Research Article

Existence Results for a System of Coupled Hybrid Differential Equations with Fractional Order

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This paper studies the existence of solutions for a system of coupled hybrid fractional differential equations. We make use of the standard tools of the fixed point theory to establish the main results. The existence and uniqueness result is elaborated with the aid of an example.

1. Introduction

Fractional calculus is the study of theory and applications of integrals and derivatives of an arbitrary (noninteger) order.

This branch of mathematical analysis, extensively investigated in the recent years, has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena. One of the key factors for the popularity of the subject is the nonlocal nature of fractional-order operators.

Due to this reason, fractional order operators are used for describing the hereditary properties of many materials and processes. It clearly reflects from the related literature that the focus of investigation has shifted from classical integer-order models to fractional order models. For applications in applied and biomedical sciences and engineering, we refer the reader to the books [1–4].

For some recent work on the topic, see [5–12] and the references therein. The study of coupled systems of fractional order differential equations is quite important as such systems appear in a variety of problems of applied nature, especially in biosciences. For details and examples, the reader is referred to the papers [13, 14] and the references cited therein.

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [15, 16].

Motivated by some recent studies on hybrid fractional differential equations, we consider the following value problem of coupled hybrid fractional differential equations:

$$\begin{cases} {}^{C}D^{\alpha}\left(\dfrac{x(t)}{f_{1}(t,x(t)y(t),)}\right) = h_{1}(t,x(t),y(t)), \quad \text{a.e. } t \in J = [0,T], \; 0 < \alpha < 1, \\[4mm] {}^{C}D^{\beta}\left(\dfrac{y(t)}{f_{2}(t,x(t)y(t),)}\right) = h_{2}(t,x(t),y(t)), \quad \text{a.e. } t \in J = [0,T], \; 0 < \beta < 1, \\[4mm] a\dfrac{x(0)}{f_{1}(0,x(0),y(0))} + b\dfrac{x(T)}{f_{1}(T,x(T),y(T))} = c, \\[4mm] a\dfrac{y(0)}{f_{2}(0,x(0),y(0))} + b\dfrac{y(T)}{f_{2}(T,x(T),y(T))} = c, \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ denote the Caputo fractional derivative of orders α and β, respectively, $a,b,c \in \mathbb{R}, f_{i} \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}/\{0\})$, and $f_{i} \in C([0,T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}/\{0\})$, $(i = 1,2)$.

The aim of this paper is to obtain some existence results for the given problem. Our first theorem describes the uniqueness of solutions for problem (1) by means of Banach's fixed point theorem. In the second theorem, we apply Leray–Schauder's alternative criterion to show the existence of solutions for the given problem. The paper is organized as follows. Section 2 contains some basic concepts and an auxiliary lemma, an important result for establishing our main results. In Section 3, we present the main results.

2. Coupled System of Hybrid Differential Equations with Fractional Order

In this section, some basic definitions on fractional calculus and an auxiliary lemma are presented [1, 2].

Definition 1 (see [6]). The fractional integral of the function $h \in L^{1}([a,b], \mathbb{R}_{+})$ of order $\alpha \in \mathbb{R}_{+}$ is defined by

$$I_{a}^{\alpha}h(t) = \int_{a}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s, \tag{2}$$

where Γ is the gamma function.

Definition 2 (see [6]). For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h is defined by

$$\left({}^{c}D_{a^{+}}^{\alpha}h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h^{(n)}(s)\mathrm{d}s, \tag{3}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α.

Lemma 1 *(Auxiliary Lemma). Given $h \in C([0,T], \mathbb{R})$ and $a, b,$ and c are real constants with $a + b \neq 0$, the integral solution of the problem*

$$\begin{cases} {}^{C}D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t),y(t))}\right) = h(t), \quad 0 < t < T, \; 0 < \alpha < 1, \\[4mm] a\dfrac{x(0)}{f(0,x(0),y(0))} + b\dfrac{x(T)}{f(T,x(T),y(T))} = c, \end{cases} \tag{4}$$

is

$$x(t) = [f(t,x(t),y(t))]\left(\frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1}h(s)\mathrm{d}s - \frac{1}{a+b}\right.$$
$$\left. \cdot \left(\frac{b}{\Gamma(\alpha)} \int_{0}^{T} (T-s)^{\alpha-1}h(s)\mathrm{d}s - c\right)\right), \quad t \in [0,T]. \tag{5}$$

Proof. Applying the Caputo integral operator of the order α, we obtain the first equation in (4).

