Research Article

Solution of Fractional Partial Differential Equations Using Fractional Power Series Method

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In this paper, we are presenting our work where the noninteger order partial differential equation is studied analytically and numerically using the noninteger power series technique, proposed to solve a noninteger differential equation. We are familiar with a coupled system of the nonlinear partial differential equation (NLPDE). Noninteger derivatives are considered in the Caputo operator. The fractional-order power series technique for finding the nonlinear fractional-order partial differential equation is found to be relatively simple in implementation with an application of the direct power series method. We obtained the solution of nonlinear dispersive equations which are used in electromagnetic and optics signal transformation. The proposed approach of using the noninteger power series technique appears to have a good chance of lowering the computational cost of solving such problems significantly. How to paradigm an initial representation plays an important role in the subsequent process, and a few examples are provided to clarify the initial solution collection.

1. Introduction

Mathematical equations containing two or more independent variables are called partial differential equations (PDEs). The highest derivative in the PDEs is called the order of the equation. The specific solution of the PDEs is a function that solves the equation, which can be verified by substituting the solution into the equation and obtaining an identity. The solution containing all the particular solutions of the equation in question is called the general solution. The term used to express a specific solution for second-order and higher-order nonlinear partial differential equations is called an exact solution. Problems involving the function of many variables, such as heat or sound propagation, static electricity, electrodynamics, and fluid flow, are solved by PDE [1, 2].

Generally, the solution with unknown coefficients is assumed to be a power series, and then the solution is substituted into the differential equation to find the recurrence relationship of the coefficients. New applications in fluid dynamics, viscoelasticity, mathematical biology, electrochemistry, and physics have led to the latest development

in fractional DE. Using data from experiments, fractional DEs have recently proved to be useful tools for simulating many physical phenomena and fractional PDEs have been suggested for filtering flows in porous media. The examples that have been studied and solved include the space-time fractional diffusion wave equation, fractional advection scattering equation, fractional telegraph equation, fractional KdV equation, and linear nonhomogeneous fractional partial differential equations [2,3].

Nowadays, the uses of nonlinear partial (or ordinary) differential equations in the practice of multidisciplinary research are considered to be an important technique for solving real-life problems that have social and public-health implications. Researchers in different scientific areas like biology, physiology, cell biology, chemistry, chemical physics, engineering, and systematic natural sciences are including applied mathematical methods in their practices. These applications are not only useful for technological advances in computer science to study cancers, hereditary diseases, tumor formation, kinematic processes, and DNA communication processes but also in mathematical controls to supply a successful cancer therapy. Indeed, there are many

biomedical engineering applications for studying biological problems involving highly nonlinear and sophisticated collaborative processes, where the most promising theory available to conduct such studies is complex dynamical systems theory. The description of this theory almost entirely depends on nonlinear partial differential equations (NLPDE). Power series solution (PSS) technique has been restricted to unravel the linear differential equations, i.e., ordinary differential equations (ODEs) [1, 2] and partial differential equations (PDEs) [4, 5].

Linear partial differential equations have traditionally been overcome using the variable separation method because it creates an ODE system that is easier to decipher with PSSM. Examples of them are the spherical harmonics used and the Legendre polynomials in the Bessel equation in cylindrical coordinates or the Laplace equation in spherical coordinates [4, 5]. It is understood that in NLPDE, as we remember, variable separation technology cannot be used. The way to unlock NLPDE is to use approximate analysis techniques to get the answer, that is, semianalytical or nondigital, direct or indirect. Directly, there are methods such as loose operator formalism [6] or inverse dispersion transformation [7]. In a direct way, for instance, the PSS can be used in an asymptotic calculation of the Hirota technique linking a bilinear operator method [8] and the homotopy analysis method (HAM) [9, 10] and the Adomian decomposition technique [11]. The latest approach requires a sequence of expansions with a nonsmall perturbation parameter estimate to change the convergence. This approach is distinct from the standard perturbation theory. Methods, also simpler, to estimate the representation in NLPDE are the Taylor polynomial approximation (TPA) technique [12, 13] and therefore for the power series solution method (PSSM). In each technique, a semianalytical representation is attained to implement the PSS technique. However, no use has been made of the PSSM to decipher the nonlinear ODE [14–17] or NLPDE [18–20]. Because NLPDEs do not have exact solutions in general, it is hard to define the most efficient method to be used; indeed, the only way to establish the efficiencies of the relevant methods involves employing experimental methods.

Further information on how to solve NLPDEs by using a point contact, Bäcklund, hodograph, Legendre, or Euler transformations or by applying Lie algebras and groups can be found in reference [21]. There is an immense work on noninteger calculus, and this has been grown exponentially in recent years, with noninteger ODEs becoming commonly used as mathematical prototypes [22, 23]. Recent studies consist of the solution of fractal media [24], viscoelastic materials [23], economics and finance [22, 25], compartment models [26], in porous media [27], epidemiology [28], nerve cell signaling [29], and anomalous diffusion system [30]. From a modeling perspective, the interpretation of the representation as an algebraic countenance involving functions of structure strictures is commonly pursued. Important progress has been made concerning linear noninteger ODEs by expanding proven techniques of elucidation for linear nonfractional ODEs. The Laplace

transform technique can be extended to noninteger linear ODEs, with constant coefficients (see [31–33]).

The disadvantage of this method depends on the complexity of the Laplace transform of such functions, regardless of whether the equations are homogeneous or not, or by inverting the Laplace transform based on a calibrated computable function. In the noninteger linear ODE, with variable coefficients, the Laplace transform method has an additional limitation, that is, the Laplace transform of the product of the function is only available in special cases. In nonfractional regular linear ODEs, with different coefficients, the series expansion method is considered to help clarify the problem. Examples of power series can be seen in [34–48]. The description of this theory is also related to the use of the residual power series method [49] to solve space-time fractional PDEs. The comparison of Aboodh transformation and differential transformation method (DTM) numerically in result solution obtained is compared the solution by DTM is rapid convergent [50]. Several papers have been dedicated to the application of series expansion approaches to linear noninteger ODEs which basically is the subject of this article.

