Research Article

The Method Based on Series Solution for Identifying an Unknown Source Coefficient on the Temperature Field in the Quasiperiodic Media

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In this paper, we consider the reconstruction of heat field in one-dimensional quasiperiodic media with an unknown source from the interior measurement. The innovation of this paper is solving the inverse problem by means of two different homotopy iteration processes. The first kind of homotopy iteration process is not convergent. For the second kind of homotopy iteration process, a convergent result is proved. Based on the uniqueness of this inverse problem and convergence results of the second kind of homotopy iteration process with exact data, the results of two numerical examples show that the proposed method is efficient, and the error of the inversion solution $r(t)$ is given.

1. Introduction

The energy demand is the hotspot in theory research, and many studies have been conducted on energy loss reduction and heat transfer enhancement [1–4]. Conduction is heat transfer by means of molecular agitation within a material without any motion of the material as a whole. Convection is heat transfer by mass motion of a fluid such as air or water when the heated fluid is caused to move away from the source of heat, carrying energy with it. The heat equation is a parabolic partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. In order to achieve an optimal thermal control system, the accurate prediction of the thermal conductivity is particularly necessary. In the process of transportation, diffusion, and conduction of natural materials, the following heat equation is a suitable approximation:

$$u_t - \Delta u = f(x, t; u), \quad (x, t) \in \Omega \times (0, t_{\max}), \quad (1)$$

where u represents the state variable, Ω is a bounded domain in R^d , and the right-hand-side f denotes physical laws in our case source terms. However, there are many inverse problems for heat equations, and it is well known that they are generally ill posed; i.e., the existence, uniqueness, and stability of their solutions are not always guaranteed. So, the regularization numerical methods have been investigated for the inverse problems.

2. Problem Description

Consider the heat conduction model

$$\begin{cases} u_t = u_{xx} + r(t)f(x, t), & (x, t) \in Q: = (0, 1) \times (0, T) \\ u(0, t) = u(1, t), -u_x(0, t) + \eta u(0, t) = 0, & t \in [0, T] \\ u(x, 0) = \varphi(x), & x \in [0, 1]. \end{cases} \quad (2)$$

for temperature field $u(x, t)$ with the unknown $r(t)$ describing the strength of heat source, where $f(x, t)$ and φ are known and $\eta > 0$ is a known constant representing the heat flux exchanges at the boundary $x = 0$. The boundary conditions in (2) arise quite often; for an example, see its nature in mathematical biology in [5]. Our aim is to determine the heat field $r(t)$ in Q_T from the given data

$$u(x_0, t) = h(t), \quad t \in [0, T], \quad (3)$$

at some interior point $x_0 \in [0, 1]$. In [6], the uniqueness of reconstruction $r(t)$ in $C[0, 1]$ for suitable $f(x, t)$ and $\varphi(x)$ is given by introducing the system

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 \leq x \leq 1, \\ X(0) = X(1), & -X'(0) + \eta X(0) = 0. \end{cases} \quad (4)$$

Lemma 1. *assumed that the known function pair $(f(x, t), \varphi(x))$ meets*

- (i) $(A1) \phi(x) \in C^4[0, 1]$, both $\varphi(x)$ and $\varphi''(x)$ meet the boundary conditions in (4)
- (ii) $(A2) f(\cdot, t) \in C^4[0, t]$, $f(x, \cdot) \in C[0, T]$ for any fixed $t \in [0, T]$ and $x \in [0, 1]$, both $f(\cdot, t)$ and $\partial_{xx}^2 f(\cdot, t)$ meet the boundary conditions in (4)

If the observation point $x_0 \in (0, 1)$ is chosen such that $f(x_0, t)|_{t \in [0, T]} \neq 0$, then there exists a unique solution $r(t) \in C[0, T]$ for inverse problems (2) and (3).

Also, in [7], the inverse problem has been solved by a regularization scheme with an explicit selection strategy for the regularization parameter. Based on the uniqueness of the inverse problem, we mainly discuss the method based on series solution in this paper. To overcome these numerical difficulties, rather than considering (2) and (3) as an inverse problem for $r(t)$, here we propose to firstly solve $u(x, t)$ in D_T from a nonstandard initial boundary value problem by deleting $r(t)$ in (2) using the extra data $h(t)$, and then, the unknown source $r(t)$ can be obtained from the function $u(x, t)$ if required. Although the uniqueness of this nonstandard problem is still established by the uniqueness of $r(t)$ for the inverse problem, its solution will be given explicitly by the homotopy analysis method (HAM) for suitably constructed homotopy transform starting from a quite general initial guess.