Again, substituting

$$D^{\alpha}\left(\frac{x(t)}{f(t,x(t),y(t))}\right) = h(t), \tag{6}$$

we get

$$\frac{x(t)}{f(t,x(t),y(t))} = \frac{x(0)}{f(0,x(0),y(0))} + I^{\alpha}h(t). \tag{7}$$

Then,

$$b\frac{x(T)}{f(T,x(T,y(T)))} = b\frac{x(0)}{f(0,x(0),y(0))}$$
$$+ \frac{b}{\Gamma(\alpha)} \int_{0}^{T} (T-s)^{\alpha-1}h(s)\mathrm{d}s. \tag{8}$$

Thus,

$$a\frac{x(0)}{f(0,x(0),y(0))} + b\frac{x(T)}{f(T,x(T),y(T))}$$

$$= (a+b)\frac{x(0)}{f(0,x(0),y(0))} + \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds, \tag{9}$$

implies that

$$\frac{x(0)}{f(0,x(0),y(0))} = \frac{1}{a+b}\left(c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds\right). \tag{10}$$

Consequently,

$$x(t) = [f(t,x(t),y(t))]\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds\right.$$
$$\left. - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - c\right)\right). \tag{11}$$

This completes the proof. $\square$

3. Main Result

Let $W = \{w(t) \in C^1([0,1])\}$ denote a Banach space equipped with the norm $\|w\| = \max\{|w(t)|, t \in [0,1]\}$, where $\mathcal{W} = \mathcal{U} \times \mathcal{V}$. Note that the product space $(\mathcal{U} \times \mathcal{V}, \|(X,Y)\|)$ with the norm $(x,y) = \|x\| + \|y\|$, $(x,y) \in \mathcal{U} \times \mathcal{V}$, is also a Banach space.

We define an operator $\mathcal{F}: \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{U} \times \mathcal{V}$ by

$$\mathcal{F}(x,y)(t) = \begin{pmatrix} \mathcal{F}_1(x,y)(t) \\ \mathcal{F}_2(x,y)(t) \end{pmatrix}, \tag{12}$$

where

$$\mathcal{F}_1(x,y)(t) = f_1(t,x(t),y(t)) \times \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds\right.$$
$$\left. - \frac{1}{a+b}\left(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h_1(t,x(t),y(t))ds - c\right)\right),$$

$$\mathcal{F}_2(x,y)(t) = f_2(t,x(t),y(t)) \times \left(\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds\right.$$
$$\left. - \frac{1}{a+b}\left(\frac{b}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}h_2(t,x(t),y(t))ds - c\right)\right). \tag{13}$$

In the sequel, we need the following assumptions:

(A_1) The functions $f_i (i = 1,2)$ are continuous and bounded; that is, there exist positive numbers μ_{f_i} such that $|f_i(t,x,y)| \le \mu_{f_i}$, $\forall (t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$.

(A_2) There exist real constants $\rho_0, \sigma_0 > 0$ and $\rho_i, \sigma_i \ge 0 (i = 1,2)$ such that

$$|h_1(t,x,y)| \le \rho_0 + \rho_1|x| + \rho_2|y|,$$
$$|h_2(t,x,y)| \le \sigma_0 + \sigma_1|x| + \sigma_2|y|, \tag{14}$$
$$\forall x, y \in \mathbb{R}$$

For brevity, let us set

$$\nu_1 = \frac{\mu_{f_1}T^\alpha}{\Gamma(\alpha+1)},$$

$$\nu_2 = \frac{\mu_{f_2}T^\beta}{\Gamma(\beta+1)}, \tag{15}$$

$$\nu_3 = \frac{|c|}{|a+b|}\left(\mu_{f_1} + \mu_{f_2}\right),$$

$$\nu_0 = \min\left\{1 - \left(\left(1 + \frac{|b|}{|a+b|}\right)(\nu_1\rho_1 + \nu_2\rho_2)\right),\right.$$
$$\left. 1 - \left(\left(1 + \frac{|b|}{|a+b|}\right)(\nu_2\sigma_1 + \nu_2\sigma_2)\right)\right\}. \tag{16}$$

3.1. First Result. Now, we are in a position to present our first result that deals with the existence and uniqueness of solutions for problem (1). This result is based on Banach's contraction mapping principle.

