Review Article

Analytical Solutions for the Equal Width Equations Containing Generalized Fractional Derivative Using the Efficient Combined Method

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In this paper, the efficient combined method based on the homotopy perturbation Sadik transform method (HPSTM) is applied to solve the physical and functional equations containing the Caputo–Prabhakar fractional derivative. The mathematical model of this equation of order $\mu \in (0, 1]$ with $\lambda \in \mathbb{Z}^+$, $\theta, \sigma \in \mathbb{R}^+$ is presented as follows: ${}^C\mathfrak{D}_t^\mu u(x, t) + \theta u^\lambda(x, t)u_x(x, t) - \sigma u_{xxt}(x, t) = 0$, where for $\lambda = 1, \theta = 1, \sigma = 1$ and $\lambda = 2, \theta = 3, \sigma = 1$, equations are changed into the equal width and modified equal width equations, respectively. The analytical method which we have used for solving this equation is based on a combination of the homotopy perturbation method and Sadik transform. The convergence and error analysis are discussed in this article. Plots of the analytical results with three examples are presented to show the applicability of this numerical method. Comparison between the obtained absolute errors by the suggested method and other methods is demonstrated.

1. Introduction

The integrals and derivatives of fractional order in fractional calculus play an important role in many branches such as mathematics, engineering, and physics [1–6]. The differential systems of fractional order are regarded as an extension of the differential systems of integer order and have important application in expressing nonlinear phenomena in many scopes such as applied sciences [7, 8], engineering [9, 10], and physics [11, 12]. One of the most important nonlinear differential equations used in many fields of mathematics, engineering, and physics is the equal width (EW) differential equation of fractional order where the equation is a partial differential equation that describes physical behaviors such as water transfer in soils [13], crystal growth [14], and shallow water waves [15]. The main reason for choosing these types of equations is due to their application in mathematics and physics, plasma waves, fluid mechanics, solid-state physics, and chemical physics [16, 17]. In this paper, we state the time-

fractional comprehensive EW equation containing Caputo–Prabhakar fractional derivative as follows [18–20]:

$${}^C\mathfrak{D}_t^\mu u(x, t) + \theta u^\lambda(x, t)u_x(x, t) - \sigma u_{xxt}(x, t) = 0, \quad (1)$$

$$u(x, 0) = u^0(x), u(x, t) \longrightarrow 0, \text{ as } x \longrightarrow \pm\infty,$$

where $0 < \mu \leq 1, \theta \geq 0, \sigma \geq 0$, and $\lambda \in \mathbb{Z}^+$ is an integer. For equation (1), we consider two cases as follows:

- (1) If $\lambda = 1, \theta = 1, \sigma = 1$ are considered, then equation (1) becomes the nonlinear equal width equation of fractional order μ . This type of equation is one of the most important equations that examine the nonlinear behaviors of physical phenomena such as hydromagnetic waves [21], optics [22], and biological systems [23].
- (2) If $\lambda = 2, \theta = 3, \sigma = 1$ are considered, then equation (1) becomes the nonlinear modified equal width equation (MEW) of fractional order μ .

Moreover, in equation (1), function $u(x, t)$ is the probability density function, x is the spatial coordinate, and t is the temporal coordinate. To find the solutions of these types of equations, the homotopy perturbation Sadik transform method is used, which is a generalization of the homotopy perturbation and Sumudu transform methods. The method presented in this article is similar to the methods discussed in [24–31]. In equation (1), ${}^C\mathfrak{D}_t^\mu$ is the Caputo–Prabhakar fractional derivative of order μ which is defined by

$$\begin{aligned} {}^C\mathfrak{D}_t^\mu u(x, t) &= \int_0^t (t-\tau)^{-\mu} \mathbb{E}_{\rho, 1-\mu}^{-\gamma}(\omega(t-\tau)^\rho) \dot{u}(x, \tau) d\tau \\ &= \mathfrak{E}^{1-\mu, -\gamma} \frac{d}{dt} u(x, t), \quad 0 < \mu \leq 1, \end{aligned} \quad (2)$$

where $\mathfrak{E}^{1-\mu, -\gamma}$ is the Prabhakar fractional integral and is defined by [32]

$$\begin{aligned} (\mathfrak{E}^{\mu, \gamma} u)(x, t) &= \int_0^t (t-\tau)^{\mu-1} \mathbb{E}_{\rho, \mu}^{\gamma}(\omega(t-\tau)^\rho) u(x, \tau) d\tau \\ &= u(x, t) * t^{\mu-1} \mathbb{E}_{\rho, \mu}^{\gamma}(\omega t^\rho), \end{aligned} \quad (3)$$

where $u(x, t) \in L^1([0, b] \times [0, b])$, $b \in \mathbb{R}$ and $u(x, t) * t^{\mu-1} \mathbb{E}_{\rho, \mu}^{\gamma}(\omega t^\rho) \in W^{m, 1}([0, b] \times [0, b])$, $m = \lceil \mu \rceil$ that $W^{m, 1}$ is the Sobolev space. Also, $\mathbb{E}_{\rho, \mu}^{\gamma}$ is the three-parameter Mittag-Leffler function which is defined by [32]

$$\mathbb{E}_{\rho, \mu}^{\gamma}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n! \Gamma(\rho n + \mu)} t^n, \quad \Re(\rho), \Re(\mu) > 0, \gamma > 0, t \in \mathbb{C}, \quad (4)$$

and $(\gamma)_n$ is the Pochhammer symbol which is given by [33]

$$(\gamma)_0 = 1, (\gamma)_n = \gamma(\gamma+1), \dots, (\gamma+n-1), n \in \mathbb{N}. \quad (5)$$

