigma}(0) = 0$ as well, the values of c_2 and d_2 will be given by

$$\begin{aligned} c_2 &= \frac{\Gamma(2\gamma)}{\Gamma(\gamma)} \left(\frac{\Gamma(\gamma) \beta_{1\sigma} ({}^C D_t^\gamma (g(t)))|_{t=0} + c_1 ({}^C D_t^\gamma (g(t) \cdot t^{\gamma-1}))|_{t=0} + f_{1\sigma}(0, c_1, d_1)}{\Gamma(2\gamma) - ({}^C D_t^\gamma (g(t) \cdot t^{2\gamma-1}))|_{t=0}} \right), \\ d_2 &= \frac{\Gamma(2\gamma)}{\Gamma(\gamma)} \left(\frac{\Gamma(\gamma) \beta_{2\sigma} ({}^C D_t^\gamma (g(t)))|_{t=0} + d_1 ({}^C D_t^\gamma (g(t) \cdot t^{\gamma-1}))|_{t=0} + f_{2\sigma}(0, c_1, d_1)}{\Gamma(2\gamma) - ({}^C D_t^\gamma (g(t) \cdot t^{2\gamma-1}))|_{t=0}} \right). \end{aligned} \quad (28)$$

For the 3rd unknown coefficients, c_3 and d_3 substitute $\varphi_{3,1\sigma}(t) = \alpha_{1\sigma} + \beta_{1\sigma}t + \sum_{n=1}^3 c_n(t^{n\gamma}/\Gamma(1+n\gamma))$ and $\varphi_{3,2\sigma}(t) = \alpha_{2\sigma} + \beta_{2\sigma}t + \sum_{n=1}^3 d_n(t^{n\gamma}/\Gamma(1+n\gamma))$ into the 3rd residual functions, $Res_{3,1\sigma}(t)$ and $Res_{3,2\sigma}(t)$ of (24), and then by

computing ${}^C D_t^{2\gamma} Res_{3,1\sigma}(t)$ and ${}^C D_t^{2\gamma} Res_{3,2\sigma}(t)$ and using the facts ${}^C D_t^{2\gamma} Res_{3,1\sigma}(0) = {}^C D_t^{2\gamma} Res_{3,2\sigma}(0) = 0$, the coefficients, c_3 and d_3 , will be given such that

$$\begin{aligned} c_3 &= \frac{\Gamma(3\gamma)}{\Gamma(\gamma)\Gamma(2\gamma)} \left(\frac{\Gamma(\gamma)\Gamma(2\gamma) \beta_{1\sigma} ({}^C D_t^{2\gamma} (g(t)))|_{t=0} + c_1 ({}^C D_t^{2\gamma} (g(t) \cdot t^{\gamma-1}))|_{t=0} + c_2 ({}^C D_t^{2\gamma} (g(t) \cdot t^{2\gamma-1}))|_{t=0} + f_{1\sigma}(0, c_2, d_2)}{\Gamma(3\gamma) - ({}^C D_t^{2\gamma} (g(t) \cdot t^{3\gamma-1}))|_{t=0}} \right), \\ d_3 &= \frac{\Gamma(3\gamma)}{\Gamma(\gamma)\Gamma(2\gamma)} \left(\frac{\Gamma(\gamma)\Gamma(2\gamma) \beta_{2\sigma} ({}^C D_t^{2\gamma} (g(t)))|_{t=0} + d_1 ({}^C D_t^{2\gamma} (g(t) \cdot t^{\gamma-1}))|_{t=0} + d_2 ({}^C D_t^{2\gamma} (g(t) \cdot t^{2\gamma-1}))|_{t=0} + f_{2\sigma}(0, c_2, d_2)}{\Gamma(3\gamma) - ({}^C D_t^{2\gamma} (g(t) \cdot t^{3\gamma-1}))|_{t=0}} \right). \end{aligned} \quad (29)$$

Using similar argument, the 4th unknown coefficients, c_4 and d_4 , will be given utilizing the facts ${}^C D_t^{3\gamma} Res_{4,1\sigma}(0) = {}^C D_t^{3\gamma} Res_{4,2\sigma}(0) = 0$. The same manner

can be repeated until we obtain on the coefficients' arbitrary order of the FPS solution for the OFDE (13).