The homotopy analysis method is developed by Liao in 1992 [8, 9] and has been successfully applied to solve many types of nonlinear problems in science and engineering by many authors [10–12]. Additional, it has been found that the

homotopy schemes can also be applied to solve some inverse problems [7, 13, 14]. Liu et al. [15] propose a homotopy-based iterative regularizing scheme to solve a backward heat conduction problem, and the convergence results of the homotopy sequence are given for noisy input data. Wang et al. [16] proposed a homotopy-based iterative regularizing scheme to solve a source distribution in heat conduction.

3. Homotopy Scheme and Iteration Process

The uniqueness of the inverse problems which recovers $r(t)$ from input data (3) ensures that we can solve $u(x, t)$ in Q in terms of (2) and (3), noticing for known $r(t)$, (2) is a well-posed problem. Here, instead of identifying $r(t)$, we firstly solve the field $u(x, t)$ directly in terms of the extra data $h(t)$. Taking $x = x_0 \in (0, 1)$ in the equation and using the fact $f(x_0, t) \neq 0$ in $[0, T]$, we have

$$r(t) = \frac{1}{f(x_0, t)} (h'(t) - u_{xx}(x_0, t)). \quad (5)$$

Also, inserting (5) into (2) again, a following nonstandard initial boundary value problem for function $u(x, t)$ is obtained as follows:

$$\begin{cases} u_t - u_{xx} - \frac{f(x, t)}{f(x_0, t)} (h'(t) - u_{xx}(x_0, t)) = 0, & (x, t) \in Q, \\ u(0, t) = u(1, t), & -u_x(0, t) + \eta u(0, t) = 0, \quad 0 \leq t \leq T, \\ u(x, 0) = \varphi(x), & 0 \leq x \leq 1. \end{cases} \quad (6)$$

We introduce the function space

$$\begin{aligned} \mathbb{D}_\varphi = \{ & \omega(x, t) \in C^{2,1}(Q): \omega(0, t) \\ & = \omega(1, t), -\omega_x(0, t) + \eta \omega(0, t) = 0, \omega(x, 0) = \varphi(x) \} \end{aligned} \quad (7)$$

and the homotopy transform by the functions $\{\phi(x, t; p): p \in [0, 1]\}$ such that

$$\begin{aligned} \phi(x, t; 0) &= u_0(x, t), \\ \phi(x, t; 1) &= u(x, t), \end{aligned} \quad (8)$$

with some known initial guess $u_0(x, t) \in \mathbb{D}_\varphi$ for the unknown $u(x, t) \in \mathbb{D}_\varphi$.

For the function $\omega(x, t)$, we define the linear operator $\mathbb{L}$ and operator $\mathbb{N}$ in two forms, respectively. The first form is

$$\begin{cases} \mathbb{L}_1[\omega](x, t) := \omega_t(x, t), \\ \mathbb{N}_1[\omega](x, t) := \omega_t(x, t) - \omega_{xx}(x, t) - \frac{f(x, t)}{f(x_0, t)} (h'(t) - \omega_{xx}(x_0, t)). \end{cases} \quad (9)$$

This kind of definition is frequently used to solve some partial differential equations [17–19]. We construct the zero-order deformation equation

$$(1 - p)\mathbb{L}_1[\phi(x, t, p) - u_0(x, t)] = \hbar \mathbb{N}_1[\phi(x, t, p)], \quad (10)$$

with some nonzero constant $\hbar$ as the convergence controlling parameter for $\{\phi(x, t, p): p \in [0, 1]\} \subset \mathbb{D}_\phi$. The homotype scheme aims to generate the zero point $u(x, t)$ at $p = 1$ from the known zero point $u_0(x, t)$ at $p = 0$ from (10). To compute $u_m(x, t)$ for $m = 1, 2, \dots$, from the known initial guess $u_0(x, t)$, we expand the function $\phi(x, t, p)$ in terms of p as a power series

$$\begin{aligned} \phi(x, t, p) &= \phi(x, t, 0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial \phi}{\partial p^m} \Big|_{p=0} p^m \\ &= u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) p^m, \end{aligned} \quad (11)$$

and insert it into (10). By comparing the coefficients of p^m for $m = 1, 2, \dots$, we obtain the m th-order deformation equations

$$\mathbb{L}_1[u_m(x, t) - \chi_m u_{m-1}(x, t)] = \hbar \mathbb{R}[\vec{u}_{m-1}] \quad (12)$$

with the constants

$$\chi_m = \begin{cases} 0, & m = 0, \\ 1, & m = 1, 2, \dots, \end{cases} \quad (13)$$

where

$$\begin{aligned} \mathbb{R}[\vec{u}_{m-1}] &= \frac{\partial u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial x^2} \\ &\quad - \frac{f(x, t)}{f(x_0, t)} \left[h'(t) - \frac{\partial^2 u_{m-1}}{\partial x^2}(x_0, t) \right]. \end{aligned} \quad (14)$$