The main reason for choosing the three-parameter Mittag-Leffler function in this paper is related to its application in many models such as disordered materials and heterogeneous models [34], Havriliak–Negami models [35, 36], viscoelasticity models [37], stochastic models [38], probability models [39], spherical stellar models [40], Poisson models [41], and fractional models or integral models [42–45]. Many research studies have been conducted in numerical fields to obtain numerical solutions of fractional differential equations such as the Adomian decomposition method [46], the q-homotopy analysis transform method (q-HATM) [47], the Laplace transform method [48], the method based on the implementation of an iterative perturbation method [49], the numerical method based on the Petrov–Galerkin method [50], the Petrov–Galerkin finite element scheme [51], the local discontinuous Galerkin method [52], and other methods [24, 27, 30, 53]. Recently, a new integral transform named the Sadik transform was introduced by Sadikali Latif Shaikh in 2018 (see [54, 55]). The Sadik transform is nothing but an unification of the Laplace transform, Sumudu transform, Elzaki transform, and all those integral transforms whose

kernels are of exponential type or similar to the kernel of the Laplace transform. The Sadik transform is a very powerful integral transform. In this paper, we apply the HPSTM to solve the nonlinear time-fractional EW equation, MEW equation, and VMEW equation. The key intention of the present work is to extend the utilization of the HPSTM to derive analytical and approximate solutions of the time-fractional EW equation, time-fractional MEW equation, and time-fractional VMEW equation.

This paper is organized as follows. Some definitions and mathematical preliminaries of the fractional calculus are presented in Section 2. In Section 3, we introduce an analytical method based on the HPSTM to the nonlinear nonhomogeneous partial differential equation. In Section 4, to show the validity of the suggested method in Section 3, three examples are simulated.

2. Important Notations

In this section, we express theorems and lemmas which are used in the next parts. where $\Re(\mu) > 0$, $\Re(\rho) > 0$ and $\rho, \mu, \omega, \gamma \in \mathbb{C}$.

Lemma 1. The Laplace transformation of function (4) is given by [33, 56]

$$\mathcal{L}(t^{\mu-1} \mathbb{E}_{\rho, \mu}^{\gamma}(\omega t^\rho); s) = s^{-\mu} (1 - \omega s^{-\rho})^{-\gamma}, \quad (6)$$

Lemma 2 (see [3]). The following relation holds:

$$(\mathfrak{E}^{\mu, \gamma} t^{\nu-1})(t) = \Gamma(\nu) t^{\mu+\nu-1} \mathbb{E}_{\rho, \mu+\nu}^{\gamma}(\omega t^\rho), \quad (7)$$

where $\Re(\mu) > 0$, $\Re(\rho) > 0$ and $\rho, \mu, \omega, \gamma \in \mathbb{C}$.

Lemma 3 (see [57]). Suppose $\Phi_1(v, \alpha, \beta)$ is a Sadik transform of $\varphi_1(t)$ and $\Phi_2(v, \alpha, \beta)$ is a Sadik transform of $\varphi_2(t)$. Then, Sadik transform of $(\varphi_1 * \varphi_2)(t)$ is given by

$$\mathcal{S}[(\varphi_1 * \varphi_2)(t)] = v^\beta \Phi_1(v, \alpha, \beta) \Phi_2(v, \alpha, \beta), \quad (8)$$

where $*$ is a convolution. Also, $\Phi_1(v, \alpha, \beta) = \mathcal{S}(\varphi_1(t)) = 1/v^\beta \int_0^{+\infty} e^{-tv^\alpha} \varphi_1(t) dt$ and $\Phi_2(v, \alpha, \beta) = \mathcal{S}(\varphi_2(t)) = 1/v^\beta \int_0^{+\infty} e^{-tv^\alpha} \varphi_2(t) dt$.

Theorem 1 (see [57]). Suppose $\Phi(v, \alpha, \beta) = \mathcal{S}(\varphi(t))$ and $\varphi(t), \varphi'(t), \dots, \varphi^{m-1}(t), m \in \mathbb{N}$, are continuous. Then,

$$\mathcal{S}[\varphi^m(t)] = v^{m\alpha} \Phi(v, \alpha, \beta) - \sum_{k=0}^{m-1} v^{k\alpha-\beta} \varphi^{m-k-1}(0). \quad (9)$$

Also, the Sadik transform of integration of $\varphi(t)$ is defined by

$$\mathcal{S}\left[\int_0^t \varphi(\tau) d\tau\right] = \frac{1}{v^\alpha} \Phi(v, \alpha, \beta). \quad (10)$$

Theorem 2. The Sadik transform of the Caputo–Prabhakar fractional derivative of order μ for $m = 1$ is obtained by

$$\mathcal{S}({}^C\mathfrak{D}_t^\mu u(x, t)) = v^{\alpha\mu} (1 - \omega v^{-\rho\alpha})^\gamma U(v, \alpha, \beta) - v^{\alpha(\mu-1)-\beta} (1 - \omega v^{-\rho\alpha})^\gamma u(x, 0^+), \quad (11)$$

where $U(v, \alpha, \beta) = \mathcal{S}(u(x, t))$ and $\rho, \mu, \omega, \gamma \in \mathbb{C}, \Re(\mu) > 0, \Re(\rho) > 0$.