2. Fundamental Concepts

Definition 1. The Caputo noninteger derivative operator of order μ with respect to “ t ” is defined in the subsequent arrangement.

$$D_t^\mu h(u) = D^\mu h(u) = J^{n-\mu} D^n h(u), \quad (1)$$

$$D^\mu h(u) = (\Gamma(n-\mu))^{-1} \int_0^u (u-t)^{-\mu+n-1} h^{(n)}(t) dt,$$

where $\mu > 0$, $u > 0$, $n \geq \mu > n-1$, $n \in \mathbb{N}$.

In a comparable manner of nonfractional-order differentiation, the Caputo noninteger derivative operator is a linear operation.

$$D^\mu (\alpha f(u) + \beta g(u)) = \alpha D^\mu f(u) + \beta D^\mu g(u), \quad (2)$$

where α and β are constants. We have $D^\mu k = 0$ for the Caputo’s derivative, if k is constant.

$$D^\mu u^m = \begin{cases} 0, & \text{for } m \in \mathbb{N}_0 \text{ and } m < \mu, \\ \frac{\Gamma(m+1)}{\Gamma(m-\mu+1)} u^{m-\mu}, & \text{for } m \in \mathbb{N}_0 \text{ and } m \geq \mu. \end{cases} \quad (3)$$

We practice the maximum function μ to signify the least integer larger than or equivalent to μ , and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Evoke that for $\mu \in \mathbb{N}$, the Caputo differential operator corresponds with the normal differential operator of nonfractional order.

Definition 2. For the variable u and coefficients a_n ($n = 0, 1, \dots, \infty$), if $u > u_0$, the fractional power series (FPS) about the point u_0 is defined as

$$\sum_{n=0}^{\infty} a_n (u - u_0)^{n\mu} = a_0 + (u - u_0)^{\mu} a_1 + (u - u_0)^{2\mu} a_2 + (u - u_0)^{3\mu} a_3 + \dots, \quad (4)$$

where $\mu > 0$, $m \geq \mu > m - 1$, $m \in \mathbb{N}^+$.

Theorem 1. Let the radius of convergence for the function with fractional power series (FPS) representation $h(u) = \sum_{n=0}^{\infty} a_n u^{n\mu}$, $0 \leq u < R$, be greater than zero (i.e., $R > 0$).

Then, for $m \in \mathbb{N}^+$ and $m - 1 < \mu \leq m$, the following expression holds true:

$$D^{\mu}(h(u)) = \sum_{m=1}^{\infty} a_m \frac{\Gamma(m\mu + 1) u^{(m-1)\mu}}{\Gamma((m-1)\mu + 1)}. \quad (5)$$

Proof. From the linearity concept of Caputo derivative and the idea of power series derivative, we have

$$\begin{aligned} D^{\mu}(h(u)) &= D^{\mu} \sum_{m=0}^{\infty} a_m u^{m\mu} = D^{\mu}(a_0 + a_1 u^{\mu} + a_2 u^{2\mu} + \dots) \\ &= D^{\mu} a_0 + a_1 D^{\mu} u^{\mu} + a_2 D^{\mu} u^{2\mu} + \dots \\ &= a_1 D^{\mu} u^{\mu} + a_2 D^{\mu} u^{2\mu} + \dots \\ D^{\mu}(h(u)) &= \sum_{m=1}^{\infty} a_m D^{\mu} u^{m\mu}. \end{aligned} \quad (6)$$

From the power rule of the Caputo derivative, we have

$$D^{\mu} u^{m\mu} = \begin{cases} 0, & \text{for } m\mu \in \mathbb{N}_0 \text{ and } m\mu < \mu, \\ \frac{\Gamma(m\mu + 1)}{\Gamma((m-1)\mu + 1)} u^{(m-1)\mu}, & \text{for } m\mu \in \mathbb{N}_0 \text{ and } m\mu \geq \mu, \end{cases}$$

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}, \mu \in \mathbb{Z}, \mu \geq \mu. \quad (7)$$

From equations (3) and (5), we arrive at

$$D^{\mu}(h(u)) = \sum_{m=1}^{\infty} a_m D^{\mu} u^{m\mu} = \sum_{m=1}^{\infty} a_m \frac{\Gamma(m\mu + 1)}{\Gamma((m-1)\mu + 1)} u^{(m-1)\mu}, \quad (8)$$

where $h(u) = \sum_{m=0}^{\infty} a_m u^{m\mu}$.

2.1. Application. In order to validate the high degree of efficiency and precision of the projected FPS approach for unraveling fractional-order systems, numerical forms and instances are pragmatic. The reader can discover a sketch and applications for this technique in [42]. Computations were accomplished by using MATLAB. Examples are taken from [32].

Example 1. Consider the succeeding noninteger order structure:

$$\begin{cases} D_0^{\mu} u(x, t) = 1 + v \frac{du}{dx} + u, \\ D_0^{\mu} v(x, t) = 1 - u \frac{dv}{dx} - v, \end{cases} \quad (9)$$

subject to the initial conditions

$$\begin{aligned} u(0, x) &= e^{-x}, \\ v(0, x) &= e^x. \end{aligned} \quad (10)$$

The closed-form result of this arrangement when $\mu = 1$ is

$$\begin{aligned} u(t, x) &= e^{t-x}, \\ v(t, x) &= e^{x-t}. \end{aligned} \quad (11)$$

Solution:

$$\begin{aligned} \text{Let } u(t, x) &= \sum_{k=0}^{\infty} a_k(x) t^{\mu k} = a_0(x) + t^{\mu} a_1(x) + t^{2\mu} a_2(x) \\ &\quad + \dots + t^{n\mu} a_n(x) + \dots, \end{aligned} \quad (12)$$

$$\begin{aligned} v(t, x) &= \sum_{k=0}^{\infty} b_k(x) t^{\mu k} = b_0(x) + t^{\mu} b_1(x) + t^{2\mu} b_2(x) \\ &\quad + \dots + t^{n\mu} b_n(x) + \dots. \end{aligned} \quad (13)$$

So using Theorem 1, equations (12) and (13) become

$$D^{\mu}[u(x, t)] = D^{\mu} \left(\sum_{k=0}^{\infty} a_k(x) t^{\mu k} \right), \quad (14)$$

$$\text{Or } D^{\mu}[u(x, t)] = \sum_{k=1}^{\infty} a_k(x) \frac{\Gamma(k\mu + 1)}{\Gamma((k-1)\mu + 1)} t^{\mu(k-1)},$$

$$D^{\mu}[v(x, t)] = D^{\mu} \left(\sum_{k=0}^{\infty} b_k(x) t^{\mu k} \right), \quad (15)$$

$$\text{Or } D^{\mu}[v(x, t)] = \sum_{k=1}^{\infty} \frac{\Gamma(k\mu + 1)}{\Gamma((k-1)\mu + 1)} b_k(x) t^{\mu(k-1)}.$$

