Hindawi
International Journal of Differential Equations
Volume 2022, Article ID 2406736, 7 pages
https://doi.org/10.1155/2022/2406736

Research Article

Oscillation of Fourth-Order Nonlinear Homogeneous Neutral Difference Equation

G. Sumitha,[1] **R. Kodeeswaran**,[1] **S. Noeiaghdam**,[2,3] **S. Balamuralitharan**,[4] **and V. Govindan**[5]

[1]*Department of Mathematics, Kandaswami Kandar's College, Paramathi Velur – 638 182, Namakkal, Tamilnadu, India*
[2]*Industrial Mathematics Laboratory, Baikal School of BRICS, Irkutsk National Research Technical University, Irkutsk 664074, Russia*
[3]*Department of Applied Mathematics and Programming, South Ural State University, Lenin Prospect 76, Chelyabinsk 454080, Russia*
[4]*Department of Mathematics, Bharath Institute of Higher Education and Research, Selaiyur, Chennai 600073, Tamilnadu, India*
[5]*Department of Mathematics, DMI St John The Baptist University Central Africa, Mangochi-409, Malawi*

Correspondence should be addressed to V. Govindan; govindoviya@gmail.com

Received 31 January 2022; Accepted 7 March 2022; Published 30 March 2022

Academic Editor: Elena Kaikina

In this paper, we establish the solution of the fourth-order nonlinear homogeneous neutral functional difference equation. Moreover, we study the new oscillation criteria have been established which generalize some of the existing results of the fourth-order nonlinear homogeneous neutral functional difference equation in the literature. Likewise, a few models are given to represent the significance of the primary outcomes.

1. Introduction

The investigation of the conduct of solutions of functional difference equations is a significant space of examination and is quickly developing because of the advancement of time scales and the time-scale analytics (see, e.g., [1, 2]). Most papers on higher-order nonlinear neutral difference equations manage the presence of positive solutions and the asymptotic conduct of solutions. We refer the per user to a portion of the works [3–9] and the references referred to in that.

In [10], Migda investigated the asymptotic properties of nonoscillatory solutions of neutral difference equation of the form

$$\Delta^m\left(x_\varrho + \mathring{p}_\varrho x_{\varrho-\xi}\right) + f\left(\varrho, x_{\varsigma(\varrho)}\right) = h_\varrho. \tag{1}$$

The papers [9, 10] are tantamount. Be that as it may, more accentuation might be given to [9], which manages the oscillatory, nonoscillatory, and asymptotic characters. Some oscillation criteria have been set up by applying the discrete Taylor series [11]. The inspiration of the current work has come from two bearings [9, 12] and the second is expected to [13].

In 2008, Tripathy [9] investigated oscillatory and asymptotic behaviour of solutions of a class of fourth-order nonlinear neutral difference equations of the form

$$\Delta^2\left(r(n)\Delta^2 t(y(n) + p(n)y(n-m))\right) + q(n)G(y(n-k)) = 0. \tag{2}$$

Also,

$$\Delta^2\left(r(n)\Delta^2 t(y(n) + p(n)y(n-m))\right) + q(n)G(y(n-k)) = f(n). \tag{3}$$

Under the condition $\sum_{n=0}^{\infty} n/r(n) < \infty$, for different ranges of $p(n)$ and $q(n) \geq 0$. Also, no super linearity or sublinearity conditions are imposed on $G \in C(R, R)$. It is interesting to observe that the nature of the function $r(n)$ influences the behaviour of solutions of (2) or (3).

In 2013, Tripathy [12] discussed oscillatory behaviour of solutions of a class of fourth-order neutral functional differences (2) and (3) under the conditions $\sum_{n=0}^{\infty} n/r(n) = \infty$ and $r(n) > 0$, $q(n) > 0$.

Parhi and Tripathy [13] studied oscillation of a class of nonlinear neutral difference equations of higher order and the behaviour of its solutions is studied separately.

In this present work, we study the oscillation behaviour of the fourth-order nonlinear homogeneous neutral functional difference equation.

2. Oscillation Behaviour of Neutral Difference Equation

In this section, we establish the solution of the fourth-order homogeneous neutral functional difference equation of the form

$$\Delta^2\left(\mathring{r}(\varrho)\Delta^2 t\left(\mathring{y}(\varrho) + \mathring{p}(\varrho)\mathring{y}(\varrho - \xi)\right)\right) + \mathring{q}(\varrho)\mathscr{G}(\mathring{y}(\varrho - \varsigma)) = 0, \tag{4}$$

where

(1) Δ is a difference operator

(2) $\Delta\mathring{y}(\varrho) = \mathring{y}(\varrho + 1) - \mathring{y}(\varrho)$

(3) $\mathring{r}(\varrho), \mathring{p}(\varrho), \mathring{q}(\varrho)$, and $\mathring{y}(\varrho)$ are real valued functions defined on $\mathscr{N}(\varrho_0) = \{\varrho_0, \varrho_0 + 1, \ldots\}$, $\varrho \geq \varrho_0 \geq 0$

(4) $\mathscr{G} \in \mathscr{C}(\mathscr{R}, \mathscr{R})$ is nondecreasing

(5) ξ, $\varsigma > 0$ are constants with

$$\sum_{\varrho=0}^{\infty} \frac{\varrho}{\mathring{r}(\varrho)} < \infty. \tag{5}$$

A solution $\mathring{y}(\varrho)$ of (4) is oscillatory if $\mathring{y}(\varrho)\mathring{y}(\varrho + 1) \neq 0$ for every integer $\varrho > \mathscr{N} > 0$. Otherwise, it is nonoscillatory. The NFDE (4) is oscillatory, if all its solutions are oscillatory.