Proof. By applying equations (2), (6), (8), and (9), we obtain

$$\begin{aligned} \mathcal{S}({}^C\mathfrak{D}_t^\mu u(x, t)) &= \frac{1}{v^\beta} \int_0^\infty e^{-tv^\beta} [{}^C\mathfrak{D}_t^\mu u(x, t)] dt \\ &= \frac{1}{v^\beta} \int_0^\infty e^{-tv^\beta} \left\{ \int_0^t (t-\tau)^{-\mu} \mathbb{E}_{\rho, 1-\mu}^{-\gamma} (\omega(t-\tau)^\rho) \frac{d}{d\tau} u(x, \tau) d\tau \right\} dt \\ &= \frac{1}{v^\beta} \int_0^\infty \int_\tau^\infty e^{-tv^\beta} (t-\tau)^{-\mu} \mathbb{E}_{\rho, 1-\mu}^{-\gamma} (\omega(t-\tau)^\rho) \frac{d}{d\tau} u(x, \tau) dt d\tau \\ &= v^\beta \mathcal{S}\left(\frac{d}{dt} u(x, t)\right) \times \mathcal{S}(t^{-\mu} \mathbb{E}_{\rho, 1-\mu}^{-\gamma} (\omega t^\rho)) \\ &= [v^\alpha U(v, \alpha, \beta) - v^{-\beta} u(x, 0^+)] \times v^{\alpha\mu-\alpha} (1 - \omega v^{-\rho\alpha})^\gamma \\ &= v^{\alpha\mu} (1 - \omega v^{-\rho\alpha})^\gamma U(v, \alpha, \beta) - v^{\alpha(\mu-1)-\beta} (1 - \omega v^{-\rho\alpha})^\gamma u(x, 0^+). \end{aligned} \quad (12)$$

The proof is proven. $\square$

$$\begin{aligned} {}^C\mathfrak{D}_t^\mu u(x, t) + \mathbf{A}u(x, t) + \mathbf{N}u(x, t) &= g(x, t), \\ u(x, 0) &= u^0(x), \end{aligned} \quad (13)$$

3. The Proposed Scheme

This section focuses on an analytical method based on the homotopy perturbation and Sadik transform method for finding the solutions of the following fractional differential equations:

where $\mathbf{A}u(x, t)$ is a linear differential operator, $\mathbf{N}u(x, t)$ is a nonlinear differential operator, and $g(x, t)$ is an indicated source term. Using equation (11) and taking the Sadik transform on both sides of equation (13), we get

$$\begin{aligned} \mathcal{S}({}^C\mathfrak{D}_t^\mu u(x, t) + \mathbf{A}u(x, t) + \mathbf{N}u(x, t) - g(x, t)) &= 0, \\ v^{\alpha\mu} (1 - \omega v^{-\rho\alpha})^\gamma U(v, \alpha, \beta) - v^{\alpha(\mu-1)-\beta} (1 - \omega v^{-\rho\alpha})^\gamma u(x, 0^+) + \mathcal{S}(\mathbf{A}u(x, t) + \mathbf{N}u(x, t)) &= \mathcal{S}(g(x, t)), \end{aligned} \quad (14)$$

$$U(v, \alpha, \beta) = v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma} \mathcal{S}(g(x, t)) + \frac{1}{v^{\alpha+\beta}} u^0(x) - (v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \times [\mathcal{S}(\mathbf{A}u(x, t) + \mathbf{N}u(x, t))].$$

Now applying Sadik inverse transform on equation (14), we obtain

$$\begin{aligned} u(x, t) &= \mathcal{S}^{-1} \left[v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma} \mathcal{S}(g(x, t)) + \frac{1}{v^{\alpha+\beta}} u^0(x) \right] \\ &\quad - \mathcal{S}^{-1} [(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \\ &\quad \times [\mathcal{S}(\mathbf{A}u(x, t) + \mathbf{N}u(x, t))]]. \end{aligned} \quad (15)$$

To get the solution of equation (15), we can state the solutions of equation (15) by the following infinite series:

$$u(x, t) = \sum_{n=0}^{\infty} \varrho^n u_n(x, t), \quad (16)$$

where $u_n(x, t)$, $n = 0, 1, 2, \dots$, are known. Also, the non-linear term $\mathbf{N}u(x, t)$ can be displayed as the following infinite series:

$$\mathbf{N}u(x, t) = \sum_{n=0}^{\infty} \varrho^n \mathbf{H}_n(u(x, t)), \quad (17)$$

where $\mathbf{H}_n(u(x, t)) = \{1/n! (\partial^n / \partial \varrho^n) [N(\sum_{i=0}^{\infty} \varrho^i u_i)]_{\varrho=0}\}$ are the Adomian polynomials which were introduced in [46]. Now, substituting equations (16) and (17) into (15), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \varrho^n u_n(x, t) &= G(x, t) - \mathcal{S}^{-1} \left[(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \times \left[\mathcal{S} \left(\mathbf{A} \sum_{n=0}^{\infty} \varrho^n u_n(x, t) + \mathbf{N} \sum_{n=0}^{\infty} \varrho^n u_n(x, t) \right) \right] \right] \\ &= G(x, t) - \mathcal{S}^{-1} \left[(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \times \left[\mathcal{S} \left(\mathbf{A} \sum_{n=0}^{\infty} \varrho^n u_n(x, t) + \sum_{n=0}^{\infty} \varrho^n \mathbf{H}_n(u(x, t)) \right) \right] \right], \end{aligned} \quad (18)$$

where $G(x, t) = \mathcal{S}^{-1} [v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma} \mathcal{S}(g(x, t)) + 1/v^{\alpha+\beta} u^0(x)]$. Therefore, we put the coefficients on powers of ϱ on both sides of relation (18) equally, which results in

$$\begin{aligned} \varrho^0: u_0(x, t) &= G(x, t), \\ \varrho^1: u_1(x, t) &= -\mathcal{S}^{-1} [(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \times [\mathcal{S}(\mathbf{A}u_0(x, t) + \mathbf{H}_0(u(x, t)))]; t], \\ \varrho^2: u_2(x, t) &= -\mathcal{S}^{-1} [(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \times [\mathcal{S}(\mathbf{A}u_1(x, t) + \mathbf{H}_1(u(x, t)))]; t], \\ &\vdots \\ \varrho^{n+1}: u_{n+1}(x, t) &= -\mathcal{S}^{-1} [(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \times [\mathcal{S}(\mathbf{A}u_n(x, t) + \mathbf{H}_n(u(x, t)))]; t]. \end{aligned} \quad (19)$$

Then, the analytical solution of equation (13) can be obtained as

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (20)$$

Theorem 3. Let $u(x, t)$ be the exact solution of equation (13) and $u_n(x, t)$ be the approximate solution of equation (13) so that $\|u_{n+1}(x, t)\| \leq \omega \|u_n(x, t)\|$, $\omega \in (0, 1)$, $n \in \mathbb{N}$. Then, the series defined by (20) converges.