Similarly,

$$S \frac{\partial u(x, t)}{\partial x} = \sum_{k=0}^{\infty} \frac{\partial a_k(x)}{\partial x} t^{\mu k}, \quad (16)$$

$$\frac{\partial v(x, t)}{\partial x} = \sum_{k=0}^{\infty} \frac{\partial b_k(x)}{\partial x} t^{\mu k}. \quad (17)$$

Apply (12)–(17) on the first and second equation of system (9), respectively, system (9) becomes

$$\begin{cases} \sum_{k=1}^{\infty} a_k \frac{\Gamma(k\mu+1)}{\Gamma((k-1)\mu+1)} t^{\mu(k-1)} = 1 + \sum_{k=0}^{\infty} t^{\mu k} a_k + \left(\sum_{k=0}^{\infty} t^{\mu k} \frac{\partial a_k}{\partial x} \right) \left(\sum_{k=0}^{\infty} b_k t^{\mu k} \right), \\ \sum_{k=1}^{\infty} \frac{\Gamma(k\mu+1)}{\Gamma((k-1)\mu+1)} b_k t^{\mu(k-1)} = 1 + \sum_{k=0}^{\infty} t^{\mu k} b_k + \left(\sum_{k=0}^{\infty} t^{\mu k} \frac{\partial b_k}{\partial x} \right) \left(\sum_{k=0}^{\infty} a_k t^{\mu k} \right). \end{cases} \quad (18)$$

Comparing the different powers of t to both sides of system (18), for t^0 ,

$$\begin{cases} a_1(x) = \frac{1}{\Gamma(\mu+1)} \left(1 + b_0 \frac{\partial a_0}{\partial x} + a_0 \right), \\ b_1(x) = \frac{1}{\Gamma(\mu+1)} \left(1 - a_0 \frac{\partial b_0}{\partial x} - b_0 \right). \end{cases} \quad (19)$$

But $a_0 = u(0, x) = e^{-x}$, $v(0, x) = b_0 = e^x$, $\partial a_0 / \partial x = -e^{-x}$, and $\partial b_0 / \partial x = -e^x$, so (19) becomes

$$\begin{cases} a_1(x) = \frac{1}{\Gamma(\mu+1)} (1 + e^x (-e^{-x}) + e^{-x}) = \frac{e^{-x}}{\Gamma(\mu+1)}, \\ b_1(x) = \frac{1}{\Gamma(\mu+1)} (1 - e^{-x} (e^x) - e^x) = \frac{-e^x}{\Gamma(\mu+1)}. \end{cases} \quad (20)$$

For t^μ ,

$$\begin{cases} a_2(x) = \frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)} \left(a_1 + b_1 \frac{\partial a_0}{\partial x} + b_0 \frac{\partial a_1}{\partial x} \right), \\ b_2(x) = \frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)} \left(-b_1 - a_1 \frac{\partial b_0}{\partial x} - a_0 \frac{\partial b_1}{\partial x} \right). \end{cases} \quad (21)$$

But $\partial a_0 / \partial x = -e^{-x}$, $\partial b_0 / \partial x = e^x$, $\partial a_1 / \partial x = e^{-x} / \Gamma(\mu+1)$, $\partial b_1 / \partial x = -e^x / \Gamma(\mu+1)$, so (21) becomes

$$\begin{cases} a_2(x) = \frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)} \left(\frac{e^{-x}}{\Gamma(\mu+1)} + e^x \left(\frac{-e^{-x}}{\Gamma(\mu+1)} \right) - \frac{e^x}{\Gamma(\mu+1)} (-e^{-x}) \right) = \frac{e^{-x}}{\Gamma(2\mu+1)}, \\ b_2(x) = \frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)} \left(\frac{e^x}{\Gamma(\mu+1)} - e^{-x} \left(\frac{-e^x}{\Gamma(\mu+1)} \right) - \frac{e^{-x}}{\Gamma(\mu+1)} (e^x) \right) = \frac{e^x}{\Gamma(2\mu+1)}. \end{cases} \quad (22)$$

For $t^{2\mu}$,

$$\begin{cases} a_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(a_2 + b_2 \frac{\partial a_0}{\partial x} + b_0 \frac{\partial a_2}{\partial x} + b_1 \frac{\partial a_1}{\partial x} \right), \\ b_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(-b_2 - a_2 \frac{\partial b_0}{\partial x} - a_0 \frac{\partial b_2}{\partial x} - a_1 \frac{\partial b_1}{\partial x} \right). \end{cases} \quad (23)$$

But $\partial a_0 / \partial x = -e^{-x}$, $\partial b_0 / \partial x = e^x$, $\partial a_1 / \partial x = e^{-x} / \Gamma(\mu+1)$, $\partial b_1 / \partial x = -e^x / \Gamma(\mu+1)$, $\partial a_2 / \partial x = -e^{-x} / \Gamma(2\mu+1)$, $\partial b_2 / \partial x = e^x / \Gamma(2\mu+1)$, so (26) becomes

$$\begin{cases} a_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(\frac{e^{-x}}{\Gamma(2\mu+1)} + \left(\frac{-e^x}{\Gamma(\mu+1)} \right) \left(\frac{-e^{-x}}{\Gamma(\mu+1)} \right) + \frac{e^x}{\Gamma(2\mu+1)} (-e^{-x}) + e^x \left(\frac{-e^{-x}}{\Gamma(2\mu+1)} \right) \right), \\ b_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(-\frac{e^x}{\Gamma(2\mu+1)} - \frac{e^{-x}}{\Gamma(2\mu+1)} (e^x) - e^{-x} \left(\frac{e^x}{\Gamma(2\mu+1)} \right) - \frac{e^{-x}}{\Gamma(\mu+1)} \left(\frac{-e^x}{\Gamma(\mu+1)} \right) \right), \end{cases} \quad (24)$$