Theorem 1. *Suppose that condition (A_1) holds and that $h_1, h_1: [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions. In addition, there exist positive constants η_i and ζ_i, $i = 1,2$, such that*

$$|h_1(t,x_2,y_2) - h_1(t,x_1,y_1)| \le \eta_1|x_1 - x_2| + \eta_2|y_1 - y_2|,$$
$$|h_2(t,x_1,y_1) - h_2(t,x_2,y_2)| \le \zeta_1|x_1 - x_2| + \zeta_2|y_1 - y_2|,$$
$$\forall t \in [0.T], x_1,x_2,y_1,y_2 \in \mathbb{R}. \tag{17}$$

If $(1 + (|b|/|a+b|))(\nu_1(\eta_1+\eta_2) + \nu_2(\zeta_1+\zeta_2)) < 1, \nu_1$ and ν_2 are given by (15): then, problem (1) has a unique solution.

Proof. Let us set $\sup_{t \in [0,T]} h_1(t,0,0) = k_1 < \infty$ and $\sup_{t \in [0,T]} h_2(t,0,0) = k_2 < \infty$ and define a closed ball:

$$\overline{B}_r = \{(x,y) \in U \times V : \|(x,y)\| \le r\}, \tag{18}$$

where

$$r \ge \frac{(1 + (|b|/|a+b|))(\nu_1 k_1 + \nu_2 k_2) + \nu_3}{1 - (1 + (|b|/|a+b|))(\nu_1(\eta_1+\eta_2) + \nu_2(\zeta_1+\zeta_2))}. \tag{19}$$

Then, we show that $\Theta\overline{B}_r \subset \overline{B}_r$. For $(x,y) \in \overline{B}_r$, we obtain

$$\left|\mathcal{F}_1(x,y)(t)\right| \le \mu_{f_1}\max_{t\in[0,T]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|h_1(s,x(s),y(s))\right|ds + \frac{1}{|a+b|}\left(\frac{|b|}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|h_1(t,x(s),y(s))\right|ds - c\right)\right\}$$

$$\le \mu_{f_1}\max_{t\in[0,T]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\times\left(\left|h_1(s,x(s),y(s)) - h_1(s,0,0)\right| + \left|h_1(s,0,0))\right|\right)ds\right.$$

$$\left. + \frac{1}{|a+b|}\left(\frac{|b|}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\times\left[\left|h_1(t,x(s),y(s)) - h_1(s,0,0)\right| + \left|h_1(s,0,0)\right|\right]ds - c\right)\right\}$$

$$\le \nu_1\left(\eta_1\|x\| + \eta_2\|y\| + k_1\right) + \frac{|b|\nu_1}{|a+b|}\left(\eta_1\|x\| + \eta_2\|y\| + k_1\right) + \frac{|c|\nu_1}{|a+b|}.$$

$$(20)$$

Hence,

$$\|\mathcal{F}(x,y)\| \le \nu_1\left(1 + \frac{|b|}{|a+b|}\right)\left(\eta_1 r + \eta_2 r + k_1\right) + \frac{|c|\nu_1}{|a+b|}. \tag{21}$$

Working in a similar manner, one can find that

$$\|\mathcal{F}(x,y)\| \le \nu_2\left(1 + \frac{|b|}{|a+b|}\right)\left(\zeta_1 r + \zeta_2 r + k_2\right) + \frac{|c|\nu_1}{|a+b|}. \tag{22}$$

From (21) and (22), it follows that $\|\mathcal{F}(x,y)\| \le r$.