Proof. From $\|u_{n+1}(x, t)\| \leq \omega \|u_n(x, t)\|$, we obtain

$$\begin{aligned} \|u_1(x, t)\| &\leq \omega \|u_0(x, t)\| = \omega \|u^0(x)\|, \\ \|u_2(x, t)\| &\leq \omega \|u_1(x, t)\| \leq \omega^2 \|u^0(x)\|, \\ &\vdots \\ \|u_{n+1}(x, t)\| &\leq \omega \|u_n(x, t)\| \leq \omega^{n+1} \|u^0(x)\|. \end{aligned} \quad (21)$$

Then,

$$\sum_{i=n+1}^{\infty} \|u_i(x, t)\| \leq \sum_{i=n+1}^{\infty} \omega^i \|u^0(x)\| = \|u^0(x)\| \sum_{i=n+1}^{\infty} \omega^i. \quad (22)$$

Therefore,

$$\begin{aligned} \|u(x, t) - u_{n+1}(x, t)\| &= \left\| \sum_{i=n+1}^{\infty} u_i(x, t) \right\| \leq \sum_{i=n+1}^{\infty} \|u_i(x, t)\| \\ &\leq \sum_{i=n+1}^{\infty} \omega^i \|u^0(x)\| = \|u^0(x)\| \sum_{i=n+1}^{\infty} \omega^i \\ &\leq \frac{\omega^{n+1}}{1 - \omega} \|u^0(x)\|. \end{aligned} \quad (23)$$

Because $\omega \in (0, 1)$, $\|u - u_n\| \rightarrow 0$ as $n \rightarrow \infty$. The proof is proven. $\square$

Theorem 4. Let $u(x, t)$ be the exact solution of equation (1) and $(u(x, t))_N$ be the best approximate solution of equation (1) so that $\|(u(x, t))_N^{\lambda+1} - u^{\lambda+1}(x, t)\|_{L^2(0,1)} \leq \xi \|(u(x, t))_N - u(x, t)\|_{L^2(0,1)}$, $\xi \in (0, 1)$, $\lambda \in \mathbb{Z}^+$. Then, the following relation for the error holds:

$$\begin{aligned} \|E(u(x, t))\|_{L^2(0,1)} &\leq \left[k_1 (t^{1-\mu} E_{\rho, 2-\mu}^{-\gamma}(\omega t^\rho)) + \frac{\theta}{\lambda+1} k_2 \xi + \sigma k_1 k_2^2 \right] c N^{-\sigma} 2^{-k\sigma} \|u^{(\sigma)}\|_{L^2(0,1)}, \\ \|E(u(x, t))\|_{L^2(0,1)} &\leq \left[k_1 (t^{1-\mu} E_{\rho, 2-\mu}^{-\gamma}(\omega t^\rho)) + \frac{\theta}{\lambda+1} k_2 \xi + \sigma k_1 k_2^2 \right] c N^{2q-(1/2)-\sigma} 2^{k(q-\sigma)} \|u^{(\sigma)}\|_{L^2(0,1)}, \quad q \geq 1. \end{aligned} \quad (24)$$

Proof. Since $u(x, t)$ is the exact solution of equation (1) and $(u(x, t))_N$ is the best approximate solution of equation (1), then we have

$$\begin{aligned}
 \|E(u(x, t))\|_{L^2(0,1)} &= \left\| {}^C\mathfrak{D}_t^\mu (u(x, t))_N + \theta (u(x, t))_N^\lambda ((u(x, t))_N)_x - \sigma (u(x, t))_N \right\|_{xx t} \\
 &\quad - {}^C\mathfrak{D}_t^\mu (u(x, t)) + \theta u^\lambda(x, t) u_x(x, t) - \sigma u_{xxt}(x, t) \Big\|_{L^2(0,1)} \\
 &= \left\| {}^C\mathfrak{D}_t^\mu [(u(x, t))_N - u(x, t)] + \theta [(u(x, t))_N^\lambda ((u(x, t))_N)_x - u^\lambda(x, t) u_x(x, t)] \right. \\
 &\quad \left. - \sigma [(u(x, t))_N]_{xxt} - u_{xxt}(x, t) \right\|_{L^2(0,1)} \\
 &= \left\| {}^C\mathfrak{D}_t^\mu [(u(x, t))_N - u(x, t)] + \frac{\theta}{\lambda + 1} \left(\frac{\partial}{\partial x} ((u(x, t))_N^{\lambda+1} - u^{\lambda+1}(x, t)) \right) \right. \\
 &\quad \left. - \sigma \left[\frac{\partial^3}{\partial x^2 \partial t} ((u(x, t))_N - u(x, t)) \right] \right\|_{L^2(0,1)} \\
 &\leq \left\| {}^C\mathfrak{D}_t^\mu [(u(x, t))_N - u(x, t)] \right\|_{L^2(0,1)} + \frac{\theta}{\lambda + 1} \left\| \left(\frac{\partial}{\partial x} ((u(x, t))_N^{\lambda+1} - u^{\lambda+1}(x, t)) \right) \right\|_{L^2(0,1)} \\
 &\quad + \sigma \left\| \left[\frac{\partial^3}{\partial x^2 \partial t} ((u(x, t))_N - u(x, t)) \right] \right\|_{L^2(0,1)}.
 \end{aligned} \tag{25}$$