Now, the solution of the m th-order deformation equation (12) for $m \geq 1$ becomes

$$u_m(x, t) = \chi_{m-1} u_{m-1}(x, t) + \hbar \mathbb{L}_1^{-1}[\mathbb{R}[\vec{u}_{m-1}]]. \quad (15)$$

However, in Section 3, we will illustrate the divergence of $\sum_{n=0}^{\infty} u_n(x, t)$ in Example 1 by means of the form in (15). So, we discuss the second form

$$\begin{cases} \mathbb{L}_2[\omega](x, t) := \omega_t(x, t) - \omega_{xx}(x, t), \\ \mathbb{N}_2[\omega](x, t) := \omega_t(x, t) - \omega_{xx}(x, t) + \frac{f(x, t)}{f(x_0, t)} \omega_{xx}(x_0, t) \end{cases} \quad (16)$$

and consider the zeroth-order decomposition equation

$$\begin{aligned} (1 - p)\mathbb{L}_2[\phi(x, t, p) - u_0(x, t)] \\ = \hbar p \left(\mathbb{N}_2[\phi(x, t, p)] - \frac{f(x, t)}{f(x_0, t)} h'(t) \right), \end{aligned} \quad (17)$$

with some nonzero constant $\hbar$ as the convergence controlling parameter for $\{\phi(x, t, p): p \in [0, 1]\} \subset \mathbb{D}_\phi$. By the same method, we obtain

$$\begin{aligned} \mathbb{L}_2[u_m - \chi_{m-1} u_{m-1}](x, t) \\ = \hbar \mathbb{N}_2[u_{m-1}](x, t) - \hbar (1 - \chi_{m-1}) \\ \cdot \frac{f(x, t)}{f(x_0, t)} h'(t), m = 1, 2, \dots, \end{aligned} \quad (18)$$

for $u_m \in \mathbb{D}_0$ by noticing the linearity of operators $\mathbb{L}_2$ and $\mathbb{N}_2$. Here, we first prove the following result.

Theorem 1. *If the series*

$$u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \quad (19)$$

converges, where $u_m(x, t)$ ($m = 1, 2, \dots$) is governed by (18), it must be an exact solution of (6) with initial condition $u(x, 0) = \varphi(x)$.

Proof. If the series is convergent, we can write

$$s = \sum_{m=0}^{\infty} u_m(x, t), \quad (20)$$

and it holds

$$\lim_{n \rightarrow \infty} u_n(x, t) = 0, \quad (21)$$

and then, using (16) and (18), we have

$$\begin{aligned} \hbar \sum_{m=1}^{\infty} \left[\mathbb{N}_2[u_{m-1}](x, t) - (1 - \chi_{m-1}) \frac{f(x, t)}{f(x_0, t)} h'(t) \right] \\ = \lim_{n \rightarrow \infty} \sum_{m=1}^n \mathbb{L}_2[u_m(x, t) - \chi_{m-1} u_{m-1}(x, t)], \\ = \mathbb{L}_2 \left[\lim_{n \rightarrow \infty} \sum_{m=1}^n [u_m(x, t) - \chi_{m-1} u_{m-1}(x, t)] \right], \\ = \mathbb{L}_2 \left[\lim_{n \rightarrow \infty} u_n(x, t) \right], \\ = 0, \end{aligned} \quad (22)$$

which gives, since $\hbar \neq 0$,

$$\sum_{m=1}^{\infty} \left[\mathbb{N}_2[u_{m-1}](x, t) - (1 - \chi_{m-1}) \frac{f(x, t)}{f(x_0, t)} h'(t) \right] = 0. \quad (23)$$