For the oscillation of (4), we define the operators

$$\left.\begin{aligned} \mathscr{L}_1 \mathring{u}(\varrho) &= \Delta\mathscr{L}_0\mathring{u}(\varrho) = \Delta\mathring{u}(\varrho) \\ \mathscr{L}_2\mathring{u}(\varrho) &= \mathring{r}(\varrho)\Delta\mathscr{L}_1\mathring{u}(\varrho) \\ \mathscr{L}_3\mathring{u}(\varrho) &= \Delta\mathscr{L}_2\mathring{u}(\varrho) \\ \mathscr{L}_4\mathring{u}(\varrho) &= \Delta\mathscr{L}_3\mathring{u}(\varrho) \end{aligned}\right]. \tag{6}$$

Also, the notations

$$\left.\begin{aligned} \mathscr{D}[\mathring{k}, \varepsilon] &= \sum_{t=\varepsilon}^{\mathring{k}-2} \frac{(\mathring{k} - \epsilon - 1)(\epsilon - \varepsilon)}{\mathring{r}(\epsilon)} \\ \mathscr{E}[\mathring{k}, \varepsilon] &= \sum_{t=\varepsilon}^{\mathring{k}-1} \frac{(\mathring{k} - \epsilon - 1)(\epsilon + 1 - \varepsilon)}{\mathring{r}(\epsilon)} \\ \mathscr{F}[\mathring{k}, \varepsilon] &= \sum_{t=\varepsilon}^{\mathring{k}-1} \frac{(\mathring{k} - \epsilon - 1)}{\mathring{r}(\epsilon)} \\ \mathscr{M}[\mathring{k}, \varepsilon] &= \sum_{t=\varepsilon}^{\mathring{k}-1} \frac{(\epsilon + 1 - \varepsilon)(\epsilon - \varepsilon)}{\mathring{r}(\epsilon)}. \end{aligned}\right]. \tag{7}$$

Lemma 1. *Let (5) hold and $\mathring{u}$ be a real valued function with $\mathscr{L}_4\mathring{u}(\varrho) \leq 0$ for large ϱ. If $\mathring{u}(\varrho) > 0$, then one of these (8) to (9) holds and if $\mathring{u}(\varrho) < 0$, then one of these (9) to (13) holds, where*

$$\mathscr{L}_1\mathring{u}(\varrho) > 0,\ \mathscr{L}_2\mathring{u}(\varrho) > 0, \text{ and } \mathscr{L}_3\mathring{u}(\varrho) > 0, \tag{8}$$

$$\mathscr{L}_1\mathring{u}(\varrho) > 0,\ \mathscr{L}_2\mathring{u}(\varrho) < 0, \text{ and } \mathscr{L}_3\mathring{u}(\varrho) > 0, \tag{9}$$

$$\mathscr{L}_1\mathring{u}(\varrho) > 0,\ \mathscr{L}_2\mathring{u}(\varrho) < 0, \text{ and } \mathscr{L}_3\mathring{u}(\varrho) < 0, \tag{10}$$

$$\mathscr{L}_1\mathring{u}(\varrho) < 0,\ \mathscr{L}_2\mathring{u}(\varrho) > 0, \text{ and } \mathscr{L}_3\mathring{u}(\varrho) > 0, \tag{11}$$

$$\mathscr{L}_1\mathring{u}(\varrho) < 0,\ \mathscr{L}_2\mathring{u}(\varrho) < 0, \text{ and } \mathscr{L}_3\mathring{u}(\varrho) > 0, \tag{12}$$

$$\mathscr{L}_1\mathring{u}(\varrho) < 0,\ \mathscr{L}_2\mathring{u}(\varrho) < 0, \text{ and } \mathscr{L}_3\mathring{u}(\varrho) < 0. \tag{13}$$

Lemma 2. *Let $\mathring{u}(\varrho) > 0$ and Lemma 1 hold. Then, $\mathscr{C}_1\mathscr{R}(\varrho) \leq \mathring{u}(\varrho) \leq \varrho\mathscr{C}_2$ for $\mathscr{C}_1 > 0$&$\mathscr{C}_2 > 0$, where*

$$\mathscr{R}(\varrho) = \sum_{s=\varrho}^{\infty} \frac{s - \varrho}{\mathring{r}(s)}, \quad \text{for large } \varrho. \tag{14}$$