Next, for (x_1,y_1), $(x_2,y_2) \in U \times V$ and for any $t \in [0,T]$, we have

$$\left|\mathcal{F}_1(x_2,y_2)(t) - \mathcal{F}_1(x_1,y_1)(t)\right| \le \mu_{f_1}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|h_1(s,x_2(s),y_2(s)) - h_1(s,x_1(s),y_1(s))\right|ds$$

$$+ \frac{1}{|a+b|}\left(\frac{|b|}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|h_1(s,x_2(s),y(s)) - h_1(s,x_1(s),y(s)))\right|ds\right)$$

$$\le \nu_1\left(\eta_1\|x_2 - x_1\| + \eta_2\|y_2 - y_1\| + k_1\right) + \frac{|b|\nu_1}{|a+b|}\left(\eta_1 x + \eta_2\|y\|\right) \tag{23}$$

$$\le \nu_1\left(1 + \frac{|b|}{|a+b|}\right)\left(\eta_1\|x_2 - x_1\| + \eta_2\|y_2 - y_1\| + k_1\right)$$

$$\le \nu_1\left(1 + \frac{|b|}{|a+b|}\right)\left(\eta_1 + \eta_2\right)\left(\|x_2 - x_1\| + \|y_2 - y_1\|\right),$$

which yields

$$\|\mathcal{F}_1(x_2,y_2) - F_1(x_1,y_1)\| \le \nu_1\left(1 + \frac{|b|}{|a+b|}\right)\left(\eta_1 + \eta_2\right)$$

$$\cdot\left(\|x_2 - x_1\| + \|y_2 - y_1\|\right). \tag{24}$$

Similarly, one can get

$$\|\mathcal{F}_2(x_2,y_2) - \mathcal{F}_2(x_1,y_1)\| \le \nu_2\left(1 + \frac{|b|}{|a+b|}\right)\left(\zeta_1 + \zeta_2\right)$$

$$\cdot\left(\|x_2 - x_1\| + \|y_2 - y_1\|\right). \tag{25}$$

From (24) and (25), we deduce that

$$\|\mathcal{F}(x_2,y_2) - \mathcal{F}(x_1,y_1)\| \le \left(1 + \frac{|b|}{|a+b|}\right)\left(\nu_2(\eta_1 + \eta_2) + \nu_2\right.$$

$$\left. \cdot(\zeta_1 + \zeta_2)\right)\times\left(\|x_2 - x_1\| + \|y_2 - y_1\|\right). \tag{26}$$

In view of condition $(1 + (|b|/va+b|))(\nu_1(\eta_1 + \eta_2) + \nu_2(\zeta_1 + \zeta_2)) < 1$, it follows that $\mathcal{F}$ is a contraction.

So Banach's fixed point theorem applies and hence the operator $\mathcal{F}$ has a unique fixed point. This, in turn, implies that problem (1) has a unique solution on $[0,T]$. This completes the proof. $\square$

3.2. Second Result. In our second result, we discuss the existence of solutions for problem (1) by means of Leray–Schauder alternative.

Lemma 2 (Leray–Schauder alternative [17]). *Let* $\mathbf{F}: \mathcal{G} \longrightarrow \mathcal{G}$ *be a completely continuous operator (i.e., a map that is restricted to any bounded set in $\mathcal{G}$ is compact). Let $\mathbb{P}(\mathbf{F}) = \{x \in \mathcal{G}: x = \lambda \mathbf{F} x \text{ for some } 0 < \lambda < 1\}$. Then, either the set $\mathbb{P}(\mathbf{F})$ is unbounded or $\mathbf{F}$ has at least one fixed point.*

Theorem 2. *Assume that conditions (A_1) and (A_2) hold. Furthermore, it is assumed that $(1 + (|b|/|a+b|))(\nu_1 \rho_1 + \nu_2 \rho_2) < 1$ and $(1 + (|b|/|a+b|))(\nu_1 \rho_1 + \nu_2 \rho_2) < 1$, where ν_1 and ν_2 are given by (15). Then, boundary value problem (1) has at least one solution.*

Proof. We will show that the operator $\mathcal{F}: \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{U} \times \mathcal{V}$ satisfies all the assumptions of Lemma 2.

In the first step, we prove that the operator $\mathcal{F}$ is completely continuous. Clearly, it follows the continuity of functions $f_1, f_2, h_1,$ and h_2 that the operator $\mathcal{F}$ is continuous.