5. Numerical Simulation and Discussion

This section aims to verify the efficiency and applicability of the proposed algorithm by applying the RPS method to a numerical example. Here, all necessary calculations and analysis are done using Mathematica 10.

For this purpose, let us consider the fuzzy fractional differential equation

$$\left({}^C D_{0+}^{\gamma} \varphi\right)(t) = \mu, \quad 0 \leq t \leq 1, \quad (30)$$

with the fuzzy initial conditions

$$\begin{aligned} \varphi(0) &= \alpha, \\ \varphi'(0) &= \beta, \end{aligned} \quad (31)$$

where $\gamma \in (1, 2]$ and μ, α, β are the fuzzy numbers whose σ -cut representation is $[\sigma - 1, 1 - \sigma]$.

Based on the type of differentiability, the FFIVPs (30) and (31) can be converted into one of the following systems.

Case 1. If $\varphi(t)$ is (1,1)-solution, then the corresponding (1,1)-system will be

$$\begin{aligned} \left({}^C D_{0+}^{\gamma} \varphi_{1\sigma}\right)(t) &= \sigma - 1, \\ \left({}^C D_{0+}^{\gamma} \varphi_{2\sigma}\right)(t) &= 1 - \sigma, \\ \varphi_{1\sigma}(0) &= \varphi'_{1\sigma}(0) = \sigma - 1, \\ \varphi_{2\sigma}(0) &= \varphi'_{2\sigma}(0) = 1 - \sigma. \end{aligned} \quad (32)$$

If $\gamma = 2$, then the exact solution of (32) is $[\varphi(t)]^{\sigma} = [\sigma - 1, 1 - \sigma](1 + t + t^2/\Gamma(\gamma + 1))$, $t \in [0, 1]$. In finding the fuzzy (1,1)-solution of FFDEs (30), let $\varphi(t)$ be Caputo [(1,1)- γ]-differentiable. Sequentially, after selecting the initial guesses as $\varphi_{0,1\sigma}(t) = (\sigma - 1) + (\sigma - 1)t$ and $\varphi_{0,2\sigma}(t) = (1 - \sigma) + (1 - \sigma)t$, the FPS expansion of solutions for OFDEs (32) can be represented as follows:

$$\begin{aligned} \varphi_{1\sigma}(t) &= (\sigma - 1) + (\sigma - 1)t + \sum_{n=1}^{\infty} c_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}, \\ \varphi_{2\sigma}(t) &= (1 - \sigma) + (1 - \sigma)t + \sum_{n=1}^{\infty} d_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}. \end{aligned} \quad (33)$$

To determine the 1st RPS approximate solution for OFDEs (32), substitute the 1st-truncated series $\varphi_{1,1\sigma}(t) = (\sigma - 1) + (\sigma - 1)t + c_1(t^{\gamma}/\Gamma(1 + \gamma))$ and $\varphi_{1,2\sigma}(t) = (1 - \sigma) + (1 - \sigma)t + d_1(t^{\gamma}/\Gamma(1 + \gamma))$ into the 1st-residual functions $Res_{1,1\sigma}(t)$ and $Res_{1,2\sigma}(t)$ such that $Res_{1,1\sigma}(t) = 1 - \sigma + c_1$ and $Res_{1,2\sigma}(t) = \sigma - 1 + d_1$. Thus, based upon the facts $Res_{1,1\sigma}(0) = 0$ and $Res_{1,2\sigma}(0) = 0$, we have $c_1 = \sigma - 1$ and $d_1 = 1 - \sigma$. Hence, the 1st RPS approximate solution for OFDEs (32) can be written in the form of

$$\begin{aligned} \varphi_{1,1\sigma}(t) &= (\sigma - 1) + (\sigma - 1)t + (\sigma - 1) \frac{t^{\gamma}}{\Gamma(1 + \gamma)}, \\ \varphi_{1,2\sigma}(t) &= (1 - \sigma) + (1 - \sigma)t + (1 - \sigma) \frac{t^{\gamma}}{\Gamma(1 + \gamma)}. \end{aligned} \quad (34)$$