Thus, according to the definition of $\mathbb{N}_2$, we have

$$\begin{aligned}
& \sum_{m=1}^{\infty} \left[\mathbb{N}_2[u_{m-1}](x, t) - (1 - \chi_{m-1}) \frac{f(x, t)}{f(x_0, t)} h'(t) \right], \\
&= \sum_{m=1}^{\infty} \left[\frac{\partial u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial x^2} + \frac{f(x, t)}{f(x_0, t)} \frac{\partial^2 u_{m-1}(x_0, t)}{\partial x^2} - (1 - \chi_{m-1}) \frac{f(x, t)}{f(x_0, t)} h'(t) \right], \\
&= \frac{\partial s}{\partial t} - \frac{\partial^2 s}{\partial x^2} - \frac{f(x, t)}{f(x_0, t)} \left[\frac{\partial^2 s(x_0, t)}{\partial x^2} - h'(t) \right], \\
&= 0.
\end{aligned} \tag{24}$$

From the initial $u_m(x, 0) = 0$ and $u_0(x, t) = \varphi(x)$ [2], it holds $s(x, 0) = \varphi(x)$. So, $s(x, t)$ satisfies the equation in (6), and therefore, it is an exact solution of problem (6). This ends the proof.

We define

$$\mathcal{K}_3[f] = \sum_{n=0}^{\infty} e^{-\lambda_n(t-\tau)} \frac{\langle f(\cdot, \tau), Y_n \rangle}{f(x_0, \tau)} X_n. \tag{25}$$

The convergence of the iteration process is stated in [6]. $\square$

Lemma 2. Assume either $0 < T < 2$ or $\|\mathcal{A}\| < 1$, where $\mathcal{A}[g](t) = \int_0^t F(t - \tau, \tau, x_0) g(\tau) d\tau$ and

$$F(t - \tau, \tau, x_0) = \sum_{n=0}^{\infty} \lambda_n X_n(x_0) e^{-\lambda_n(t-\tau)} \frac{\langle f(\cdot, \tau), Y_n \rangle}{f(x_0, \tau)}. \tag{26}$$

Then, for any initial guess $u_0(x, t) \in \mathbb{D}_\varphi$ and $h \in (-1, 0)$, the iteration process $\{u_m(x, t): m = 0, 1, 2, 3, \dots\}$ will give the exact solution as

$$u(x, t) = \sum_{m=0}^{\infty} u_m(x, t). \tag{27}$$

Moreover, $u(x, t)$ has the representation

$$\begin{aligned}
u(x, t) &= u_0(x, t) - \frac{1}{h} u_1(x, t) + \frac{1}{h} \mathcal{K}_3[f](t - \tau, \tau, x) \\
&\quad \cdot (\mathcal{J} + \mathcal{A})^{-1} \left[\partial_{xx}^2 u_1(x_0, \cdot) \right](\tau) d\tau,
\end{aligned} \tag{28}$$

which means we can compute $u(x, t)$ in terms of u_0 , noticing $u_1 = h(\mathcal{M}[u_0] - \mathcal{Q})$. Also, at N iteration step, the error can be obtained as follows:

$$\begin{aligned}
& u(x, t) - \sum_{m=0}^N u_m(x, t), \\
&= (1 + h)^N (\mathcal{M}[u_0](x, t) - \mathcal{Q}[f, h](x, t)) + \\
&\quad \cdot (1 + h)^N \int_0^t \mathcal{K}_3[f](t - \tau, \tau, x) \left(\mathcal{A}^{-1}[\mathcal{W}^1](\tau) - \sum_{k=0}^N C_N^k \left(\frac{h}{1+h} \right)^k \mathcal{A}^k[\mathcal{R}](\tau) \right) d\tau.
\end{aligned} \tag{29}$$

Remark 1. The means of symbols $\mathcal{A}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{M}$, $\mathcal{W}^1$, and $\mathcal{R}$ are the same as those in [6].

4. Test Examples

In this section, we present two examples to show the efficiency of the present method described in the previous part. The inverse solution $r(t)$ is shown in the figures. We define the numerical relative error in the form

$$\text{Error} = \frac{\|r(t) - r_{\text{num}}(t)\|_{L_2[0,1]}}{\|r(t)\|_{L_2[0,1]}}. \tag{30}$$

All the computations were performed using MATLAB 2016a on a personal computer with Intel Core i5 and 8.00 GB memory.