Theorem 1. *Let $\varsigma \geq 2\xi 0 \leq \mathring{p}(\varrho) \leq d < \infty$. If (5) and*

$$\frac{\mathscr{G}(\mathring{u})}{\mathring{u}} \geq \varphi > 0,\ \mathring{u} \neq 0 \in \mathscr{R}, \tag{15}$$

$$\begin{gathered} \mathscr{G}(\mathring{u}\mathring{v}) \geq \mathscr{G}(\mathring{u})\mathscr{G}(\mathring{v}), \\ \mathscr{G}(-\mathring{u}) = -\mathscr{G}(\mathring{u}),\ \mathring{u}, \mathring{v} > 0 \in \mathscr{R}, \end{gathered} \tag{16}$$

$$\mathscr{G}(\mathring{u}) + \mathscr{G}(\mathring{v}) \geq \lambda\mathscr{G}(\mathring{u} + \mathring{v}),\ \lambda > 0, \tag{17}$$

$$\mathscr{Q}(\varrho) = \min\{\mathring{q}(\varrho), \mathring{q}(\varrho - \xi)\},\ \varrho \geq \xi, \tag{18}$$

$$\limsup_{\mathring{k} \longrightarrow \infty} \sum_{\mathring{j}=\mathring{k}-\xi}^{\mathring{k}} \mathscr{Q}(\mathring{j})\mathscr{G}(\mathscr{D}[\mathring{j} - \varsigma, \mathring{k} - \varsigma]) > \frac{1 + \mathscr{G}(d)}{\lambda\varphi}, \tag{19}$$

$$\limsup_{\mathring{k} \longrightarrow \infty} \sum_{\mathring{j}=\mathring{k}-\xi}^{\mathring{k}} \mathscr{Q}(\mathring{j})\mathscr{G}(\mathrm{E}[\mathring{j} - \varsigma, \mathring{k}\varsigma]) > \frac{1 + \mathscr{G}(d)}{\lambda\varphi}, \tag{20}$$

$$\limsup_{\varepsilon \longrightarrow \infty} \sum_{\mathring{j}=\varepsilon-\varsigma-2}^{\varepsilon-\varsigma-1} \mathscr{Q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j} - \varsigma, \mathring{k} - \varsigma]) > \frac{1 + \mathscr{G}(d)}{\lambda\varphi}, \tag{21}$$

$$\limsup_{\varepsilon \longrightarrow \infty} \sum_{\mathring{k}=\mathring{j}+\xi-\varsigma-2}^{\mathring{j}+\xi-\varsigma} \mathscr{Q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j} - \varsigma, \mathring{k} - \varsigma]) > \frac{1 + \mathscr{G}(d)}{\lambda\varphi}, \tag{22}$$

hold, then (4) is oscillatory.

Proof. Let $\mathring{y}(\varrho)$ be a nonoscillatory solution of (4) and $\mathring{y}(\varrho) > 0$ for $\varrho \geq \varrho_0 > \mu$.

If we set

$$z(\varrho) = \mathring{y}(\varrho) + \mathring{p}(\varrho)\mathring{y}(\varrho - \xi), \tag{23}$$

then (4) becomes

$$\mathscr{L}_4 z(\varrho) = -\mathring{q}(\varrho)\mathscr{G}(\mathring{y}(\varrho - \varsigma)) < 0. \tag{24}$$

Hence, we find $\varrho_1 \geq \varrho_0$ such that $\mathscr{L}_1 z(\varrho)$, $1 = 1, 2, 3$ are eventually of one sign on $[\varrho_1, \infty)$.

Consider the potential cases 2.3 to 2.6 of Lemma 1.

Case 2.5. For $\mathring{k} - 1 \geq \varepsilon \geq \varrho_1$, it follows from the Taylor series that

$$\begin{aligned} -z(\mathring{k}) &= -z(\varepsilon) - (\mathring{k} - \varepsilon)\Delta z(\mathring{k}) + \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon)\Delta^2 z(\epsilon) \\ &\leq \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon)\Delta^2 z(\epsilon). \end{aligned} \tag{25}$$

Also,

$$\mathscr{L}_2 z(\epsilon) - \mathscr{L}_2 z(\varepsilon) = \sum_{s=\varepsilon}^{\epsilon-1} \mathscr{L}_3 z(s) \leq (\epsilon - \varepsilon)\mathscr{L}_3 z(\varepsilon), \tag{26}$$

which implies

$$\mathscr{L}_2 z(\epsilon) \leq (\epsilon - \varepsilon)\mathscr{L}_3 z(\varepsilon). \tag{27}$$

That is,

$$\Delta^2 z(\epsilon) \leq \frac{(\epsilon - \varepsilon)\mathscr{L}_3 z(\varepsilon)}{\mathring{r}(\epsilon)}. \tag{28}$$

Consequently,

$$z(\mathring{k}) \geq -\sum_{\epsilon=\varepsilon}^{\mathring{k}-\epsilon} (\epsilon + 1 - \varepsilon)\frac{(\epsilon - \varepsilon)}{\mathring{r}(\epsilon)}\mathscr{L}_3 z(\varepsilon) = -\mathscr{L}_3 z(\varepsilon)\mathscr{M}[\mathring{k}, \varepsilon]. \tag{29}$$

For $\mathring{j} - \varsigma \geq \mathring{k} - \varsigma + 2 \geq \varrho_1 + 2$, the above disparity can be composed as

$$z(\mathring{j} - \varsigma) \geq -\mathscr{L}_3 z(\varepsilon - \varsigma)\mathscr{M}[\mathring{j} - \varsigma, \mathring{k} - \varsigma]. \tag{30}$$

Using (4), it is not difficult to check that

$$\mathscr{L}_4 z(\varrho) + \mathring{q}(\varrho)\mathscr{G}(\mathring{y}(\varrho - \varsigma)) + \mathscr{G}(d)\mathscr{L}_4 z(\varrho - \xi) + \mathscr{G}(d)\mathring{q}(\varrho - \xi)\mathscr{G}(\mathring{y}(\varrho - \varsigma - \xi)) = 0. \tag{31}$$