Similarly, to find out the 2nd RPS approximate solution for OFDEs (32), substitute the 2nd truncated series $\varphi_{2,1\sigma}(t) = (\sigma - 1) + (\sigma - 1)t + c_1(t^{\gamma}/\Gamma(1 + \gamma)) + c_2(t^{2\gamma}/\Gamma(1 + 2\gamma))$ and $\varphi_{2,2\sigma}(t) = (1 - \sigma) + (1 - \sigma)t + d_1(t^{\gamma}/\Gamma(1 + \gamma)) + d_2(t^{2\gamma}/\Gamma(1 + 2\gamma))$ into the 2nd residual functions $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$ such that $Res_{2,1\sigma}(t) = ({}^C D_{0+}^{\gamma} \varphi_{2,1\sigma})(t) - (\sigma - 1) = 1 - \sigma + c_1 + c_2(t^{\gamma}/\Gamma(1 + \gamma))$ and $Res_{2,2\sigma}(t) = ({}^C D_{0+}^{\gamma} \varphi_{2,2\sigma})(t) - (1 - \sigma) = -1 + \sigma + d_1 + d_2(t^{\gamma}/\Gamma(1 + \gamma))$. Now, applying the fractional derivative ${}^C D_t^{\gamma}$ on both sides of $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$ yields the following: ${}^C D_t^{\gamma} Res_{2,1\sigma}(t) = c_2$ and ${}^C D_t^{\gamma} Res_{2,2\sigma}(t) = d_2$. So, the 2nd unknown coefficients are $c_2 = 0$ and $d_2 = 0$ through using the facts ${}^C D_t^{\gamma} Res_{2,1\sigma}(0) = {}^C D_t^{\gamma} Res_{2,2\sigma}(0) = 0$. Therefore, the 2nd RPS approximate solution for OFDEs (32) is given by

$$\begin{aligned} \varphi_{2,1\sigma}(t) &= (\sigma - 1) + (\sigma - 1)t + (\sigma - 1) \frac{t^{\gamma}}{\Gamma(1 + \gamma)}, \\ \varphi_{2,2\sigma}(t) &= (1 - \sigma) + (1 - \sigma)t + (1 - \sigma) \frac{t^{\gamma}}{\Gamma(1 + \gamma)}. \end{aligned} \quad (35)$$

Accordingly, the unknown coefficients c_n and d_n will be vanished for $n \geq 3$ by continuing in the similar approach, that is, $\sum_{n=3}^{\infty} c_n(t^{n\gamma}/\Gamma(1 + n\gamma)) = 0$ and $\sum_{n=3}^{\infty} d_n(t^{n\gamma}/\Gamma(1 + n\gamma)) = 0$. Hence, the RPS approximate solutions corresponding to (1,1)-system are coinciding well with the exact solutions $\varphi_{1\sigma}(t) = (1 + t + (t^{\gamma}/\Gamma(1 + \gamma)))(\sigma - 1)$ and $\varphi_{2\sigma}(t) = (1 + t + t^{\gamma}/\Gamma(1 + \gamma))(1 - \sigma)$. Here, $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{1\sigma}(t), \varphi'_{2\sigma}(t)]$, and $[\varphi''_{1\sigma}(t), \varphi''_{2\sigma}(t)]$ are valid level sets for $\sigma \in [0, 1]$ and $t \in [0, 1]$. Moreover, $\varphi(t) = \mu(1 + t + t^{\gamma}/\Gamma(1 + \gamma))$ is a (1,1)-solution for FFIVPs (30) and (31) on $[0, 1]$.

Case 2. If $\varphi(t)$ is (1,2)-solution, then the corresponding (1,2)-system will be

$$\begin{aligned} \left({}^C D_{0+}^{\gamma} \varphi_{1\sigma}\right)(t) &= 1 - \sigma, \\ \left({}^C D_{0+}^{\gamma} \varphi_{2\sigma}\right)(t) &= \sigma - 1, \\ \varphi_{1\sigma}(0) &= \varphi'_{1\sigma}(0) = \sigma - 1, \\ \varphi_{2\sigma}(0) &= \varphi'_{2\sigma}(0) = 1 - \sigma, \end{aligned} \quad (36)$$