Using Lemma 2, equation (3), and $\|f_1 * f_2\|_{L^2(0,1)} \leq \|f_1\|_1 \|f_2\|_{L^2(0,1)}$, $f_1 \in L^1(0, 1)$, $f_2 \in L^2(0, 1)$, for $\|{}^C\mathfrak{D}_t^\mu [(u(x, t))_N - u(x, t)]\|_{L^2(0,1)}$, we get

$$\begin{aligned}
 \|{}^C\mathfrak{D}_t^\mu [(u(x, t))_N - u(x, t)]\|_{L^2(0,1)}^2 &= \|\mathfrak{E}^{1-\mu, -\gamma} ((u(x, t))_N - u(x, t))\|_{L^2(0,1)}^2 \\
 &= \left\| \mathfrak{E}^{1-\mu, -\gamma} \frac{d}{dt} ((u(x, t))_N - u(x, t)) \right\|_{L^2(0,1)}^2 = \left\| t^{-\mu} \mathbb{E}_{\rho, 1-\mu}^{-\gamma} (\omega t^\rho) * \frac{d}{dt} ((u(x, t))_N - u(x, t)) \right\|_{L^2(0,1)}^2 \\
 &\leq \|t^{-\mu} \mathbb{E}_{\rho, 1-\mu}^{-\gamma} (\omega t^\rho)\|_1 \times \left\| \frac{d}{dt} ((u(x, t))_N - u(x, t)) \right\|_{L^2(0,1)}^2,
 \end{aligned} \tag{26}$$

and by using the Lipschitz condition with constant $0 < k_1 < 1$ for derivative of the first order with respect to the variable t , we have

$$\begin{aligned}
 \|{}^C\mathfrak{D}_t^\mu [(u(x, t))_N - u(x, t)]\|_{L^2(0,1)}^2 &\leq k_1^2 (t^{1-\mu} \mathbb{E}_{\rho, 2-\mu}^{-\gamma} (\omega t^\rho))^2 \| (u(x, t))_N - u(x, t) \|_{L^2(0,1)}^2, \\
 \|{}^C\mathfrak{D}_t^\mu [(u(x, t))_N - u(x, t)]\|_{L^2(0,1)}^2 &\leq k_1 (t^{1-\mu} \mathbb{E}_{\rho, 2-\mu}^{-\gamma} (\omega t^\rho)) \| (u(x, t))_N - u(x, t) \|_{L^2(0,1)}.
 \end{aligned} \tag{27}$$

By using the Lipschitz condition with constant $0 < k_2 < 1$ for derivative of the first order with respect to the variable x and theorem assumptions, from equation (25), we obtain

$$\begin{aligned} \|E(u(x, t))\|_{L^2(0,1)} &\leq k_1 \left(t^{1-\mu} \mathbb{E}_{\rho, 2-\mu}^{-\gamma}(\omega t^\rho) \right) \| (u(x, t))_N - u(x, t) \|_{L^2(0,1)} + \frac{\theta}{\lambda+1} k_2 \xi \| (u(x, t))_N - u(x, t) \|_{L^2(0,1)} \\ &\quad + \sigma k_1 k_2^2 \| (u(x, t))_N - u(x, t) \|_{L^2(0,1)}, \end{aligned} \quad (28)$$

and according to Theorem 2 introduced in [58], we have

$$\begin{aligned} \|E(u(x, t))\|_{L^2(0,1)} &\leq \left[k_1 \left(t^{1-\mu} \mathbb{E}_{\rho, 2-\mu}^{-\gamma}(\omega t^\rho) \right) + \frac{\theta}{\lambda+1} k_2 \xi + \sigma k_1 k_2^2 \right] \| (u(x, t))_N - u(x, t) \|_{L^2(0,1)} \\ &\leq \left[k_1 \left(t^{1-\mu} \mathbb{E}_{\rho, 2-\mu}^{-\gamma}(\omega t^\rho) \right) + \frac{\theta}{\lambda+1} k_2 \xi + \sigma k_1 k_2^2 \right] c N^{-\sigma} 2^{-k\sigma} \| u^{(\sigma)} \|_{L^2(0,1)}, \\ \|E(u(x, t))\|_{L^2(0,1)} &\leq \left[k_1 \left(t^{1-\mu} \mathbb{E}_{\rho, 2-\mu}^{-\gamma}(\omega t^\rho) \right) + \frac{\theta}{\lambda+1} k_2 \xi + \sigma k_1 k_2^2 \right] c N^{2q-(1/2)-\sigma} 2^{k(q-\sigma)} \| u^{(\sigma)} \|_{L^2(0,1)}, \quad q \geq 1, \end{aligned} \quad (29)$$

where $u(x, t), (u(x, t))_N \in \mathfrak{H}^\sigma(0, 1), \sigma \geq 0, \quad \xi > 0,$ that $\mathfrak{H}^\sigma(0, 1), \sigma \geq 0,$ given [58]. $\square$

4. Numerical Results

In this section, we show the approximate solution obtained by the proposed method which is named the homotopy perturbation Sadik transform method with three examples to express its performance. In this section, we consider the absolute error as follows:

$$E(x, t) = |u(x, t) - u_n(x, t)|, \quad (30)$$

where $u(x, t)$ is the exact solution of equation (1) and $u_n(x, t)$ is the numerical solution of equation (1).