Expected to (16), (17) and (31) give

$$\begin{aligned} 0 &\geq \mathscr{L}_4 z(\mathring{j}) + \mathscr{G}(d)\mathscr{L}_4 z(\mathring{j} - \xi) + \lambda \mathscr{Q}(\mathring{j})\mathscr{G}(z(\mathring{j} - \varsigma)) \\ &\geq \mathscr{L}_4 z(\mathring{j}) + \mathscr{G}(d)\mathscr{L}_4 z(\mathring{j} - \xi) + \lambda \mathscr{Q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j} - \varsigma, \mathring{k} - \varsigma])\mathscr{G}(-\mathscr{L}_3 z(\varepsilon - \varsigma)). \end{aligned} \tag{32}$$

That is,

$$\begin{aligned} &\lambda \sum_{\mathring{j}=\varepsilon-\varsigma-2}^{\varepsilon-\varsigma-1} \mathscr{Q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j} - \varsigma, \mathring{k} - \varsigma])\mathscr{G}(-\mathscr{L}_3 z(\varepsilon - \varsigma)) \\ &\leq -\sum_{\mathring{j}=\varepsilon-\varsigma-2}^{\varepsilon-\varsigma-1} \left(\mathscr{L}_4 z(\mathring{j}) + \mathscr{G}(d)\mathscr{L}_4 z(\mathring{j} - \xi)\right) \\ &\leq -\mathscr{L}_3 z(\varepsilon - \varsigma) - \mathscr{G}(d)\mathscr{L}_3 z(\varepsilon - \varsigma - \xi) \\ &\leq -(1 + \mathscr{G}(d))\mathscr{L}_3 z(\varepsilon - \varsigma) \end{aligned} \tag{33}$$

As a result,

$$\begin{aligned} &\sum_{\mathring{j}=\varepsilon-\varsigma-2}^{\varepsilon-\varsigma-1} \mathscr{Q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j} - \varsigma, \mathring{k} - \varsigma]) \\ &\leq \frac{1 + \mathscr{G}(d)}{\lambda} \frac{-\mathscr{L}_3 z(\varepsilon - \varsigma)}{\mathscr{G}(-\mathscr{L}_3 z(\varepsilon - \varsigma))} \\ &\leq \frac{1 + \mathscr{G}(d)}{\lambda\varphi} \end{aligned}, \tag{34}$$

which is logical inconsistency to (21) due to (15).

Case 2.6. For $\mathring{k} - 1 \geq \varepsilon \geq \varrho_1$, it follows from (25) that

$$z(\varepsilon) = z(\mathring{k}) - (\mathring{k} - \varepsilon)\Delta z(\mathring{k}) + \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon)\Delta^2 z(\epsilon) \geq \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon)\Delta^2 z(\epsilon). \tag{35}$$

Since

$$\left.\mathscr{L}_2 z(\epsilon) - \mathscr{L}_2 z(\varepsilon) = \sum_{s=\varepsilon}^{\epsilon-1} \mathscr{L}_3 z(s) \geq (\epsilon-\varepsilon)\mathscr{L}_3 z(\epsilon-1),\right] \tag{36}$$

we have

$$\mathscr{L}_2 z(\epsilon) \geq (\epsilon-\varepsilon)\mathscr{L}_3 z(\epsilon-1). \tag{37}$$

That is,

$$\Delta^2 z(\epsilon) \leq \frac{(\epsilon-\varepsilon)\mathscr{L}_3 z(\varepsilon)}{\mathring{r}(\epsilon)}. \tag{38}$$

Consequently,

$$\left. z(\varepsilon) \geq \sum_{\epsilon=\varepsilon}^{\mathring{k}-\varepsilon} (\epsilon+1-\varepsilon)\frac{(\epsilon-\varepsilon)}{\mathring{r}(\epsilon)}\mathscr{L}_3 z(\epsilon-1) = \mathscr{L}_3 z(\mathring{k}-2)\mathscr{M}[\mathring{k},\varepsilon]\right]. \tag{39}$$

For $\mathring{j}-\varsigma \geq \mathring{k}-\varsigma+2 \geq \varrho_1+2$, the above disparity can be composed as

$$z(\mathring{k}-\varsigma) \geq \mathscr{L}_3 z(\mathring{j}-\varsigma-2)\mathscr{M}[\mathring{j}-\varsigma,\mathring{k}-\varsigma]. \tag{40}$$

Applying (16) and (17) in (31) yields that

$$0 \geq \mathscr{L}_4 z(\mathring{k}) + \mathscr{G}(d)\mathscr{L}_4 z(\mathring{k}-\xi) + \lambda\mathscr{Q}(\mathring{k})\mathscr{G}(z(\mathring{k}-\varsigma)). \tag{41}$$

Expected to (40), the above equation becomes

$$\left.\begin{aligned}
&\lambda \sum_{\mathring{k}=\mathring{j}+\xi-\varsigma-2}^{\mathring{j}+\xi-\varsigma} \mathscr{Q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j}-\varsigma,\mathring{k}-\varsigma])\mathscr{G}\left(-\mathscr{L}_3 z(\mathring{j}-\varsigma-2)\right)\\
&\leq -\sum_{\mathring{k}=\mathring{j}+\xi-\varsigma-2}^{\mathring{j}+\xi-\varsigma}\left(\mathscr{L}_4 z(\mathring{k}) + \mathscr{G}(d)\mathscr{L}_4 z(\mathring{k}-\xi)\right)\\
&\leq \mathscr{L}_3 z(\mathring{j}+\xi+\varsigma-2) + \mathscr{G}(d)\mathscr{L}_3 z(\mathring{j}-\varsigma-2)\\
&\leq (1+\mathscr{G}(d))\mathscr{L}_3 z(\mathring{j}-\varsigma-2)
\end{aligned}\right]. \tag{42}$$