If $\gamma = 2$, then the exact solution of (36) is $[\varphi(t)]^{\sigma} = [\sigma - 1, 1 - \sigma](1 + t - t^2/\Gamma(\gamma + 1))$, $t \in [0, 1]$. In finding the fuzzy (1,2)-solution of FFDEs (30), let $\varphi(t)$ be Caputo [(1,2)- γ]-differentiable. Sequentially, after selecting the initial guesses as in case 1, the FPS expansion of solutions for OFDEs (36) can be represented by

$$\begin{aligned} \varphi_{1\sigma}(t) &= (\sigma - 1) + (\sigma - 1)t + \sum_{n=1}^{\infty} c_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}, \\ \varphi_{2\sigma}(t) &= (1 - \sigma) + (1 - \sigma)t + \sum_{n=1}^{\infty} d_n \frac{t^{n\gamma}}{\Gamma(1 + n\gamma)}. \end{aligned} \quad (37)$$

To determine the 1st RPS approximate solution for OFDEs (36), substitute the 1st truncated series $\varphi_{1,1\sigma}(t) =$

$(\sigma - 1) + (\sigma - 1)t + c_1(t^\gamma/\Gamma(1 + \gamma))$ and $\varphi_{1,2\sigma}(t) = (1 - \sigma) + (1 - \sigma)t + d_1(t^\gamma/\Gamma(1 + \gamma))$ into the 1st residual functions $Res_{1,1\sigma}(t)$ and $Res_{1,2\sigma}(t)$ such that $Res_{1,1\sigma}(t) = \sigma - 1 + c_1$ and $Res_{1,2\sigma}(t) = 1 - \sigma + d_1$. Thus, based upon the facts $Res_{1,1\sigma}(0) = Res_{1,2\sigma}(0) = 0$, we have $c_1 = 1 - \sigma$ and $d_1 = \sigma - 1$. Hence, the 1st RPS approximate solution for OFDEs (36) can be written in the form of

$$\begin{aligned}\varphi_{1,1\sigma}(t) &= (\sigma - 1) + (\sigma - 1)t + (1 - \sigma) \frac{t^\gamma}{\Gamma(1 + \gamma)}, \\ \varphi_{1,2\sigma}(t) &= (1 - \sigma) + (1 - \sigma)t + (\sigma - 1) \frac{t^\gamma}{\Gamma(1 + \gamma)}.\end{aligned}\quad (38)$$

Similarly, to find out the 2nd RPS approximate solution for OFDEs (36), substitute the 2nd truncated series $\varphi_{2,1\sigma}(t) = (\sigma - 1) + (\sigma - 1)t + c_1(t^\gamma/\Gamma(1 + \gamma)) + c_2(t^{2\gamma}/\Gamma(1 + 2\gamma))$ and $\varphi_{2,2\sigma}(t) = (1 - \sigma) + (1 - \sigma)t + d_1(t^\gamma/\Gamma(1 + \gamma)) + d_2(t^{2\gamma}/\Gamma(1 + 2\gamma))$ into the 2nd residual functions $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$ such that $Res_{2,1\sigma}(t) = 1 - \sigma + c_1 + c_2(t^\gamma/\Gamma(1 + \gamma))$ and $Res_{2,2\sigma}(t) = -1 + \sigma + d_1 + d_2(t^\gamma/\Gamma(1 + \gamma))$. Then, applying the fractional derivative ${}^C D_t^\gamma$ on both sides of $Res_{2,1\sigma}(t)$ and $Res_{2,2\sigma}(t)$ yields the following: ${}^C D_t^\gamma Res_{2,1\sigma}(t) = c_2$ and ${}^C D_t^\gamma Res_{2,2\sigma}(t) = d_2$. So, the 2nd unknown coefficients are $c_2 = 0$ and $d_2 = 0$ through using the facts ${}^C D_t^\gamma Res_{2,1\sigma}(0) = {}^C D_t^\gamma Res_{2,2\sigma}(0) = 0$. Therefore, the 2nd RPS approximate solution for OFDEs (36) is given by

$$\begin{aligned}\varphi_{2,1\sigma}(t) &= (\sigma - 1) + (\sigma - 1)t + (1 - \sigma) \frac{t^\gamma}{\Gamma(1 + \gamma)}, \\ \varphi_{2,2\sigma}(t) &= (1 - \sigma) + (1 - \sigma)t + (\sigma - 1) \frac{t^\gamma}{\Gamma(1 + \gamma)}.\end{aligned}\quad (39)$$

