1. Introduction
In this paper, we investigate the oscillatory behavior of even-order half-linear neutral differential equation
\[
\left[ p(y)\left( y^{(n-1)}(y)\right)^{\alpha}\right] + q(y)y^{n}(\sigma(y)) = 0,
\]
where \( n \geq 2 \) is an even integer and \( y(y) = x(y) + h(y)x(\zeta(y)) \) with \( \alpha \) is the ratio of two odd positive integers.

Throughout this paper, we assume that the following conditions are satisfied:

\((K_1)\) \( p, q, r \in C([y_0, \infty), \mathbb{R}), p(y) \geq 0, q(y) \geq 0, r(y) \geq 0 \) and
\[
E(y) = \int_{y_0}^{y} \left[ \frac{1}{p(s)} \exp\left( -\int_{y_0}^{s} \frac{q(t)}{p(t)} \, dt \right) \right]^{1/\alpha} \, ds = \infty, \quad \text{as } y \to \infty. \]

\((K_2)\) \( h: ([0, \infty], \mathbb{R}), 0 \leq h(y) < 1 \) for all \( y \geq y_0 \). \((K_3)\) \( \zeta, \sigma: ([y_0, \infty), \mathbb{R}) \) with \( \zeta(y) \leq y, \sigma(y) \leq y \) and \( \lim_{y \to \infty} \zeta(y) = \lim_{y \to \infty} \sigma(y) = \infty. \)

By a solution \( x \) of (1), we mean a function \( x \in C([y_x, \infty), \mathbb{R}) \) for some \( y_x \geq y_0 \) which has a property \( y \in C^2([y_x, \infty), \mathbb{R}) \) and satisfies (1) on \([y_x, \infty)\). We consider the solution \( x \) of (1) on some half-line \([y_x, \infty)\) and satisfy the condition as follows:
\[
\sup\{|x(y)|: y \leq y_x \geq y_0\} = 0 \quad \text{for any } y \geq y_x. \tag{3}
\]

Such solutions can be called as oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is non-oscillatory.

The oscillation problem for neutral differential (1) has been studied by many authors with various techniques. Most of the results are obtained for the case \( 0 \leq h(i) < \infty \), see [1–6], \( h(i) \geq 1 \), see [7, 8], and \( a \geq 1 \), see [2, 5, 6, 8–12]. The purpose of this paper is to extend the main results for \( a \leq 1 \) using the Philos function method. Furthermore, the oscillation conditions are obtained in order to make it easy to apply. Also, it is our hope that the present paper will be a good contribution to study the oscillatory behavior of solutions of the even-order neutral differential equations. An example is provided to illustrate the main result.

2. Main Results
To prove our main results, we use the following notations:
\( y(y) = (1 - h(y)). \)  

\( (4) \)

We introduce a function class \( P \) by following Philos [13]. Let \( G_0 = \{(y, i) \in R^2 : y > i \geq y_0 \} \) and \( G = \{(y, i) \in R^2 : y \geq i \geq y_0 \} \). We say that the function \( H \in C(G, R) \) belongs to the class \( P \), denoted by \( He \) if

(i) \( H(y, y) = 0 \) for \( y \geq y_0 \), and \( H(y, i) > 0 \) on \( (y, i) \in G_0 \)

(ii) \( H \) has a continuous and nonpositive partial derivative on \( G_0 \) with respect to the second variable

\[
\lim_{y \to \infty} \sup 1 \int_0^\infty \frac{H(y, i)\Psi(i)}{\Phi(a^+(y, i)H^a(y, i)(n-2))} di = \infty,
\]

for sufficiently large \( y \geq 1 \), where

\[
\beta(y) = -\frac{\rho(y)}{\rho(y)-q(y)}, F(y) = \frac{\rho}{{\rho}^{\frac{1}{\alpha}}(y)\rho({\rho}^{\frac{1}{\alpha}}(y))(n-2)},
\]

\[
\Psi(y) = \rho(y)\psi(y)(y)\left(\frac{\sigma(y)}{\gamma}\right)^{\alpha(n-1)}, \quad (7)
\]

\[
\Phi(y, i) = (-h(y, i)\sqrt{H(y, i) + \beta(i)H(y, i))}
\]

for some \( \mu \in (0, 1) \), then every solution of (1) is oscillatory.