or

$$\begin{cases} a_3(x) = \frac{1}{\Gamma(3\mu+1)} \left(e^{-x} - 2 + \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right), \\ b_3(x) = \frac{1}{\Gamma(3\mu+1)} \left(-e^x - 2 + \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right). \end{cases} \quad \text{For } t^{3\mu}, \quad (25)$$

$$\begin{cases} a_4(x) = \frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)} \left(a_3 + \frac{\partial a_3}{\partial x} b_0 + \frac{\partial a_2}{\partial x} b_1 + \frac{\partial a_1}{\partial x} b_2 + \frac{\partial a_0}{\partial x} b_3 \right), \\ b_4(x) = \frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)} \left(-b_3 - \frac{\partial b_3}{\partial x} a_0 - \frac{\partial b_2}{\partial x} a_1 - \frac{\partial b_1}{\partial x} a_2 - \frac{\partial b_0}{\partial x} a_3 \right), \end{cases}$$

$$\begin{cases} a_4(x) = \frac{-0.5}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -6\pi e^{-x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) + 4\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ e^{-x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} \, 16^\mu - \\ \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} \, 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right), \\ b_4(x) = \frac{-0.5}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -6\pi e^{-x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) + 4\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ e^{-x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} \, 16^\mu + \\ \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} \, 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right). \end{cases} \quad (26)$$

In a similar way, we can compare the values for $t^{4\mu}$, $t^{5\mu}$, So,

...

$$u(t, x) = e^{-x} + \frac{e^{-x}}{\Gamma(\mu+1)} t^\mu + \frac{e^{-x}}{\Gamma(2\mu+1)} t^{2\mu} + \frac{1}{\Gamma(3\mu+1)} \left(e^{-x} - 2 + \frac{\Gamma(2\mu+1)}{(\Gamma(\mu+1))^2} \right) t^{3\mu} \\ + \left[\frac{-0.5}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -6\pi e^{-x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) + 4\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ e^{-x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu - \\ \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right) \right] t^{4\mu} + \dots, \quad (27)$$

$$v(t, x) = e^{-x} + \frac{e^{-x}}{\Gamma(\mu+1)} t^\mu + \frac{e^{-x}}{\Gamma(2\mu+1)} t^{2\mu} + \frac{1}{\Gamma(3\mu+1)} \left(e^{-x} - 2 + \frac{\Gamma(2\mu+1)}{(\Gamma(\mu+1))^2} \right) t^{3\mu} \\ + \left[\frac{-0.5}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -6\pi e^{-x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) + 4\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ e^{-x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu + \\ \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right) \right] t^{4\mu} + \dots, \quad (28)$$

If $\mu = 1$, then equations (27) and (28) give

$$u(t, x) = e^{-x} \left(1 + \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} + \frac{t^3}{\Gamma(4)} + \frac{t^4}{\Gamma(5)} + \dots \right) = e^{t-x}, \\ v(t, x) = e^x \left(1 - \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} - \frac{t^3}{\Gamma(4)} + \frac{t^4}{\Gamma(5)} + \dots \right) = e^{-t+x}. \quad (29)$$

Example 2. Consider the following fractional-order system:

$$\begin{cases} D_0^\mu u(t, x) = 2 - 2v \frac{du}{dx} + u, \\ D_0^\mu v(t, x) = 3 + 3u \frac{dv}{dx} - v, \end{cases} \quad (30)$$

subject to the initial conditions

$$\begin{aligned} u(0, x) &= e^x, \\ v(0, x) &= e^{-x}. \end{aligned} \quad (31)$$

The closed-form result of this system when $\mu = 1$ is

$$\begin{aligned} u(t, x) &= e^{x+t}, \\ v(t, x) &= e^{-x-t}. \end{aligned} \quad (32)$$

Solution:

Let

$$u(t, x) = \sum_{k=0}^{\infty} a_k(x) t^{\mu k} = a_0 + t^\mu a_1 + t^{2\mu} a_2 + \dots + t^{n\mu} a_n + \dots, \quad (33)$$

$$v(t, x) = \sum_{k=0}^{\infty} b_k(x) t^{\mu k} = b_0 + t^\mu b_1 + t^{2\mu} b_2 + \dots + t^{n\mu} b_n + \dots. \quad (34)$$

Apply (12)–(17) on the first and second equations of the system (30), respectively. So, system (30) becomes

$$\begin{cases} \sum_{k=1}^{\infty} a_k \frac{\Gamma(k\mu+1)}{\Gamma((k-1)\mu+1)} t^{\mu(k-1)} = 2 - 2 \left(\sum_{k=0}^{\infty} b_k t^{\mu k} \right) \left(\sum_{k=0}^{\infty} \frac{\partial a_k}{\partial x} t^{\mu k} \right) + \sum_{k=0}^{\infty} a_k t^{\mu k}, \\ \sum_{k=1}^{\infty} b_k \frac{\Gamma(k\mu+1)}{\Gamma((k-1)\mu+1)} t^{\mu(k-1)} = 3 + 3 \left(\sum_{k=0}^{\infty} a_k t^{\mu k} \right) \left(\sum_{k=0}^{\infty} \frac{\partial b_k}{\partial x} t^{\mu k} \right) + \sum_{k=0}^{\infty} b_k t^{\mu k}. \end{cases} \quad (35)$$

Comparing the different powers of t to both sides of the system (35), for t^0 ,

$$\begin{cases} a_1(x) = \frac{1}{\Gamma(\mu+1)} \left(2 + a_0 - 2b_0 \frac{\partial a_0}{\partial x} \right), \\ b_1(x) = \frac{1}{\Gamma(\mu+1)} \left(3 - b_0 + 3a_0 \frac{\partial b_0}{\partial x} \right). \end{cases} \quad (36)$$

After substituting the values a_0 , b_0 , $\partial a_0/\partial x$, and $\partial b_0/\partial x$, we obtain

$$\begin{cases} a_1(x) = \frac{e^x}{\Gamma(\mu+1)}, \\ b_1(x) = \frac{-e^{-x}}{\Gamma(\mu+1)}. \end{cases} \quad (37)$$

For t^μ ,

$$\begin{cases} a_2(x) = \frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)} \left(a_1 - 2b_0 \frac{\partial a_1}{\partial x} - 2b_1 \frac{\partial a_0}{\partial x} \right), \\ b_2(x) = \frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)} \left(-b_1 + 3a_1 \frac{\partial b_0}{\partial x} + 3a_0 \frac{\partial b_1}{\partial x} \right). \end{cases} \quad (38)$$