By continuing in the similar manner, the unknown coefficients c_n and d_n will be vanished for $n \geq 3$, that is, $\sum_{n=3}^{\infty} c_n(t^{n\gamma}/\Gamma(1 + n\gamma)) = 0$ and $\sum_{n=3}^{\infty} d_n(t^{n\gamma}/\Gamma(1 + n\gamma)) = 0$. Hence, the RPS approximate solutions corresponding to (1,2)-system are coinciding well with the exact solutions $\varphi_{1\sigma}(t) = (1 + t - t^\gamma/\Gamma(1 + \gamma))(\sigma - 1)$ and $\varphi_{2\sigma}(t) = (1 + t - t^\gamma/\Gamma(1 + \gamma))(1 - \sigma)$. Here, $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{1\sigma}(t), \varphi'_{2\sigma}(t)]$, and $[\varphi''_{1\sigma}(t), \varphi''_{2\sigma}(t)]$ are valid level sets for $\sigma \in [0, 1]$ and $t \in [0, 1]$. On the other hand, $\varphi(t) = \mu(1 + t - t^\gamma/\Gamma(1 + \gamma))$ is a (1,2)-solution for FFIVPs (30) and (31) on $[0, 1]$.

Case 3. If $\varphi(t)$ is (2,1)-solution, then the corresponding (2,1)-system will be

$$\begin{aligned}({}^C D_{0+}^\gamma \varphi_{1\sigma})(t) &= 1 - \sigma, \\ ({}^C D_{0+}^\gamma \varphi_{2\sigma})(t) &= \sigma - 1, \\ \varphi_{1\sigma}(0) &= \varphi'_{2\sigma}(0) = \sigma - 1, \\ \varphi_{2\sigma}(0) &= \varphi'_{1\sigma}(0) = 1 - \sigma,\end{aligned}\quad (40)$$

If $\gamma = 2$, then the exact solution of (40) is $[\varphi(t)]^\sigma = [\sigma - 1, 1 - \sigma](1 - t - t^\gamma/\Gamma(1 + \gamma))$, $t \in (0, \sqrt{3} - 1)$. To obtain the fuzzy (2,1)-solution of FFDEs (30), let $\varphi(t)$ is Caputo [(2,1)- γ]-differentiable. By using the same manner in previous cases,

the solutions for (2,1)-system can be obtained such as $\varphi_{1\sigma}(t) = (1 - t - t^\gamma/\Gamma(1 + \gamma))(\sigma - 1)$ and $\varphi_{2\sigma}(t) = (1 - t - t^\gamma/\Gamma(1 + \gamma))(1 - \sigma)$. It is easy to check that $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{2\sigma}(t), \varphi'_{1\sigma}(t)]$ and $[\varphi''_{2\sigma}(t), \varphi''_{1\sigma}(t)]$ are also valid level sets for $\sigma \in [0, 1]$ and $t \in [0, \sqrt{3} - 1]$. Thus, $\varphi(t) = \mu(1 - t - t^\gamma/\Gamma(1 + \gamma))$ is a (2,1)-solution for FFIVPs (30) and (31) on $(0, \sqrt{3} - 1]$.

Case 4. If $\varphi(t)$ is (2,2)-solution, then the corresponding (2,2)-system will be

$$\begin{aligned}({}^C D_{0+}^\gamma \varphi_{1\sigma})(t) &= \sigma - 1, \\ ({}^C D_{0+}^\gamma \varphi_{2\sigma})(t) &= 1 - \sigma, \\ \varphi_{1\sigma}(0) &= \varphi'_{2\sigma}(0) = \sigma - 1, \\ \varphi_{2\sigma}(0) &= \varphi'_{1\sigma}(0) = 1 - \sigma,\end{aligned}\quad (41)$$