4.1. Example 1. We consider the following fractional EW equation with $\lambda = 1, \theta = 1, \sigma = 1$:

$$\begin{aligned} {}^C \mathfrak{D}_t^\mu u(x, t) u(x, t) + u(x, t) u_x(x, t) \\ - u_{xxt}(x, t) = 0, \quad 0 < \mu \leq 1, \\ u(x, 0) = 3 \sec h^2\left(\frac{x-15}{2}\right), \end{aligned} \quad (31)$$

when $\gamma = 0,$ and the exact solution is $u(x, t) = 3 \sec h^2(x - 15 - t/2)$. By applying the proposed method given in Section 3, we solve equation (31); then,

$$\begin{aligned} \sum_{n=0}^{\infty} \varrho^n u_n(x, t) &= 3 \sec h^2\left(\frac{x-15}{2}\right) + \mathcal{S}^{-1} \left[(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \right. \\ &\quad \left. \times \left[\mathcal{S} \left(\left(\sum_{n=0}^{\infty} \varrho^n u_n(x, t) \right)_{xxt} - \sum_{n=0}^{\infty} \varrho^n \mathbf{H}_n(u(x, t)) \right) \right]; t \right], \end{aligned} \quad (32)$$

where $\mathbf{H}_n(u(x, t))$ is the nonlinear uu_x . The polynomials $\mathbf{H}_n(u(x, t))$ are calculated as follows:

$$\begin{aligned} \mathbf{H}_0(u(x, t)) &= u_0(u_0)_x, \\ \mathbf{H}_1(u(x, t)) &= u_0(u_1)_x + u_1(u_0)_x, \\ &\vdots \end{aligned} \quad (33)$$

Comparing coefficients of same powers of p in equation (32), we obtain

$$\begin{aligned} \varrho^0: u_0(x, t) &= 3 \sec h^2\left(\frac{x-15}{2}\right), \\ \varrho^1: u_1(x, t) &= \mathcal{S}^{-1} \left[(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \right. \\ &\quad \left. \times [\mathcal{S}((u_0(x, t))_{xxt} - \mathbf{H}_0(u(x, t)))] \right]; t \\ &= 9 \sec h^4\left(\frac{x-15}{2}\right) \tanh\left(\frac{x-15}{2}\right) t^\mu \mathbb{E}_{\rho, 1+\mu}^\gamma(\omega t^\rho), \\ &\vdots \end{aligned} \quad (34)$$

FIGURE 1: Graphs of approximate solution and the exact solution and absolute error for different values of μ when $\mu = 0.5, 0.85, 0.99, 1$ and $\rho = \omega = \gamma = 1$ for example 1.

Then, the approximate solution of $u(x, t)$ is obtained as

$$\begin{aligned}
 u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\
 &= 3 \sec h^2\left(\frac{x-15}{2}\right) + 9 \sec h^4\left(\frac{x-15}{2}\right) \tanh\left(\frac{x-15}{2}\right) t^\mu \mathbb{E}_{\rho, 1+\mu}^\gamma(\omega t^\rho) + \dots
 \end{aligned}
 \tag{35}$$

FIGURE 2: Comparison of exact solution and approximate solution with homotopy perturbation Sadik transform method when $t = 0.0001$ and $\rho = \omega = \gamma = 1$ for example 1.

TABLE 1: Comparison between the absolute error in [2] and the absolute error for homotopy perturbation Sadik transform method with different values of μ for example 1.

x	$ u - u_n $ $\mu = 0.5$	$ u - u_n $ $\mu = 0.85$	$ u - u_n $ $\mu = 0.99$	Method [2]
10	$1.4006135487E - 09$	$1.4210717446E - 09$	$1.4412222851E - 09$	$7.242560880E - 06$
11	$1.4804436993E - 09$	$1.4994344786E - 09$	$1.5179577058E - 09$	$1.610108943E - 05$
12	$1.5534369379E - 09$	$1.5703105528E - 09$	$1.5865522230E - 09$	$2.246615682E - 05$
13	$1.16169787676E - 09$	$1.6310851118E - 09$	$1.6444028957E - 09$	$2.494373852E - 05$
14	$1.16685291313E - 09$	$1.6792699469E - 09$	$1.6890873673E - 09$	$1.482055499E - 04$
15	$1.17271757893E - 09$	$1.7298751108E - 09$	$1.7314989455E - 09$	$7.000000000E - 09$
16	$1.7315032733E - 09$	$1.7298837530E - 09$	$1.7271887190E - 09$	$1.482115835E - 04$
17	$1.7186088774E - 09$	$1.7127505267E - 09$	$1.7058693781E - 09$	$2.493848058E - 05$
18	$1.16891247436E - 09$	$1.6793110759E - 09$	$1.6685738938E - 09$	$2.247007139E - 05$
19	$1.6444545342E - 09$	$1.6311399784E - 09$	$1.6170367143E - 09$	$1.610299997E - 05$
20	$1.5866158664E - 09$	$1.5703768054E - 09$	$1.5535056371E - 09$	$7.243321962E - 06$

The comparison between the exact and the approximate solutions and the absolute error are expressed in Figure 1 for different values of μ when $\mu = 0.5, 0.85, 0.99, 1$. Also, in Figure 2, the exact and the numerical solutions at $t = 0.0001$ are given. In Table 1, a comparison between the absolute error in [2] and the absolute error for HPSTM with different values of μ is given.