After substituting the values a_0 , a_1 , b_0 , b_1 , $\partial a_0/\partial x$, $\partial a_1/\partial x$, $\partial b_0/\partial x$, and $\partial b_1/\partial x$, we obtain

$$\begin{cases} a_2(x) = \frac{e^x}{\Gamma(2\mu+1)}, \\ b_2(x) = \frac{e^{-x}}{\Gamma(2\mu+1)}. \end{cases} \quad (39)$$

For $t^{2\mu}$,

$$\begin{cases} a_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(a_2 - 2 \frac{\partial a_2}{\partial x} b_0 - 2 \frac{\partial a_1}{\partial x} b_1 - 2 \frac{\partial a_0}{\partial x} b_2 \right), \\ b_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(-b_2 + 3 \frac{\partial b_2}{\partial x} a_0 + 3 \frac{\partial b_1}{\partial x} a_1 + 3 \frac{\partial b_0}{\partial x} a_2 \right). \end{cases} \quad (40)$$

After substituting the values a_0 , a_1 , a_2 , b_0 , b_1 , b_2 , $\partial a_0/\partial x$, $\partial a_1/\partial x$, $\partial a_2/\partial x$, $\partial b_0/\partial x$, $\partial b_1/\partial x$, and $\partial b_2/\partial x$, we obtain

$$\begin{cases} a_3(x) = \frac{1}{\Gamma(3\mu+1)} \left(e^x - 4 + 2 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right), \\ b_3(x) = \frac{1}{\Gamma(3\mu+1)} \left(-e^{-x} - 6 + 3 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right). \end{cases} \quad (41)$$

For $t^{3\mu}$,

$$\begin{cases} a_4(x) = \frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)} \left(a_3 - 2 \frac{\partial a_3}{\partial x} b_0 - 2 \frac{\partial a_2}{\partial x} b_1 - 2 \frac{\partial a_1}{\partial x} b_2 - 2 \frac{\partial a_0}{\partial x} b_3 \right), \\ b_4(x) = \frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)} \left(-b_3 + 3 \frac{\partial b_3}{\partial x} a_0 + 3 \frac{\partial b_2}{\partial x} a_1 + 3 \frac{\partial b_1}{\partial x} a_2 + 3 \frac{\partial b_0}{\partial x} a_3 \right). \end{cases} \quad (42)$$

After inserting the values a_0 , a_1 , a_2 , a_3 , b_0 , b_1 , b_2 , b_3 , $\partial a_0/\partial x$, $\partial a_1/\partial x$, $\partial a_2/\partial x$, $\partial a_3/\partial x$, $\partial b_0/\partial x$, $\partial b_1/\partial x$, $\partial b_2/\partial x$, and $\partial b_3/\partial x$, we obtain

$$\left\{ \begin{array}{l} a_4(x) = \frac{-1}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -13\pi e^x \Gamma(\mu)\mu \Gamma\left(2\mu+\frac{1}{2}\right) + 4\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ 3e^x \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu - \\ \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right), \\ b_4(x) = \frac{-0.5}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -26\pi e^{-x} \Gamma(\mu)\mu \Gamma\left(2\mu+\frac{1}{2}\right) - 12\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ 6e^{-x} \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu + \\ 3\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right). \end{array} \right. \quad (43)$$

In a similar way, we can compare the values for $t^{4\mu}$, $t^{5\mu}$,
...

So, by inserting the values of $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3$, and b_4 in (33 and 34), we have

$$\begin{aligned} u(t, x) = & e^x + \frac{e^x}{\Gamma(\mu+1)} t^\mu + \frac{e^x}{\Gamma(2\mu+1)} t^{2\mu} + \frac{1}{\Gamma(3\mu+1)} \left(e^x - 4 + 2 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right) t^{3\mu} \\ & + \left[\frac{-1}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -13\pi e^x \Gamma(\mu)\mu \Gamma\left(2\mu+\frac{1}{2}\right) + 4\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ 3e^x \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu - \\ \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right) \right] t^{4\mu} + \dots, \end{aligned} \quad (44)$$

$$\begin{aligned} v(t, x) = & e^{-x} - \frac{e^{-x}}{\Gamma(\mu+1)} t^\mu + \frac{e^{-x}}{\Gamma(2\mu+1)} t^{2\mu} + \frac{1}{\Gamma(3\mu+1)} \left(-e^{-x} - 6 + 3 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right) t^{3\mu} \\ & + \left[\frac{-0.5}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(\begin{pmatrix} -26\pi e^{-x} \Gamma(\mu)\mu \Gamma\left(2\mu+\frac{1}{2}\right) - 12\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \\ 6e^{-x} \Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu + \\ 3\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \end{pmatrix} \Gamma(\mu)\mu \right) \right] t^{4\mu} + \dots. \end{aligned} \quad (45)$$

If $\mu = 1$, then equations (44) and (45) give

$$\begin{aligned}
 u(t, x) &= \left(1 + \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} + \frac{t^3}{\Gamma(4)} + \frac{t^4}{\Gamma(5)} + \dots \right) e^x = e^{x+t}, \\
 v(t, x) &= \left(1 - \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} - \frac{t^3}{\Gamma(4)} + \frac{t^4}{\Gamma(5)} + \dots \right) e^{-x} = e^{-x-t}.
 \end{aligned}
 \tag{46}$$

Example 3. Consider the following fractional-order system:

$$\begin{cases} D_0^\mu u(t, x) = 2 + 3u - v \frac{du}{dx}, \\ D_0^\mu v(t, x) = 2 - 3v + u \frac{dv}{dx}, \end{cases}
 \tag{47}$$

subject to the initial conditions

$$\begin{aligned}
 u(0, x) &= e^{2x}, \\
 v(0, x) &= e^{-2x}.
 \end{aligned}
 \tag{48}$$

The exact solution of this system when $\mu = 1$ is

$$\begin{aligned}
 u(t, x) &= e^{2x+3t}, \\
 v(t, x) &= e^{-2x-3t}.
 \end{aligned}
 \tag{49}$$

Solution:

Let

$$u(x, t) = \sum_{k=0}^{\infty} a_k(x) t^{\mu k} = a_0(x) + t^\mu a_1(x) + t^{2\mu} a_2(x) + \dots + t^{n\mu} a_n(x) + \dots,
 \tag{50}$$

$$v(x, t) = \sum_{k=0}^{\infty} b_k(x) t^{\mu k} = b_0(x) + t^\mu b_1(x) + t^{2\mu} b_2(x) + \dots + t^{n\mu} b_n(x) + \dots.
 \tag{51}$$

Apply (12)–(17) on the first and second equations of the system (47), respectively, system (47) becomes

$$\begin{cases} \sum_{k=1}^{\infty} a_k \frac{\Gamma(k\mu + 1)}{\Gamma((k-1)\mu + 1)} t^{\mu(k-1)} = 2 - \left(\sum_{k=0}^{\infty} b_k t^{\mu k} \right) \left(\sum_{k=0}^{\infty} \frac{\partial a_k}{\partial x} t^{\mu k} \right) + 3 \sum_{k=0}^{\infty} a_k t^{\mu k}, \\ \sum_{k=1}^{\infty} b_k \frac{\Gamma(k\mu + 1)}{\Gamma((k-1)\mu + 1)} t^{\mu(k-1)} = 2 + \left(\sum_{k=0}^{\infty} a_k t^{\mu k} \right) \left(\sum_{k=0}^{\infty} \frac{\partial b_k}{\partial x} t^{\mu k} \right) + 3 \sum_{k=0}^{\infty} b_k t^{\mu k}. \end{cases}
 \tag{52}$$

Comparing the different powers of t to both sides of the system (52), for t^0 ,

$$\begin{cases} a_1(x) = \frac{1}{\Gamma(\mu + 1)} \left(2 - b_0 \frac{\partial a_0}{\partial x} + 3a_0 \right), \\ b_1(x) = \frac{1}{\Gamma(\mu + 1)} \left(2 + a_0 \frac{\partial b_0}{\partial x} - 3b_0 \right). \end{cases}
 \tag{53}$$

After substituting the values $a_0(x)$, $b_0(x)$, $\partial a_0(x)/\partial x$, and $\partial b_0(x)/\partial x$, we obtain

$$\begin{cases} a_1(x) = \frac{3e^{2x}}{\Gamma(\mu + 1)}, \\ b_1(x) = \frac{-3e^{-2x}}{\Gamma(\mu + 1)}. \end{cases}
 \tag{54}$$

For t^μ ,

$$\begin{cases} a_2(x) = \frac{\Gamma(\mu + 1)}{\Gamma(2\mu + 1)} \left(3a_1 - \frac{\partial a_0}{\partial x} b_1 - \frac{\partial a_1}{\partial x} b_0 \right), \\ b_2(x) = \frac{\Gamma(\mu + 1)}{\Gamma(2\mu + 1)} \left(-3b_1 + \frac{\partial b_0}{\partial x} a_1 + \frac{\partial b_1}{\partial x} a_0 \right). \end{cases}
 \tag{55}$$

After substituting the values a_0 , a_1 , b_0 , b_1 , $\partial a_0/\partial x$, $\partial a_1/\partial x$, $\partial b_0/\partial x$, and $\partial b_1/\partial x$, we obtain

$$\begin{cases} a_2(x) = \frac{9e^{2x}}{\Gamma(2\mu + 1)}, \\ b_2(x) = \frac{9e^{-2x}}{\Gamma(2\mu + 1)}. \end{cases}
 \tag{56}$$

For $t^{2\mu}$,

$$\left\{ \begin{array}{l} a_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(3a_2 - \frac{\partial a_2}{\partial x} b_0 - \frac{\partial a_1}{\partial x} b_1 - \frac{\partial a_0}{\partial x} b_2 \right), \\ b_3(x) = \frac{\Gamma(2\mu+1)}{\Gamma(3\mu+1)} \left(-3b_2 + \frac{\partial b_2}{\partial x} a_0 + \frac{\partial b_1}{\partial x} a_1 + \frac{\partial b_0}{\partial x} a_2 \right). \end{array} \right. \quad (57)$$

For $t^{3\mu}$,

$$\left\{ \begin{array}{l} a_3(x) = \frac{9}{\Gamma(3\mu+1)} \left(3e^{2x} - 4 + 2 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right), \\ b_3(x) = \frac{9}{\Gamma(3\mu+1)} \left(-3e^{-2x} - 4 + 2 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right). \end{array} \right. \quad (58)$$

After substituting the values $a_0, a_1, a_2, b_0, b_1, b_2, \partial a_0/\partial x, \partial a_1/\partial x, \partial a_2/\partial x, \partial b_0/\partial x, \partial b_1/\partial x$, and $\partial b_2/\partial x$, we obtain

$$\left\{ \begin{array}{l} a_4(x) = \frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)} \left(3a_3 - \frac{\partial a_3}{\partial x} b_0 - \frac{\partial a_2}{\partial x} b_1 - \frac{\partial a_1}{\partial x} b_2 - \frac{\partial a_0}{\partial x} b_3 \right), \\ b_4(x) = \frac{\Gamma(3\mu+1)}{\Gamma(4\mu+1)} \left(-3b_3 + \frac{\partial b_3}{\partial x} a_0 + \frac{\partial b_2}{\partial x} a_1 + \frac{\partial b_1}{\partial x} a_2 + \frac{\partial b_0}{\partial x} a_3 \right). \end{array} \right. \quad (59)$$

After inserting the values $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, \partial a_0/\partial x, \partial a_1/\partial x, \partial a_2/\partial x, \partial a_3/\partial x, \partial b_0/\partial x, \partial b_1/\partial x, \partial b_2/\partial x$, and $\partial b_3/\partial x$, we obtain

$$\left\{ \begin{array}{l} a_4(x) = \frac{1}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(9 \left(\begin{array}{l} 17\pi e^{2x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) - 12\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi- \\ 2e^{2x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2}16^\mu+ \\ 3\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2}16^\mu \end{array} \right) \Gamma(\mu)\mu \right), \\ b_4(x) = \frac{1}{\Gamma(4\mu+1)\Gamma(2\mu+1/2)\Gamma(\mu+1)^2\pi} \left(9 \left(\begin{array}{l} -17\pi e^{2x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) - 12\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi+ \\ 2e^{2x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2}16^\mu+ \\ 3\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2}16^\mu \end{array} \right) \Gamma(\mu)\mu \right). \end{array} \right. \quad (60)$$

In a similar way, we can compare the values for $t^{4\mu}, t^{5\mu}$,

.....