Example 1. The inverse problem is considered in the form [20]

$$\begin{cases} u_t = u_{xx} + r(t) \sin\left(2\pi x + \frac{\pi}{4}\right), \\ u(0, t) = u(1, t), \quad -u_x(0, t) + \eta u(0, t) = 0, \\ \varphi(x) = 0, \end{cases} \tag{31}$$

with the exact solution

$$u(x, t) = \sin\left(2\pi x + \frac{\pi}{4}\right) \sin(8t). \tag{32}$$

We set $x_0 = 1/4$, $\eta = 2\pi$, then $h(t) = u(1/4, t) = \sqrt{2}/2 \sin(8t)$, and the coefficient $r(t) = 8 \cos(8t) + 4\pi^2 \sin(8t)$.

FIGURE 1: Reconstructions of Example 1. (a) First line: exact solution of $u(x,t)$ (left); approximate solution of $u(x,t)$ when $N = 5$ (right); (b) second line: approximate solutions of $u(x,t)$ when $N = 15$ (left) and $N = 20$ (right), respectively.

FIGURE 2: Reconstructions of $r(t)$ of Example 1.

TABLE 1: The error of solution $r(t)$ ($t \in [0, 1]$) of Example 1.

x	0.1	0.2	0.3	0.4	0.5
Error	0.032 033	0.021 254	0.063 216	0.011 208	0.037 257
x	0.6	0.7	0.8	0.9	1
Error	0.020 058	0.030 674	0.004 583	0.034 396	0.046 099

FIGURE 3: Reconstructions of Example 2. (a) First line: exact solution of $u(x, t)$ (left); approximate solution of $u(x, t)$ when $N = 5$ (right); (b) second line: approximate solutions of $u(x, t)$ when $N = 15$ (left) and $N = 20$ (right), respectively. FIGURE 4: Reconstructions of $r(t)$ of Example 2.

TABLE 2: The error of solution $r(t)$ ($t \in [0, 1]$) of Example 2.

x	0.1	0.2	0.3	0.4	0.5
Error	0.008 681 51	0.042 939 7	0.025 728 3	0.029 755 5	0.042 485 8
x	0.6	0.7	0.8	0.9	1
Error	0.011 920 7	0.049 506 7	0.070 640 7	0.024 357 3	0.078 507

According to (15), when selecting the initial guess $u_0(x, t) = \varphi(x) = 0$, some terms of series solution $u(x, t)$ are as follows:

$$\begin{aligned}
 u_0(x, t) &= 0, \\
 u_1(x, t) &= -\hbar \sin\left(2\pi x + \frac{\pi}{4}\right) \sin(8t), \\
 u_2(x, t) &= -(\hbar^2 + 2\hbar) \sin\left(2\pi x + \frac{\pi}{4}\right) \sin(8t), \\
 u_3(x, t) &= -(\hbar^3 + 3\hbar^2 + 3\hbar) \sin\left(2\pi x + \frac{\pi}{4}\right) \sin(8t), \\
 u_4(x, t) &= -(\hbar^4 + 4\hbar^3 + 6\hbar^2 + 4\hbar) \sin\left(2\pi x + \frac{\pi}{4}\right) \sin(8t), \\
 &\vdots
 \end{aligned} \tag{33}$$

So,

$$\sum_{k=0}^n u_k(x, t) = \frac{(1 + \hbar)[1 - (1 + \hbar)^n] + n\hbar}{\hbar} \cdot \sin\left(2\pi x + \frac{\pi}{4}\right) \sin(8t), \tag{34}$$

when set $\hbar = -1$, $u_1(x, t)$, $u_2(x, t)$, $\dots$, $u_n(x, t)$ are all the exact solutions of problem (31). Thus, the limit of the sum (34) does not exist. Even if $\hbar$ takes other values, it is impossible to get convergence results. So, the form in (15) is unsuitable for calculating $u_n(x, t)$, $n = 1, 2, \dots$.

According to the convergence results in Lemma 2, the form in (18) can be used to calculate $u_n(x, t)$, $n = 1, 2, \dots$. Similarly, we select the initial guess $u_0(x, t) = \varphi(x) = 0$. By (18), we can obtain

$$\begin{aligned}
 u_1(x, t) &= -\hbar \sin\left(2\pi x + \frac{\pi}{4}\right) \left[\sin(8t) - \frac{2}{4\pi^2} \cos(8t) \right], \\
 \partial_{xx}^2 u_1(x_0, t) &= \hbar \left[2\sqrt{2} \pi^2 \sin(8t) - 4\sqrt{2} \cos(8t) \right].
 \end{aligned} \tag{35}$$