If $\gamma = 2$, then the exact solution of OFDEs (41) is $[\varphi(t)]^\sigma = [\sigma - 1, 1 - \sigma](1 - t + t^\gamma/\Gamma(\gamma + 1))$, $t \in [0, 1]$. Finally, to determine the fuzzy (2,2)-solution of FFDEs (30), let $\varphi(t)$ be Caputo [(2,2)- γ]-differentiable. By using the same manner in previous cases, the solutions for (2,2)-system can be obtained such as $\varphi_{1\sigma}(t) = (\sigma - 1)(1 - t + t^\gamma/\Gamma(1 + \gamma))$ and $\varphi_{2\sigma}(t) = (1 - \sigma)(1 - t + t^\gamma/\Gamma(1 + \gamma))$. Here, $[\varphi_{1\sigma}(t), \varphi_{2\sigma}(t)]$, $[\varphi'_{2\sigma}(t), \varphi'_{1\sigma}(t)]$ and $[\varphi''_{2\sigma}(t), \varphi''_{1\sigma}(t)]$ are also valid level sets for $\sigma \in [0, 1]$ and $t \in [0, 1]$. However, $\varphi(t) = \mu(1 - t + t^\gamma/\Gamma(\gamma + 1))$ defines as a (2,2)-solution for FFIVPs (30) and (31) on $(0, 1]$.

To demonstrate the agreement between the exact and approximate solution, Table 1 shows the absolute error of the 10th PRS approximate solution for FFIVPs (30) and (31) obtained for different values of σ -cut representations and nodes with fractional order $\gamma = 1.9$. Some graphical results are also presented in Figures 1 and 2. The numerical results obtained indicate that the RPS approximate solutions are in good agreement with each other and with the exact solutions for all cases of differentiability.

6. Conclusion

In this paper, the RPS algorithm is successfully developed, investigated, and applied to solve the fuzzy differential equation of fractional order $1 < \gamma \leq 2$ with fuzzy initial constraints under the fuzzy concept of Caputo H-differentiability. The fuzziness is represented using upper semicontinuous membership function of bounded support, convex, and normalized fuzzy numbers based on its single parametric form. The behavior of approximate solution for different values of fractional order γ is discussed quantitatively as well as graphically. The numerical results in this paper demonstrate the efficiency of the algorithm. We conclude that the proposed scheme is highly accurate in solving widely array of fuzzy fractional issues.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