So, by inserting the values of $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3$, and b_4 in (51) and (52), we have

TABLE 1: Numerical values for $u(t, x)$ of equation (27) for different values of t and x .

x	t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.2	0.25	1.051264145	1.177800483	1.376647053	1.690057985	2.190896007
	0.50	1.349626476	1.552782962	1.825654773	2.178405687	2.598177788
	0.75	1.731407662	1.995745862	2.305360890	2.636753607	2.913559185
	1	2.217395789	2.524361162	2.833631770	3.091314344	3.183670371
0.5	0.25	0.778795634	0.871298353	1.010006624	1.199502045	1.416439046
	0.50	0.999827884	1.143252515	1.314743942	1.481305224	1.592001081
	0.75	1.282658343	1.458459258	1.623729060	1.723827348	1.717670616
	1	1.642687203	1.827732063	1.949469341	1.949842216	1.819993633
0.8	0.25	0.576945996	0.644235991	0.738392714	0.836089264	0.842707220
	0.50	0.740690714	0.839864902	0.936251891	0.964880499	0.846607044
	0.75	0.950216672	1.060427554	1.118763783	1.047514936	0.831734576
	1	1.216932611	1.311656534	1.294465704	1.104218862	0.809757060

$$\begin{aligned}
u(x, t) = & e^{2x} + \frac{3e^{2x}}{\Gamma(\mu+1)} t^\mu + \frac{9e^{2x}}{\Gamma(2\mu+1)} t^{2\mu} + \frac{9}{\Gamma(3\mu+1)} \left(3e^{2x} - 4 + 2 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right) t^{3\mu} \\
& + \left[\frac{1}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(9 \left(\left(17\pi e^{2x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) - 12\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi - \right. \right. \right. \right. \right. \\
& \left. \left. \left. 2e^{2x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu + \right. \right. \right. \right. \\
& \left. \left. \left. 3\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \right) \right) \Gamma(\mu)\mu \right] t^{4\mu} + \dots,
\end{aligned} \tag{61}$$

$$\begin{aligned}
v(x, t) = & e^{-2x} - \frac{3e^{-2x}}{\Gamma(\mu+1)} t^\mu + \frac{9e^{-2x}}{\Gamma(2\mu+1)} t^{2\mu} + \frac{9}{\Gamma(3\mu+1)} \left(-3e^{-2x} - 4 + 2 \frac{\Gamma(2\mu+1)}{\Gamma(\mu+1)^2} \right) t^{3\mu} \\
& + \left[\frac{1}{\Gamma(4\mu+1)\Gamma(2\mu+(1/2))\Gamma(\mu+1)^2\pi} \left(9 \left(\left(-17\pi e^{2x}\Gamma(\mu)\mu\Gamma\left(2\mu+\frac{1}{2}\right) - 12\Gamma\left(2\mu+\frac{1}{2}\right)\Gamma(\mu)\mu\pi + \right. \right. \right. \right. \right. \\
& \left. \left. \left. 2e^{2x}\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu + \right. \right. \right. \right. \\
& \left. \left. \left. 3\Gamma\left(\mu+\frac{1}{4}\right)\Gamma\left(\mu+\frac{3}{4}\right)\Gamma\left(\mu+\frac{1}{2}\right)\sqrt{2} 16^\mu \right) \right) \Gamma(\mu)\mu \right] t^{4\mu} + \dots.
\end{aligned} \tag{62}$$

If $\mu = 1$, then equations (61) and (62) give

$$\begin{aligned}
u(x, t) = & e^{2x} \left(1 + \frac{3t}{\Gamma(2)} + \frac{9t^2}{\Gamma(3)} + \frac{27t^3}{\Gamma(4)} + \frac{81t^4}{\Gamma(5)} + \dots \right) = e^{2x+3t}, \\
v(x, t) = & e^{-2x} \left(1 - \frac{3t}{\Gamma(2)} + \frac{9t^2}{\Gamma(3)} - \frac{27t^3}{\Gamma(4)} + \frac{81t^4}{\Gamma(5)} - \dots \right) = e^{-2x-3t}.
\end{aligned} \tag{63}$$

3. Results and Discussion

Tables 1 and 2 show the values of $u(t, x)$ and $v(t, x)$ for Example 1 with different values of x , t , and μ . Afterward, the

graphical simulations are drawn with the help of MATLAB. The graphs are shown in 2D and 3D, and these can be seen in Figures 1 and 2. Tables 3 and 4 elaborate the values of $u(t, x)$ and $v(t, x)$ for Example 2 and graphical imitations can be

TABLE 2: Numerical values for $v(t, x)$ of equation (28) for different values of t and x .

x	t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.2	0.25	0.951238964	0.865889787	0.777138934	0.743944913	1.125397015
	0.50	0.741111569	0.678152261	0.651799698	0.826638198	1.556286290
	0.75	0.579092812	0.557061297	0.638677023	1.066582027	1.987000806
	1	0.458026034	0.494869468	0.736296772	1.429900092	2.411626265
0.5	0.25	1.284038295	1.170123705	1.057085400	1.031611960	1.563206998
	0.50	1.000395979	0.921508202	0.903205690	1.162805382	2.141116646
	0.75	0.781693532	0.766339970	0.902730012	1.497237306	2.711609503
	1	0.618270477	0.693434087	1.050515555	1.989643341	3.270439580
0.8	0.25	1.733270400	1.580796538	1.434973603	1.419921858	2.154188653
	0.50	1.350393324	1.250004362	1.242568283	1.616583612	2.930555054
	0.75	1.055175900	1.048836629	1.259164265	2.078561123	3.689728934
	1	0.834577848	0.961468288	1.474666548	2.745217689	4.429716292

FIGURE 1: 2D graphical representation for $u(t, x)$ and $v(t, x)$ of equation (9). FIGURE 2: 3D graphical representation for $u(t, x)$ and $v(t, x)$ of equation (9).