**Proof.** Let \( x(y) \) be a positive solution of (1). Then, \( y(y) \) is positive for all \( y \geq y_0 \). From equation (1), we get that

\[
\left[ p(y)(y^{(n-1)}(y))^a \right]' + q(y)(y^{(n-1)}(y))^a < 0.
\]

\( (8) \)

\[
\exp\left(\int_{y_0}^{y} \frac{q(i)}{p(i)} di\right)p(y)[y^{(n-1)}(y)]^a \leq \exp\left(\int_{y_0}^{y} \frac{q(i)}{p(i)} di\right)p(1)[y^{(n-1)}(1)]^a
\]

\[
= -M^a \exp\left(\int_{y_0}^{y} \frac{q(i)}{p(i)} di\right).
\]

\( (10) \)

where \( M = p^{\frac{1}{\alpha}}(1)\left[ y^{(n-1)}(1) \right] > 0 \). Now, we have

\[
y^{(n-1)}(y) \leq \left[ -\frac{M^a}{p(y)} \exp\left( -\int_{y_0}^{y} \frac{q(i)}{p(i)} di \right) \right]^{1/\alpha}.
\]

\( (11) \)

Integrating the last inequality with respect to \( y \), we get

\[
\int_{y_0}^{y} y^{(n-1)}(y) dt \leq -M \int_{y_0}^{y} \frac{1}{p(y)} \exp\left( -\int_{y_0}^{y} \frac{q(u)}{p(u)} du \right) \frac{1}{\alpha} dt.
\]

\( (12) \)

As \( y \to \infty \), we have \( \lim_{y \to \infty} y^{(n-2)}(y) = -\infty \), and consequently, we see that \( \lim_{y \to \infty} y(y) = -\infty \) which is a contradiction.

Now, let us take Case (II).

Since \( y^{(n-1)} > 0 \), from Lemma 1 of [5], we have \( y' > 0 \). From the definition of \( y(y) \), we see that

\[
\psi(y) = (1 - h(y)).
\]

\( (4) \)

**Theorem 1.** Let the conditions \((K_1) - (K_3)\) hold. Let \( h, H : G \to \mathbb{R} \) be continuous functions such that \( H \) belongs to the class \( P \) and

\[
\frac{\partial H}{\partial i}(y, i) \equiv h(y, i)\sqrt{H(y, i)f(i)} \text{ for all } (y, i) \in G_0.
\]

If there exists a positive function \( \rho \in C^1([y_0, \infty), \mathbb{R}) \) such that for some \( \beta \geq 1, \)

\[
\frac{\rho(y)}{\rho(y)-q(y)} \equiv -\frac{1}{\alpha} + \frac{1}{\beta(y)}.
\]

Let \( u(y) = p(y)(y^{(n-1)}(y))^a \); then, from (8), we get that

\[
\left(\exp\left(\int_{y_0}^{y} \frac{q(i)}{p(i)} di\right)u(y)\right)' < 0,
\]

which says that \( \exp\left(\int_{y_0}^{y} (q(i)/p(i)) di\right)u(y) \) is decreasing and does not change its sign which leads to two cases:

(I) \( y^{(n-1)}(y) < 0 \)

(II) \( y^{(n-1)}(y) > 0 \)

Now, we consider that Case (I) holds. If \( y \geq \Gamma \), then we obtain

\[
x(y) = y(y) - h(y)x(\zeta(y)) > (1 - h(y))y(y),
\]

\( (13) \)

\[
x(\sigma(y)) > \psi(\sigma(y))y(\sigma(y)).
\]

(14)

Then, (1) becomes

\[
\left[ p(y)(y^{(n-1)}(y))^a \right]' + q(y)(y^{(n-1)}(y))^a + r(y)\psi(\sigma(y))y^{(n-1)}(\sigma(y)) \leq 0.
\]

(15)

We consider the Riccati transformation

\[
w(y) = \rho(y)p(y)(y^{(n-1)}(y))^a.
\]

(16)