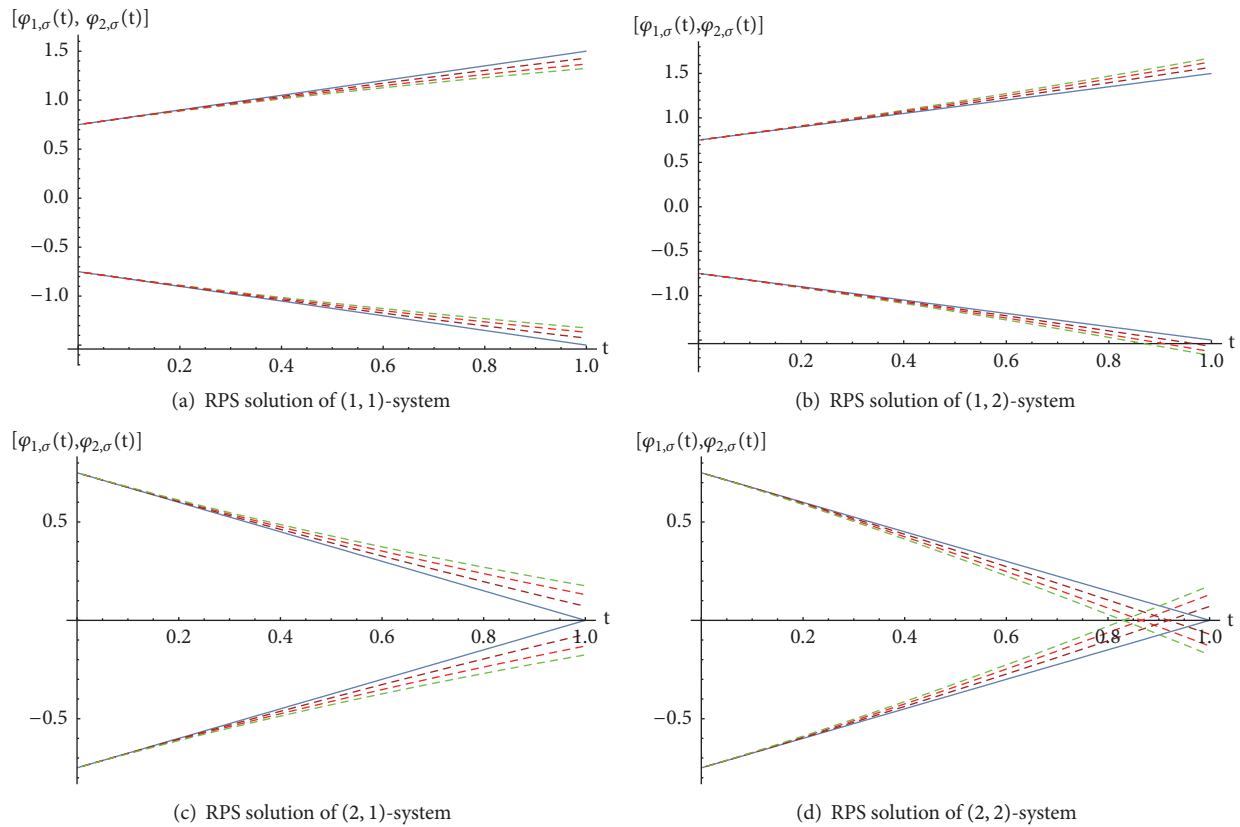


FIGURE 1: Plots of α -cut representations of $\varphi_{k,1\sigma}(t), \varphi_{k,2\sigma}(t)$ with $k = 10, \sigma = 0.25$, and different values of $\gamma \in \{2, 1.9, 1.8, 1.7\}$ (--- Exact, ... RPS-approximation).