Let $\mathcal{M} \subset \mathcal{U} \times \mathcal{V}$ be bounded. Then, we can find positive constants N_1 and N_2 such that

$$
\begin{aligned}
h_1(t, x(s), y(s)) &\leq N_1, \\
h_2(t, x(s), y(s)) &\leq N_2, \\
\forall (x, y) &\in \mathcal{M}.
\end{aligned}
\tag{27}
$$

Thus, for any $x, y \in \mathcal{M}$, we can get

$$
\begin{aligned}
\left| \mathcal{F}_1(x, y)(t) \right| &\leq \mu_{f_1} \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| h_1(s, x(s), y(s)) \right| ds \right. \\
&\quad + \frac{1}{|a+b|} \left(\frac{|b|}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left| h_1(t, x(s), y(s)) \right| ds - c \right) \Bigg\} \\
&\leq \mu_{f_1} N_1 \left(1 + \frac{|b|}{|a+b|} \right) + \frac{|c| \mu_{f_1}}{|a+b|},
\end{aligned}
\tag{28}
$$

which yields

$$
\left\| \mathcal{F}_1(x, y) \right\| \leq \mu_{f_1} N_1 \left(1 + \frac{|b|}{|a+b|} \right) + \frac{|c| \mu_{f_1}}{|a+b|}.
\tag{29}
$$

In a similar manner, one can show that

$$
\left\| \mathcal{F}_2(x, y) \right\| \leq \mu_{f_2} N_2 \left(1 + \frac{|b|}{|a+b|} \right) + \frac{|c| \mu_{f_2}}{|a+b|}.
\tag{30}
$$

From inequalities (29) and (30), we deduce that the operator $\mathcal{F}$ is uniformly bounded.

Now, we show that the operator $\mathcal{F}$ is equicontinuous. For that, we take $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and obtain

$$
\begin{aligned}
\left| \mathcal{F}_1(x, y)(\tau_2) - \mathcal{F}_1(x, y)(\tau_1) \right| &\leq \frac{\mu_{f_1} N_1 T^\alpha}{\Gamma(\alpha+1)} \left| \int_0^{\tau_1} (\tau_1 - s)^{\alpha-1} ds - \int_0^{\tau_2} (\tau_2 - s)^{\alpha-1} ds \right| \\
&= \mu_{f_1} N_1 T^\alpha \left| \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha+1)} ds - \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha+1)} ds \right|.
\end{aligned}
\tag{31}
$$

Similarly, one can get

$$
\begin{aligned}
\left| \mathcal{F}_2(x(\tau_2), y(\tau_2)) - \mathcal{F}_2(x(\tau_1), y(\tau_1)) \right| &\leq \frac{\mu_{f_2} N_2 T^\beta}{\Gamma(\beta+1)} \left| \int_0^{\tau_1} (\tau_1 - s)^{\beta-1} ds \right. \\
&\left. - \int_0^{\tau_2} (\tau_2 - s)^{\beta-1} ds \right| = \mu_{f_2} N_2 T^\beta \left| \int_0^{\tau_1} \frac{(\tau_1 - s)^{\beta-1} - (\tau_2 - s)^{\beta-1}}{\Gamma(\beta+1)} ds - \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - s)^{\beta-1}}{\Gamma(\beta+1)} ds \right|,
\end{aligned}
\tag{32}
$$

which tends to 0 independent of (x, y).

This implies that the operator $\mathcal{F}(x, y)$ is equicontinuous. Thus, by the above findings, the operator $\mathcal{F}(x, y)$ is completely continuous.

In the next step, it will be established that the set

$$\mathcal{P} = \{(x, y) \in \mathcal{U} \times \mathcal{V} \mid (x, y) = \lambda \mathcal{F}(x, y), 0 \leq \lambda \leq 1\}, \quad (33)$$

is bounded.

Let $(x, y) \in \mathcal{P}$; then we have $(x, y) = \lambda \mathcal{F}(x, y)$. Thus, for any $t \in [0, T]$, we can write

$$\begin{aligned}
x(t) &= \lambda \mathcal{F}_1(x, y)(t), \\
y(t) &= \lambda \mathcal{F}_2(x, y)(t).
\end{aligned} \quad (34)$$