4.2. Example 2. Consider the following MEW equation with $\lambda = 2, \theta = 3, \sigma = 1$:

$$\begin{aligned}
 {}^C \mathfrak{D}_t^\mu u(x, t) u(x, t) + 3u^2(x, t) u_x(x, t) \\
 - u_{xxt}(x, t) = 0, \quad 0 < \mu \leq 1, \\
 u(x, 0) = \frac{1}{4} \sec h(x - 30),
 \end{aligned} \tag{36}$$

for $\gamma = 0$, and the exact solution is $u(x, t) = 1/4 \sec h(x - 30 - t/4)$. Applying the HPSTM on equation (36), we obtain

FIGURE 3: Graphs of the approximation and exact solutions and absolute error for different values of μ when $\mu = 0.5, 0.85, 0.99, 1$ and $\rho = \omega = \gamma = 1$ for example 2.

$$\begin{aligned}
 \sum_{n=0}^{\infty} \varrho^n u_n(x, t) &= \frac{1}{4} \sec h(x - 30) + \mathcal{S}^{-1} \left[(v^{-\alpha\mu} (1 - \omega v^{-\rho\alpha})^{-\gamma}) \right. \\
 &\quad \left. \times \left[\mathcal{S} \left(\left(\sum_{n=0}^{\infty} \varrho^n u_n(x, t) \right)_{xxt} - \sum_{n=0}^{\infty} \varrho^n \mathbf{H}_n(u(x, t)) \right) \right]; t \right],
 \end{aligned} \tag{37}$$

FIGURE 4: Comparison of exact solution and approximate solution with homotopy perturbation Sadik transform method when $t = 0.0001$ and $\rho = \omega = \gamma = 1$ for example 2.

TABLE 2: Comparison between the absolute error in [2] and the absolute error for homotopy perturbation Sadik transform method with different values of μ for example 2.

x	$ u - u_n $ $\mu = 0.5$	$ u - u_n $ $\mu = 0.85$	$ u - u_n $ $\mu = 0.99$	Method [2]
25	$4.9233966126E - 09$	$4.8905216623E - 09$	$4.8523133049E - 09$	$8.419085488E - 07$
26	$4.7611171679E - 09$	$4.7088017083E - 09$	$4.6524848045E - 09$	$2.284567360E - 06$
27	$4.5293288600E - 09$	$4.4632329663E - 09$	$4.3946122324E - 09$	$6.130812328E - 06$
28	$4.2511941226E - 09$	$4.1770584430E - 09$	$4.1017168885E - 09$	$1.516464963E - 05$
29	$3.9485345529E - 09$	$3.8712015990E - 09$	$3.7936781419E - 09$	$2.113002231E - 05$
30	$3.6388500689E - 09$	$3.5618921913E - 09$	$3.4854390699E - 09$	$7.800000000E - 09$
31	$3.3345479504E - 09$	$3.2603221763E - 09$	$3.1870278987E - 09$	$2.113176846E - 05$
32	$3.04351478950E - 09$	$2.9734094032E - 09$	$2.9044644353E - 09$	$1.516821359E - 05$
33	$2.7701849765E - 09$	$2.7048973756E - 09$	$2.6408655090E - 09$	$6.132358065E - 06$
34	$2.5166006751E - 09$	$2.4563726541E - 09$	$2.3974113232E - 09$	$2.285133776E - 06$
35	$2.2832615675E - 09$	$2.2280532855E - 09$	$2.1740726737E - 09$	$8.421208912E - 07$

where the polynomials $\mathbf{H}_n(u(x, t))$ are represented as the nonlinear terms which are denoted as

$$\sum_{n=0}^{\infty} \mathbf{Q}^n \mathbf{H}_n(u(x, t)) = 3u^2 u_x. \quad (38)$$

The polynomials $\mathbf{H}_n(u(x, t))$ are calculated as follows:

$$\begin{aligned} \mathbf{H}_0(u(x, t)) &= 3u_0^2(u_0)_x, \\ \mathbf{H}_1(u(x, t)) &= 3u_0^2(u_1)_x + 6u_0 u_1(u_0)_x, \\ &\vdots \end{aligned} \quad (39)$$

Comparing coefficients of same powers of p in equation (37), we obtain

$$\begin{aligned} \mathbf{Q}^0: u_0(x, t) &= \frac{1}{4} \sec h(x - 30), \\ \mathbf{Q}^1: u_1(x, t) &= \mathcal{S}^{-1}[(v^{-\alpha\mu}(1 - \omega v^{-\rho\alpha})^{-\gamma}) \\ &\quad \times [\mathcal{S}((u_0(x, t))_{xxt} - \mathbf{H}_0(u(x, t)))]; t] \\ &= \frac{3}{64} \sec h^3(x - 30) \tanh(x - 30) t^\mu \mathbb{E}_{\rho, 1+\mu}^\gamma(\omega t^\rho), \\ &\vdots \end{aligned} \quad (40)$$

Finally, the approximate solution of $u(x, t)$ is given by

FIGURE 5: Graphs of the approximation and exact solutions and the absolute error for different values of μ when $\mu = 0.5, 0.85, 0.99, 1$ and $\rho = \omega = \gamma = 1$ for example 3.

FIGURE 6: Comparison of exact solution and approximate solution with homotopy perturbation and Sadik transform method when $t = 0.0001$ and $\rho = \omega = \gamma = 1$ for example 3.