TABLE 3: Numerical values for $u(t, x)$ of equation (44) for different values of t and x .

x	t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.2	0.25	1.568301816	1.757926562	2.080012379	2.798019675	4.912818944
	0.50	2.013406109	2.332758728	2.890179752	4.141145675	6.791936646
	0.75	2.582956711	3.048410210	3.910944981	5.685815446	8.418001219
	1	3.307965802	3.959963255	5.203632802	7.434136691	9.906507037
0.5	0.25	2.116986021	2.376294107	2.834284463	3.918730674	7.189418490
	0.50	2.717813970	3.168001876	4.003206759	5.947667531	10.06655772
	0.75	3.486626866	4.168986944	5.506335361	8.294708222	12.55300584
	1	4.465286775	5.459753157	7.428416392	10.95364021	14.82625765
0.8	0.25	2.857632224	3.211002984	3.852445275	5.431532282	10.26250643
	0.50	3.668665124	4.295462192	5.505636061	8.386216959	14.48683382
	0.75	4.706453984	5.681607311	7.659887116	11.81634511	18.13467822
	1	6.027506680	7.484257760	10.43156010	15.70447300	21.46722633

TABLE 4: Numerical values for $v(t, x)$ of equation (45) for different values of t and x .

x	t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.2	0.25	0.637634547	0.574001290	0.497834882	0.570826497	1.834670972
	0.50	0.496781941	0.434184061	0.442592615	1.013033316	3.166481657
	0.75	0.388177520	0.351599173	0.584512413	1.808277898	4.444360353
	1	0.307024032	0.348139360	0.967799336	2.901554369	5.680755396
0.5	0.25	0.472371290	0.422353092	0.350897578	0.362002598	1.261192050
	0.50	0.368025114	0.308098047	0.275946070	0.646105894	2.256111494
	0.75	0.287568980	0.228499125	0.342771994	1.211809268	3.227036363
	1	0.227448997	0.201390564	0.591130501	2.017332508	4.174882213
0.8	0.25	0.349941259	0.310009344	0.242043747	0.207302048	0.836348419
	0.50	0.272639710	0.214691230	0.152491273	0.374279373	1.581692701
	0.75	0.213036340	0.137304367	0.163686284	0.766993444	2.325220576
	1	0.168498362	0.092676381	0.312087366	1.362284840	3.059303926

FIGURE 3: 2D graphical representation for $u(t, x)$ and $v(t, x)$ of equation (30).

seen in Figures 3 and 4 for different values of t , x , and μ . Besides these, Tables 5 and 6 represent the values of t , x , and μ and simulations can be viewed in Figures 5 and 6.

Fractional physical equations can be solved successfully by using analytical and approximate solutions method called fractional power series method (FPSM). The Caputo

FIGURE 4: 3D graphical representation for $u(t, x)$ and $v(t, x)$ of equation (30).TABLE 5: Numerical values for $u(x, t)$ of equation (61) for different values of x and t .

x	t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.2	0.25	3.154830453	4.691959285	8.866135776	22.72166785	67.89003771
	0.50	6.561697696	11.20075599	22.75683011	51.52487616	108.6954101
	0.75	13.04982396	23.14398543	44.97510977	87.11237439	144.2819487
	1	24.42862944	42.63967701	76.52840205	128.5195969	176.9305414
0.5	0.25	5.748475877	8.635877638	16.93781919	46.24073749	145.4330455
	0.50	11.95619272	20.95933710	44.78616917	107.0281709	234.3493828
	0.75	23.77832957	43.84935668	89.83870690	182.5069498	311.9830971
	1	44.51186494	81.45614191	154.1476188	270.5167864	383.2455096
0.8	0.25	10.47440597	15.82216542	31.64538531	89.09527647	286.7256176
	0.50	21.78560356	38.74063122	84.92624204	208.1617677	463.3058486
	0.75	43.32694134	81.57700295	171.5855108	356.3271994	617.5545123
	1	81.10590594	152.1843524	295.5790527	529.2525354	759.1758921

TABLE 6: The numerical value for $v(x, t)$ of equation (62) for different values of x and t .

x	t	$\mu = 1$	$\mu = 0.8$	$\mu = 0.6$	$\mu = 0.4$	$\mu = 0.2$
0.2	0.25	0.317812874	0.247943726	0.525872181	4.855612927	31.24156799
	0.50	0.183290638	0.416943067	3.086090208	16.96966349	57.09462535
	0.75	0.302102248	1.709140335	9.532343390	34.85940052	81.01323439
	1	0.921690063	5.080281660	20.90355020	57.73320775	103.7181425
0.5	0.25	0.174419403	0.114746644	0.222808556	3.038942776	21.57352776
	0.50	0.100592035	0.167462940	1.791174215	11.28469761	39.88834029
	0.75	0.165797228	0.897605834	5982261590	23.65380070	56.90494333
	1	0.505834231	2.934839500	13.53544442	39.58514034	73.09600369
0.8	0.25	0.095723398	0.041646536	0.056483713	2.041933069	16.26759478
	0.50	0.055206079	0.030545346	1.080509256	8.164722200	30.44533086
	0.75	0.090991448	0.452226258	4.033935410	17.50403717	43.67403279
	1	0.277607711	1.757395880	9.491742250	29.62526973	56.29021759

operator is presented as fractional derivatives. As compared to other methods to solve nonlinear equations, the power series method is employed efficiently to obtain the solution.

Results thus obtained show that FPSM can be applied to solve the system of nonlinear partial differential equations (PDEs) with accuracy and effectiveness.

FIGURE 5: 2D graphical representation for $u(x,t)$ and $v(x,t)$ of equation (47). FIGURE 6: 3D graphical representation for $u(x,t)$ and $v(x,t)$ of equation (47).

4. Conclusion

Numerical simulation results prove the validity and reliability of the power series method. Therefore, it is more convenient to describe that PSM is very powerful in constructing fractional power series solutions for fractional PDEs in scientific problems of any order and time of space. As per the work done, we had offered the fractional-order power series method for finding the nonlinear fractional-order partial differential equation with comparable easy implementation. In the tables, reasonable values are presented for fractional and integer cases. 2D and 3D graphical presentations are given for fractional-order $\mu = 0.1$ for

$t = 0.25, 0.5, 0.75$, and 1 . The proposed method is quite promising in reducing the computational cost of solving such problems to a great extent.

Data Availability

All the data are available in the article and cited wherever required.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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