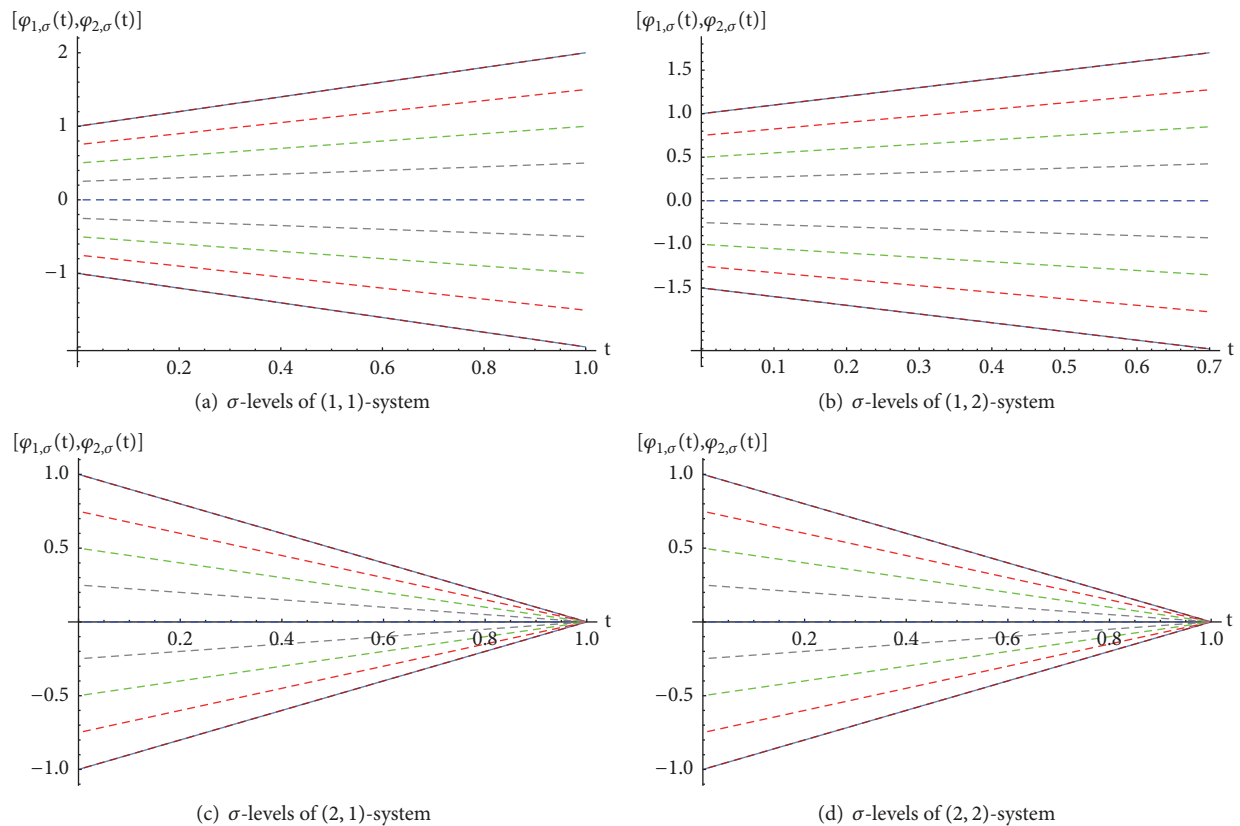


FIGURE 2: Plots of exact and RPS-approximation at $\gamma = 2$ with different values of σ -levels, $\sigma \in \{0, 0.25, 0.5, 0.75, 1\}$ (--- Exact, ... RPS-approximation).

TABLE 1: The absolute error of 10^{th} approximation of FFIVPs (30) and (31).

(n, m) -solution	t	$\sigma = 0$	$\sigma = 0.25$	$\sigma = 0.5$	$\sigma = 0.75$
(1, 1)-system	0.2	2.0578567×10^{-4}	1.5433927×10^{-4}	1.0289283×10^{-4}	5.1446419×10^{-5}
	0.4	3.8465419×10^{-4}	2.8849064×10^{-4}	1.9232709×10^{-4}	9.6163547×10^{-5}
	0.6	5.5428820×10^{-4}	4.1571615×10^{-4}	2.7714410×10^{-4}	1.3857205×10^{-4}
	0.8	7.1821803×10^{-4}	5.3866352×10^{-4}	3.5910901×10^{-4}	1.7955450×10^{-4}
(1, 2)-system	0.2	2.9676224×10^{-4}	2.2257168×10^{-4}	1.4838112×10^{-4}	5.1446419×10^{-5}
	0.4	4.7033136×10^{-4}	3.5274852×10^{-4}	2.3516568×10^{-4}	9.6163547×10^{-5}
	0.6	6.3684312×10^{-4}	4.7763234×10^{-4}	2.7714410×10^{-4}	1.3857205×10^{-4}
	0.8	7.9857642×10^{-4}	5.3866352×10^{-4}	3.5910901×10^{-4}	1.7955450×10^{-4}
(2, 1)-system	0.2	2.0578567×10^{-4}	1.5433925×10^{-4}	1.0289283×10^{-4}	5.1446419×10^{-5}
	0.4	3.8465419×10^{-4}	2.8849064×10^{-4}	1.9232709×10^{-4}	9.6163547×10^{-5}
	0.6	5.5428820×10^{-4}	4.1571615×10^{-4}	2.7714410×10^{-4}	1.3857205×10^{-4}
	0.8	6.3684312×10^{-4}	5.3746353×10^{-4}	3.5910901×10^{-4}	1.7955450×10^{-4}
(2, 2)-system	0.2	2.0578567×10^{-4}	1.5433925×10^{-4}	1.0289283×10^{-4}	5.1446419×10^{-5}
	0.4	3.8465419×10^{-4}	2.8849064×10^{-4}	1.9232709×10^{-4}	9.6163547×10^{-5}
	0.6	5.5428820×10^{-4}	4.1571615×10^{-4}	2.7714410×10^{-4}	1.3857205×10^{-4}
	0.8	7.1821803×10^{-4}	5.3866352×10^{-4}	3.5910901×10^{-4}	1.7955450×10^{-4}

Conflicts of Interest

The authors declare that they have no conflicts of interest.

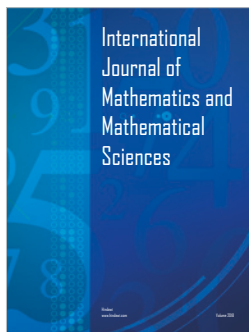
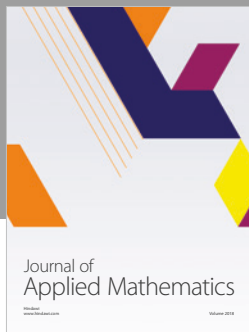
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