TABLE 3: Comparison between the absolute error in [2] and the absolute error for homotopy perturbation Sadik transform method with different values of μ for example 3.

x	$ u - u_n $		$ u - u_n $	
	$\mu = 0.5$	$\mu = 0.85$	$\mu = 0.99$	Method [2]
0	4.5709477859E - 09	4.5330148328E - 09	4.4953966746E - 09	1.000000000E - 09
1	4.4210943150E - 09	4.3844049537E - 09	4.3480200672E - 09	2.563406968E - 05
2	4.2761536328E - 09	4.2406670942E - 09	4.2054750487E - 09	4.601380631E - 05
3	4.1359646818E - 09	4.1016415335E - 09	4.0676032238E - 09	6.668778565E - 05
4	4.0003716853E - 09	3.9671737878E - 09	3.9342513912E - 09	9.224742588E - 05
5	3.8692239749E - 09	3.8371144396E - 09	3.8052713736E - 09	1.244701205E - 04
6	3.7423758233E - 09	3.7113189714E - 09	3.6805198536E - 09	1.615598280E - 04
7	3.6196862832E - 09	3.5896476064E - 09	3.5598582149E - 09	1.923074240E - 04
8	3.5010190312E - 09	3.4719651533E - 09	3.4431523893E - 09	1.781977647E - 04
9	3.3862422172E - 09	3.3581408576E - 09	3.3302727086E - 09	3.522674800E - 06
10	3.2752283194E - 09	3.2480482574E - 09	3.2210937625E - 09	6.615336660E - 04

$$\begin{aligned}
 u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\
 &= \frac{1}{4} \sec h(x-30) + \frac{3}{64} \sec h^3(x-30) \tanh \\
 &\quad (x-30) t^{\mu} E_{\rho, 1+\mu}^{\gamma}(\omega t^{\rho}) + \dots
 \end{aligned} \quad (41)$$

The comparison between exact solution and the approximate solution and absolute error are expressed in Figure 3 for different values of μ when $\mu = 0.5, 0.85, 0.99, 1$. Figure 4 shows the comparison of exact solution with homotopy perturbation Sadik transform method solution at $t = 0.0001$. In Table 2, a comparison between the absolute error in [2] and the absolute error for HPSTM with different values of μ is given.

In the example 2, we study the solution of a variant of fractional modified equal width equation (VMEW) which is obtained by the proposed method.

4.3. Example 3. Consider the following VMEW equation:

$$\begin{aligned}
 {}^C \mathfrak{D}_t^{\mu} u(x, t) u(x, t) + \frac{12}{7} u_x^6(x, t) \\
 - \frac{3}{7} u_{xxt}^6(x, t) = 0, \quad 0 < \mu \leq 1, \\
 u(x, 0) = \cosh^{5/2}\left(\frac{5x}{6}\right),
 \end{aligned} \quad (42)$$

and the exact solution is $u(x, t) = \cosh^{5/2}(5(x-t)/6)$ when $\gamma = 0$. Applying the homotopy perturbation and Sadik transform method on equation (42), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \varrho^n u_n(x, t) &= \cosh^{5/2}\left(\frac{5x}{6}\right) + \mathcal{S}^{-1}[(v^{-\alpha\mu}(1 - \omega v^{-\rho\alpha})^{-\gamma}) \\
 &\quad \times \left[\mathcal{S} \left(\sum_{n=0}^{\infty} \varrho^n \mathbf{H}_n(u(x, t)) \right) \right]; t],
 \end{aligned} \quad (43)$$

where $\mathbf{H}_n(u(x, t))$ are obtained as

$$\sum_{n=0}^{\infty} \varrho^n \mathbf{H}_n(u(x, t)) = \frac{3}{7} u_{xxt}^6 - \frac{12}{7} u_x^6, \quad (44)$$

and $\mathbf{H}_0(u(x, t)), \dots, \mathbf{H}_n(u(x, t))$ are calculated as follows:

$$\mathbf{H}_0(u(x, t)) = \frac{3}{7} (u_0^6)_{xxt} - \frac{12}{7} (u_0^6)_x, \quad (45)$$

$\vdots$

Comparing coefficients of same powers of p in equation (43), we obtain

$$\begin{aligned}
 \varrho^0: u_0(x, t) &= \cosh^{5/2}\left(\frac{5x}{6}\right), \\
 \varrho^1: u_1(x, t) &= \mathcal{S}^{-1}[(v^{-\alpha\mu}(1 - \omega v^{-\rho\alpha})^{-\gamma}) \\
 &\quad \times [\mathcal{S}(\mathbf{H}_0(u(x, t)))]; t] \\
 &= -\frac{24}{7} \cosh^{7/5}\left(\frac{5x}{6}\right) \sinh\left(\frac{5x}{6}\right) t^{\mu} E_{\rho, 1+\mu}^{\gamma}(\omega t^{\rho}), \\
 &\quad \vdots
 \end{aligned} \quad (46)$$

So, the approximate solution of $u(x, t)$ is calculated as

$$\begin{aligned}
 u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\
 &= \cosh^{5/2}\left(\frac{5x}{6}\right) - \frac{24}{7} \cosh^{7/5}\left(\frac{5x}{6}\right) \sinh\left(\frac{5x}{6}\right) t^{\mu} E_{\rho, 1+\mu}^{\gamma}(\omega t^{\rho}) + \dots
 \end{aligned} \quad (47)$$

Comparison between the exact and the approximate solutions and the absolute error are shown in Figure 5 for different values of μ when $\mu = 0.5, 0.85, 0.99, 1$. Also, in Figure 6, comparison of the exact solution and the approximate solution at $t = 0.0001$ is illustrated. In Table 3, a comparison between the absolute error in [2] and the absolute error for homotopy perturbation Sadik transform method with different values of μ is given.

5. Conclusion

This paper describes a numerical method based on the homotopy perturbation Sadik transform method for solving the equations such as EW, MEW, and VMEW. The fractional derivatives are used in this paper in the Caputo–Prabhakar sense. This paper demonstrates that the HPSTM is sufficient, easy, and suitable for various other nonlinear models. It can be seen that as x increases, it leads to variation in $u(x, t)$ for waves in plasma at specific value of t . All tables and figures show that the proposed method produces numerical solutions with more accuracy. Comparison between the obtained absolute errors by the suggested method and method presented in [2] is demonstrated. Discussion on the convergence and error analysis of the proposed method is presented. Three numerical examples are given to show the applicability of the suggested method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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