Therefore,

$$\left.\begin{aligned}
&\sum_{\mathring{k}=\mathring{j}+\xi-\varsigma-2}^{\mathring{j}+\xi-\varsigma} \mathscr{Q}(\mathring{k})\mathscr{G}(\mathscr{M}[\mathring{j}-\varsigma,\mathring{k}-\varsigma])\\
&\leq \frac{1+\mathscr{G}(d)}{\lambda}\frac{\mathscr{L}_3 z(\mathring{j}-\varsigma-2)}{\mathscr{G}\left(\mathscr{L}_3 z(\mathring{j}-\varsigma-2)\right)}\\
&\leq \frac{1+\mathscr{G}(d)}{\lambda\varphi},
\end{aligned}\right], \tag{43}$$

which is logical inconsistency to (22) due to (15).

Cases 2.3 and 2.4 can be managed also as the above cases.

Finally, $\mathring{y}(\varrho) < 0$ for $\varrho \geq \varrho_0$. Using (16) and $x(\varrho) = -\mathring{y}(\varrho)$ in (4), we obtain $x(\varrho) > 0$ and

$$\Delta^2\left(\mathring{r}(\varrho)\Delta^2 t(x(\varrho) + \mathring{p}(\varrho)x(\varrho-\xi))\right) + \mathring{q}(\varrho)\mathscr{G}(x(\varrho-\varsigma)) = 0. \tag{44}$$

Proceeding as above, we see that every solution of (44) oscillates.

This completes the proof of theorem. □

Theorem 2. *Let* $-1 \leq \mathring{p}(\varrho) \leq 0$ *and* $\varsigma \geq 2\xi$. *If (5) to (16) and*

$$\limsup_{\mathring{k}\longrightarrow\infty} \sum_{\mathring{j}=\mathring{k}-\xi}^{\mathring{k}} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{D}[\mathring{j}-\varsigma,\mathring{k}-\varsigma]) > \frac{1}{\varphi}, \tag{45}$$

$$\limsup_{\mathring{k}\longrightarrow\infty} \sum_{\mathring{j}=\mathring{k}-\xi}^{\mathring{k}} \mathring{q}(\mathring{j})\mathscr{G}(\mathrm{E}[\mathring{j}-\varsigma,\mathring{k}-\varsigma]) > \frac{1}{\varphi}, \tag{46}$$

$$\limsup_{\varepsilon\longrightarrow\infty} \sum_{\mathring{j}=\varepsilon-\varsigma-2}^{\varepsilon-\varsigma-1} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j}-\varsigma,\mathring{k}-\varsigma]) > \frac{1}{\varphi}, \tag{47}$$

$$\limsup_{\mathring{j}\longrightarrow\infty} \sum_{\mathring{k}=\mathring{j}+\xi-\varsigma-2}^{\mathring{j}+\xi-\varsigma} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j}-\varsigma,\mathring{k}-\varsigma]) > \frac{1}{\varphi}, \tag{48}$$

$$\limsup_{\mathring{k}\longrightarrow\infty} \sum_{\mathring{j}=\mathring{k}+\xi-\varsigma}^{\mathring{k}+\xi-\varsigma} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{F}[\mathring{k},+\xi-\varsigma,\mathring{j}+\xi-\varsigma]) > \frac{1}{\varphi}, \tag{49}$$

$$\limsup_{l\longrightarrow\infty} \sum_{\mathring{j}=\varepsilon+\xi-\varsigma}^{\varepsilon+\xi-\varsigma+1} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{k},+\xi-\varsigma,\mathring{j}+\xi-\varsigma]) > \frac{1}{\varphi}, \tag{50}$$

$$\limsup_{\mathring{k}\longrightarrow\infty} \sum_{\mathring{j}=\mathring{k}+\xi-\varsigma-2}^{\mathring{k}} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{M}[\mathring{j}+\xi-\varsigma,\mathring{k}+\xi-\varsigma]) > \frac{1}{\varphi}, \tag{51}$$

hold, then every solution of (4) oscillates.

Proof. Let the opposition that $\mathring{y}(\varrho)$ is a nonoscillatory solution of (4) and $\mathring{y}(\varrho) > 0$ for $\varrho \geq \varrho_0 > \mu$.

The case $\mathring{y}(\varrho) < 0$ for $\varrho \geq \varrho_0 > \mu$ can be dealt with similarly.

Setting $z(\varrho)$ by (23), we get (24).

Consequently, we find $\varrho_1 > \varrho_0$ such that $z(\varrho)$ and $\mathscr{L}_i z(\varrho)$, $i = 1, 2, 3$ are eventually of one sign on $[\varrho_1, \infty)$.

Let $z(\varrho) > 0$. Then, $z(\varrho) \leq \mathring{y}(\varrho)$ for $\varrho \geq \varrho_2 > \varrho_1$ and (24) becomes

$$\mathscr{L}_2 z(\varrho) + \mathring{q}(\varrho)\mathscr{G}(z(\varrho-\varsigma)) \leq 0. \tag{52}$$

Applying Lemma 1 to (52) and Theorem 1, we get contradictions to (45) to (48) expected to $\varsigma > 2\xi > \xi$.