Then,

$$|x(t)| \leq \nu_1 \left(1 + \frac{|b|}{|a+b|}\right)(\rho_0 + \rho_1 \|x\| + \rho_2 \|y\|) + \frac{|c|\mu_{f_1}}{|a+b|},$$

$$|y(t)| \leq \nu_2 \left(1 + \frac{|b|}{|a+b|}\right)(\sigma_0 + \sigma_1 \|x\| + \sigma_2 \|y\|) + \frac{|c|\mu_{f_2}}{|a+b|},$$

$$\quad (35)$$

which imply that

$$\|x\| \leq \nu_1 \left(1 + \frac{|b|}{|a+b|}\right)(\rho_0 + \rho_1 \|x\| + \rho_2 \|y\|) + \frac{|c|\mu_{f_1}}{|a+b|},$$

$$\|y\| \leq \nu_2 \left(1 + \frac{|b|}{|a+b|}\right)(\sigma_0 + \sigma_1 \|x\| + \sigma_2 \|y\|) + \frac{|c|\mu_{f_2}}{|a+b|}.$$

$$\quad (36)$$

Consequentlyd, we have

$$\|x\| + \|y\| = \left(1 + \frac{|b|}{|a+b|}\right)((\nu_1 \rho_0 + \nu_2 \sigma_0)$$

$$+ (\nu_1 \rho_1 + \nu_2 \sigma_1)\|x\| + (\nu_1 \rho_2 + \nu_2 \sigma_2)\|y\|), \quad (37)$$

which, in view of (16), can be expressed as

$$\|(x, y)\| \leq \frac{(1 + (|b|/|a+b|))(\nu_1 \rho_0 + \nu_2 \sigma_0) + \nu_3}{\nu_0}. \quad (38)$$

This shows that the set $\mathcal{P}$ is bounded. Hence, all the conditions of Lemma 2 are satisfied and, consequently, the operator $\mathcal{F}$ has at least one fixed point, which corresponds to a solution of problem (1). This completes the proof. □

4. Example

An example is provided as follows:

$$\begin{cases}
{}^C D^{1/2}\left(\dfrac{x(t)}{(1/2)(|\sin x(t)| + 1)}\right) = \dfrac{1}{4(t+2)^2}\dfrac{|x(t)|}{1+|x(t)|} + 1 + \dfrac{1}{32}\sin^2 y(t), & t \in [0, 1], \\[4mm]
{}^C D^{1/2}\left(\dfrac{y(t)}{(1/2)(|\cos x(t)| + 1)}\right) = \dfrac{1}{32\pi}\sin(2\pi x(t)) + \dfrac{|y(t)|}{16(1+|y(t)|)} + \dfrac{1}{2}, & t \in [0, 1], \\[4mm]
\dfrac{x(0)}{f_1(0, x(0), y(0))} + \dfrac{x(1)}{f_1(1, x(1), y(1))} = 1, \\[4mm]
\dfrac{y(0)}{f_2(0, x(0), y(0))} + \dfrac{y(1)}{f_2(1, x(1), y(1))} = 1.
\end{cases} \quad (39)$$

Here, $\alpha = (1/2)$, $\beta = (1/2)$, $a = b = c = 1$,

$$f_1(t, x, y) = \left(\frac{1}{2}\right)(|\sin x(t)| + 1),$$

$$f_2(t, x, y) = \left(\frac{1}{2}\right)(|\cos x(t)| + 1),$$

(40)

$$h_1(t, x, y) = \frac{1}{4(t+2)^2} \frac{|x(t)|}{1+|x(t)|} + 1 + \frac{1}{32}\sin^2 y(t),$$

$$h_2(t, x, y) = \frac{1}{32\pi}\sin(2\pi x(t)) + \frac{|y(t)|}{16(1+|y(t)|)} + \frac{1}{2}.$$

Note that

$$\left|h_1(t, x_1, y_1) - h_1(t, x_2, y_2)\right| \le \frac{1}{16}|x_2 - x_1| + \frac{1}{16}|y_2 - y_1|,$$

$$\left|h_2(t, x_1, y_1) - h_1(t, x_2, y_2)\right| \le \frac{1}{16}|x_2 - x_1| + \frac{1}{16}|y_2 - y_1|,$$

(41)

$$\forall t \in [0.1], x_1, x_2, y_1, y_2 \in \mathbb{R},$$

$$\left(1 + \frac{|b|}{|a+b|}\right)(\nu_1(\eta_1 + \eta_2) + \nu_2(\zeta_1 + \zeta_2)) = 0.12394093 < 1.$$

Thus, all the conditions of Theorem 1 are satisfied and, consequently, there exists a unique solution for problem (39) on $[0, 1]$.

Data Availability

There is no data used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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