It should be noted that we truncate to construct $\mathcal{K}_3[f]$ in (25) and $F(t - \tau, \tau, x_0)$ in (26):

$$\mathcal{K}_3[f] = \sum_{n=0}^{M_1} e^{-\lambda_n(t-\tau)} \frac{\langle f(\cdot, \tau), Y_n \rangle}{f(x_0, \tau)} X_n, \tag{36}$$

$$F(t - \tau, \tau, x_0) = \sum_{n=0}^{M_2} \lambda_n X_n(x_0) e^{-\lambda_n(t-\tau)} \frac{\langle f(\cdot, \tau), Y_n \rangle}{f(x_0, \tau)}. \tag{37}$$

We set $\hbar = -0.5$, $M_1 = 10$ in (36), and $M_2 = 10$ in (37) and substitute $u_0(x, t)$, $u_1(x, t)$, $\partial_{xx}^2 u_1(x_0, t)$, and (36) into

(28). For the exact input of $u(x_0, t) = \sqrt{2}/2 \sin(8t)$, we can approximate $u(x, t)$ using our iterative process by $\sum_{m=0}^N u_m(x, t)$ directly. Our results of $u(x, t)$ in $[0, 1] \times [0, 1]$ using exact input data with $N = 5, 15, 20$ are shown in Figure 1. Therefore, according to (5), we can obtain the approximate solution $r(t)$ of the inverse problem by means of the approximate solution $u(x, t)$ when $N = 20$, see Figure 2. By (30), the error of inverse solution $r(t)$ ($t \in [0, 1]$) is shown in Table 1. Moreover, the running time of the scheme based on the homotopy method in this paper is 1m30s, while the algorithm in [20] takes 7m10s.

Example 2. We consider the inverse problem in the form in [20], which is the same as with Example 1 with the exact solution

$$u(x, t) = \sin\left(2\pi x + \frac{\pi}{4}\right) e^{-(\pi^2 - (3/4))t}. \tag{38}$$

We set $x_0 = 1/4$, $\eta = 2\pi$, then $h(t) = u(1/4, t) = \sqrt{2}/2 e^{-(\pi^2 - (3/4))t}$, and the coefficient $r(t) = e^{-\pi^2 t}$.

Same as Example 1, when selecting the initial guess $u_0(x, t) = \varphi(x) = \sin(2\pi x + \pi/4)$, we can obtain

$$\begin{aligned}
 u_1(x, t) &= \hbar \left[1 - \frac{1}{4\pi^2} \left(\pi^2 - \frac{3}{4} \right) \right] \\
 &\quad \cdot \sin\left(2\pi x + \frac{\pi}{4}\right) e^{-(\pi^2 - (3/4))t},
 \end{aligned} \tag{39}$$

$$\partial_{xx}^2 u_1(x_0, t) = \frac{\sqrt{2}}{2} \hbar \left(3\pi^2 + \frac{3}{4} \right) e^{-(\pi^2 - (3/4))t}.$$

For the exact input of $u(x_0, t) = \sqrt{2}/2 \sin(8t)$, we can approximate $u(x, t)$ using our iterative process by $\sum_{m=0}^N u_m(x, t)$ directly. Our results of $u(x, t)$ in $[0, 1] \times [0, 1]$ using exact input data with $N = 5, 15, 20$ are shown in Figure 3. Similarly, according to (5), the approximate solution $r(t)$ of the inverse problem can be obtained by means of the approximate solution $u(x, t)$ when $N = 20$, see Figure 4. By (30), the error of inverse solution $r(t)$ ($t \in [0, 1]$) is shown in Table 2. Moreover, the running time of the scheme in this paper is 1m48s, while the algorithm in [20] takes 7m25s.

5. Discussion and Conclusions

In fact, the inverse problem is frequently proposed by using the noisy data $h^\delta(t)$ satisfying $\|h^\delta - h\|_{C^1[0, T]} \leq \delta$. For noisy inversion input data, the number of iterations in the homotopy scheme is taken as the regularizing parameter, for which Wang and Liu [6] proposed a choice strategy in terms of the noisy level, and then, the convergence rate for the regularizing solution is given. The choice of the convergence

parameter h can also be determined by means of the results of Lemma 2. In [8], Liao showed that whatever a solution series converges, it will be one of the solutions of the considered problem. To investigate the influence of convergence parameter h on the convergence of the solution series given by the HAM, according to the h -curves, it is easy to find the valid region of h , which corresponds to the line segment nearly parallel to the horizontal axis [8]. Additionally, some modified homotopy analysis methods are for dealing with different types of problems with strong nonlinearity [21–24].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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