Next, $z(\varrho) < 0$ for $\varrho \geq \varrho_1$ and $z(\varrho) \geq -\mathring{y}(\varrho-\xi)$ for $\varrho \geq \varrho_2 \geq$ implies $\mathring{y}(\varrho-\varsigma) \geq -z(\varrho+\xi-\varsigma)$.

By Lemma 1, the cases of 2.4 to 2.8 hold for.

Case 2.4. Since

$$-\mathscr{L}_2 z(\epsilon) \geq \mathscr{L}_2 z(\mathring{k}-1) - \mathscr{L}_2 z(\epsilon) = \sum_{s=l}^{\mathring{k}-2} \mathscr{L}_3 z(s) \geq (\mathring{k}-\epsilon-1)\mathscr{L}_3 z(\mathring{k}-2).\Big], \tag{53}$$

it follows that

$$-\Delta^2 z(\epsilon) \geq \frac{\mathring{k}-\epsilon-1}{\mathring{r}(\epsilon)} \mathscr{L}_3 z(\mathring{k}-2). \tag{54}$$

For $\mathring{k} \geq \epsilon + 2 > \varrho_1$, summing the above inequality from $\epsilon = \varepsilon$ to $\mathring{k}-1$, we have

$$\Delta z(\varepsilon) \geq \mathscr{L}_3 z(\mathring{k}-2) \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} \frac{\mathring{k}-\epsilon-1}{\mathring{r}(\epsilon)}. \tag{55}$$

That is,

$$z(\varepsilon+1) - z(\varepsilon) \geq \mathscr{L}_3 z(\mathring{k}-2) \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} \frac{\mathring{k}-\epsilon-1}{\mathring{r}(\epsilon)}, \tag{56}$$

which implies

$$-z(\varepsilon) \geq \mathscr{L}_3 z(\mathring{k}-2) \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} \frac{\mathring{k}-\epsilon-1}{\mathring{r}(\epsilon)} = \mathscr{L}_3 z(\mathring{k}-2)\mathscr{F}[\mathring{k},\varepsilon] \geq \mathscr{L}_3 z(\mathring{k})\mathscr{F}[\mathring{k},\varepsilon], \tag{57}$$

for $\mathring{k} \geq \epsilon + 2 > \varrho_2$.

Therefore, for $\mathring{k} + \xi - \varsigma \geq \mathring{j} + \xi - \varsigma \geq \epsilon + \xi - \varsigma + 2 > \varrho_2$,

$$-z(\mathring{j}+\xi-\varsigma) \geq \mathscr{L}_3 z(\mathring{k}+\xi-\varsigma)\mathscr{F}[\mathring{k}, +\xi-\varsigma, \mathring{j}+\xi-\varsigma]. \tag{58}$$

Since (4) can be viewed as

$$\mathscr{L}_4 z(\mathring{j}) + \mathring{q}(\mathring{j})\mathscr{G}(-z(\mathring{j}+\xi-\varsigma)) \leq 0, \tag{59}$$

using (58), (59), and (16) yields

$$\mathscr{L}_4 z(\mathring{j}) + \mathring{q}(\mathring{j})\mathscr{G}\big(\mathscr{L}_3 z(\mathring{k}+\xi-\varsigma)\big)\mathscr{G}(\mathscr{F}[\mathring{k}, +\xi-\varsigma, \mathring{j}+\xi-\varsigma]) \leq 0. \tag{60}$$

Summing the last inequality form $\mathring{j} = \mathring{k}+\xi-\varsigma$ to $\mathring{k}+\varsigma-\xi$, we get

$$\mathscr{G}\big(\mathscr{L}_3 z(\mathring{k}+\xi-\varsigma)\big) \sum_{\mathring{j}=\mathring{k}+\xi-\varsigma}^{\mathring{k}+\varsigma-\xi} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{F}[\mathring{k}, +\xi-\varsigma, \mathring{j}+\xi-\varsigma]) \leq \mathscr{L}_3 z(\mathring{k}+\xi-\varsigma). \tag{61}$$

Hence,

$$\limsup_{\mathring{k}\longrightarrow\infty} \sum_{\mathring{j}=\mathring{k}+\xi-\varsigma}^{\mathring{k}+\varsigma-\xi} \mathring{q}(\mathring{j})\mathscr{G}(\mathscr{F}[\mathring{k}, +\xi-\varsigma, \mathring{j}+\xi-\varsigma]) \leq \frac{1}{\varphi}, \tag{62}$$

which gives the contradiction to (49).

Case 2.5. From (25), we have

$$-z(\varepsilon) = -z(\mathring{k}) + (\mathring{k}-\varepsilon)\Delta z(\mathring{k}) - \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon+1-\varepsilon)\Delta^2 z(\epsilon) \geq -\sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon+1-\varepsilon)\Delta^2 z(\epsilon), \tag{63}$$

for $\mathring{k}-1 \geq \varepsilon \geq \varrho_1$ and

$$\mathscr{L}_2 z(\epsilon) - \mathscr{L}_2 z(\varepsilon) = \sum_{s=\varepsilon}^{\epsilon-1} \mathscr{L}_3 z(s) \leq (\epsilon-\varepsilon)\mathscr{L}_3 z(\varepsilon), \tag{64}$$

which implies that

$$\mathscr{L}_2 z(\epsilon) \leq (\epsilon-\varepsilon)\mathscr{L}_3 z(\varepsilon). \tag{65}$$

That is,

$$-\Delta^2 z(\epsilon) \geq \frac{\epsilon - \varepsilon}{\mathring{r}(\epsilon)} \mathscr{L}_2 z(\varepsilon). \qquad (66)$$

Consequently,

$$-z(\varepsilon) \geq -\sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon) \frac{\epsilon - \varepsilon}{\mathring{r}(\epsilon)} \mathscr{L}_3 z(\varepsilon) = -\mathscr{L}_3 z(\varepsilon) \mathscr{M}[\mathring{k}, \varepsilon]. \qquad (67)$$

Also hence,

$$-z(\mathring{j} + \xi - \varsigma) \geq -\mathscr{L}_3 z(\mathring{j} + \xi - \varsigma) \mathscr{M}[\mathring{k}, +\xi - \varsigma, \mathring{j} + \xi - \varsigma] \geq -\mathscr{L}_3 z(\mathring{j} + \xi - \varsigma + 2) \mathscr{M}[\mathring{k}, +\xi - \varsigma, \mathring{j} + \xi - \varsigma], \qquad (68)$$

holds for $\mathring{k} + \xi - \varsigma \geq \mathring{j} + \xi - \varsigma \geq \epsilon + \xi - \varsigma + 2 > \varrho_2$, using (16) and (68) in (59) and summing from $\epsilon + \xi - \varsigma$ to $\epsilon + \xi - \varsigma + 1$, we obtain

$$\mathscr{G}(-\mathscr{L}_3 z(\epsilon + \xi - \varsigma + 2)) \sum_{\mathring{j}=\epsilon+\xi-\varsigma}^{\epsilon+\xi-\varsigma+1} \mathring{q}(\mathring{j}) \mathscr{G}(\mathscr{M}[\mathring{k}, +\xi - \varsigma, \mathring{j} + \xi - \varsigma]) \leq -\mathscr{L}_3 z(\epsilon + \xi - \varsigma + 2). \qquad (69)$$

Therefore,

$$\sum_{\mathring{j}=\epsilon+\xi-\varsigma}^{\epsilon+\xi-\varsigma+1} \mathring{q}(\mathring{j}) \mathscr{G}(\mathscr{M}[\mathring{k}, +\xi - \varsigma, \mathring{j} + \xi - \varsigma]) \leq \frac{-\mathscr{L}_3 z(\epsilon + \xi - \varsigma + 2)}{\mathscr{G}(-\mathscr{L}_3 z(\epsilon + \xi - \varsigma + 2))} \leq \frac{1}{\varphi}, \qquad (70)$$

which is logical inconsistency to (50).

Case 2.6. We use (25) and it follows that

$$-z(\mathring{k}) = -z(\varepsilon) + (\mathring{k} - \varepsilon)\Delta z(\mathring{k}) + \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon) \Delta^2 z(\epsilon) \geq \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon) \Delta^2 z(\epsilon), \qquad (71)$$

for $\mathring{k} - 1 \geq \varepsilon \geq \varrho_1$. Since

$$\left.\mathscr{L}_2 z(\epsilon) - \mathscr{L}_2 z(\varepsilon) = \sum_{s=\varepsilon}^{\epsilon-1} \mathscr{L}_3 z(s) \leq (\epsilon - \varepsilon) \mathscr{L}_3 z(\epsilon - 1)\right], \qquad (72)$$

we have

$$\Delta^2 z(\epsilon) \geq \frac{\mathscr{L}_3 z(\epsilon - 1)(\epsilon - \varepsilon)}{\mathring{r}(\epsilon)}, \qquad (73)$$

which implies that

$$-z(\mathring{k}) \geq \sum_{\epsilon=\varepsilon}^{\mathring{k}-1} (\epsilon + 1 - \varepsilon) \mathscr{L}_3 z(\epsilon - 1) \frac{\epsilon - \varepsilon}{\mathring{r}(\epsilon)} = \mathscr{L}_3 z(\mathring{k} - 2) \mathscr{M}[\mathring{k}, \varepsilon]. \qquad (74)$$

Hence, for $\mathring{j} + \xi - \varsigma \geq \mathring{k} + \xi - \varsigma + 2 > \varrho_1 + 2$, it follows that

$$-z(\mathring{j} + \xi - \varsigma) \geq \mathscr{L}_3 z(\mathring{j} + \xi - \varsigma - 2) \mathscr{M}[\mathring{j} + \xi - \varsigma, \mathring{k} + \xi - \varsigma]. \qquad (75)$$

Consequently, (59) becomes

$$\sum_{\mathring{j}=\epsilon+\xi-\varsigma-2}^{\mathring{k}} \mathring{q}(\mathring{j}) \mathscr{G}(\mathscr{M}[\mathring{j} + \xi - \varsigma, \mathring{k} + \xi - \varsigma]) \mathscr{G}\left(\mathscr{L}_3 z(\mathring{j} + \xi - \varsigma - 2)\right) \leq \mathscr{L}_3 z(\mathring{k} + \xi - \varsigma - 2). \qquad (76)$$

As a result,

$$\sum_{\mathring{j}=\epsilon+\xi-\varsigma-2}^{\mathring{k}} \mathring{q}(\mathring{j}) \mathscr{G}(\mathscr{M}[\mathring{j} + \xi - \varsigma, \mathring{k} + \xi - \varsigma]) \leq \frac{\mathscr{L}_3 z(\mathring{k} + \xi - \varsigma - 2)}{\mathscr{G}\left(\mathscr{L}_3 z(\mathring{j} + \xi - \varsigma - 2)\right)} \leq \frac{1}{\varphi}, \qquad (77)$$

which is logical inconsistency to (51).

In both cases 2.7 and 2.8, we see that $\lim_{n\longrightarrow\infty} z(\varrho) = -\infty$.

On the other hand, $z(\varrho) < 0$ for $\varrho \geq \varrho_1$ implies $\mathring{y}(\varrho) \leq \mathring{y}(\varrho - \xi)$ for $\varrho \geq \varrho_1$.

That is,

$$\mathring{y}(\varrho) \leq \mathring{y}(\varrho - \xi) \leq \mathring{y}(\varrho - 2\xi) \leq \cdots \leq \mathring{y}\varrho_1. \quad (78)$$

Hence, $\mathring{y}(\varrho)$ is bounded and $\Rightarrow z(\varrho)$ is bounded, which is a contradiction.

This completes the proof of the theorem. □

Example 1. Consider the NFDE,

$$\Delta^2\left(\varrho e^{\varrho}\Delta^2\left(\mathring{y}(\varrho) + \mathring{p}(\varrho)\mathring{y}(\varrho - 1)\right)\right) + \mathring{q}(\varrho)G\left(\mathring{y}(\varrho - 3)\right) = 0,\ \varrho > 3. \quad (79)$$

Here,

$$\begin{aligned}\mathring{p}(\varrho) &= e^{-2} + e^{-\varrho};\ \mathring{q}(\varrho)\\ &= \left(e^2 - 1\right)^2 (e+1)(2e + \varrho e + \varrho)e^{\varrho} - (e+1)^2(\varrho + 1);,\\ \mathring{r}(\varrho) &= \varrho e^{\varrho} \text{ and } g(\mathring{u})\\ &= \frac{4u}{e^2}\\ &= \varphi u.\end{aligned} \quad (80)$$

Clearly, all conditions of Theorem 1 are satisfied.

Hence, (79) is oscillatory and $\mathring{y}(\varrho) = (-1)^{\varrho}$ is one of the oscillatory solutions of (79).

3. Conclusion

In this paper, we inferred new properties of the non-oscillatory solutions and using these outcomes, some new adequate are introduced for the concentrated on NFDE to have the purported property oscillatory. Our outcomes improve and supplement many known outcomes for NFDEs as well as for ordinary functional difference equations also. At last, we give two models that show the meaning of the fundamental results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkh user Boston, Inc., Boston, MA, 2001.

[2] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkh user Boston, Inc., Boston, MA, 2003.

[3] R. P. Agarwal, S. R. Grace, and P. J. Y. Wong, "Oscillation of fourth order nonlinear difference equations," *Int. J. Difference Equ*, vol. 2, no. 2, pp. 123–137, 2007.

[4] J. R. Graef, A. Miciano, P. W. Spikes, P. Sundaram, and E. Thandapani, "Oscillatory and asymptotic behavior of solutions of nonlinear neutral-type difference equations," *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, vol. 38, no. 2, pp. 163–171, 1996.

[5] J. R. Graef and E. Thandapani, "Oscillatory and asymptotic behavior of fourth order nonlinear delay difference equations," *Fasc. Math. No*, vol. 31, pp. 23–36, 2001.

[6] M. Migda and J. Migda, "Oscillatory and asymptotic properties of solutions of even order neutral difference equations," *Journal of Difference Equations and Applications*, vol. 15, no. 11-12, pp. 1077–1084, 2009.

[7] E. Thandapani and I. M. Arockiasamy, "Oscillatory and asymptotic behaviour of fourth order nonlinear neutral delay difference equations," *Indian Journal of Pure and Applied Mathematics*, vol. 32, no. 1, pp. 109–123, 2001.

[8] E. Thandapani, P. Sundaram, J. R. Graef, A. Miciano, and P. W. Spikes, "Classification of nonoscillatory solutions of higher order neutral type difference equations," *Archivum Mathematicum*, vol. 31, no. 4, pp. 263–277, 1995.

[9] A. Tripathy, "Oscillation of fourth order nonlinear neutral difference equations-II," *Mathematica Slovaca*, vol. 58, no. 5, pp. 581–604, 2008.

[10] M. Migda, "Asymptotic properties of nonoscillatory solutions of higher order neutral difference equations," *Opuscula Math*, vol. 26, no. 3, pp. 507–514, 2006.

[11] R. P. Agarwal, "Difference equations and inequalities. Theory, methods, and applications," *Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 228, Marcel Dekker, New York, NY, USA, Second edition, 2000.

[12] A. K. Tripathy, "New oscillation criteria for fourth-order difference equations," *Advances in Dynamical Systems and Applications*, vol. 8, no. 2, pp. 387–399, 2013.

[13] N. Parhi and A. K. Tripathy, "Oscillation of a class of nonlinear neutral difference equations of higher order," *Journal of Mathematical Analysis and Applications*, vol. 284, no. 2, pp. 756–774, 2003.