We see that \( w(y) > 0 \), and from (16), we get
Integrating by parts yields
\[
\int_{\Gamma}^{y} H(y, i)\Psi(i)\,di \leq -\int_{\Gamma}^{y} H(y, i)w'(i)\,di + \int_{\Gamma}^{y} H(y, i)\beta(i)w(i)\,di
\]

\[
-\frac{\alpha}{\beta} \int_{\Gamma}^{y} H(y, i)w^{1+\alpha}(i)F(i)\,di
\]

\[
-\frac{\alpha(\beta - 1)}{\beta} \int_{\Gamma}^{y} H(y, i)w^{1+\alpha}(i)F(i)\,di.
\]

Substituting (5) and (21) into (20), we get
\[
\int_{\Gamma}^{y} H(y, i)w'(i)\,di = -H(y, \Gamma)w(\Gamma) - \int_{\Gamma}^{y} \frac{\partial H(y, i)}{\partial t}w(i)\,di.
\]

Integrating by parts yields
\[
\int_{\Gamma}^{y} H(y, i)\Psi(i)\,di \leq -\int_{\Gamma}^{y} H(y, i)w'(i)\,di + \int_{\Gamma}^{y} H(y, i)\beta(i)w(i)\,di
\]

\[
-\frac{\alpha}{\beta} \int_{\Gamma}^{y} H(y, i)w^{1+\alpha}(i)F(i)\,di
\]

\[
-\frac{\alpha(\beta - 1)}{\beta} \int_{\Gamma}^{y} H(y, i)w^{1+\alpha}(i)F(i)\,di.
\]

Now, by applying the inequality, \( Au - Bu^{1+\alpha} \leq \alpha^{\alpha/1} (\alpha + 1)^{\alpha+1} B^{\alpha} \) (22), we get
\[
\int_{\Gamma}^{y} H(y, i)\Psi(i) \leq \frac{\beta^{\alpha}}{(\alpha + 1)^{\alpha+1}} \frac{\Phi^{\alpha+1}(y, i)}{H^\alpha(y, i)F^\alpha(i)} \,di
\]

\[
\leq H(y, \Gamma)w(\Gamma)
\]

\[
-\frac{\alpha(\beta - 1)}{\beta} \int_{\Gamma}^{y} H(y, i)w^{1+\alpha}(i)F(i)\,di.
\]

So, for every \( y \geq \Gamma \), we obtain
\[
\limsup_{y \to \infty} \frac{1}{H(y, \Gamma)} \int_{\Gamma}^{y} H(y, i)\Psi(i) \leq \frac{\beta^{\alpha}}{(\alpha + 1)^{\alpha+1}} \frac{\Phi^{\alpha+1}(y, i)}{H^\alpha(y, i)F^\alpha(i)} \,di \geq \Phi(\Gamma).
\]

The above inequality can be written as
\[
w'(y) \leq \beta(y)w(y) - \Psi(y) - \alpha F(y)w^{1+\alpha}(y).
\]

Multiplying (19) by \( H(y, i) \) and integrating from \( \Gamma \) to \( y \), we have for some \( \beta \geq 1 \) and for all \( y \geq \Gamma \leq y_0 \)

\[
\int_{\Gamma}^{y} H(y, i)\Psi(i)\,di \leq -\int_{\Gamma}^{y} H(y, i)w'(i)\,di + \int_{\Gamma}^{y} H(y, i)\beta(i)w(i)\,di
\]

\[
-\frac{\alpha}{\beta} \int_{\Gamma}^{y} H(y, i)w^{1+\alpha}(i)F(i)\,di
\]

\[
-\frac{\alpha(\beta - 1)}{\beta} \int_{\Gamma}^{y} H(y, i)w^{1+\alpha}(i)F(i)\,di.
\]

which is a contradiction.

Next, oscillation result follows immediately from Theorem 1.

\[\square\]

**Corollary 1.** Let the assumptions of Theorem 1 be helded except that the condition (4) is replaced by

\[
\lim_{y \to \infty} \sup \frac{1}{H(y, i)} \int_{\Gamma}^{y} H(y, i)\Psi(i)\,di = \infty,
\]

\[
\lim_{y \to \infty} \sup \frac{1}{H(y, i)} \int_{\Gamma}^{y} \Phi^{\alpha+1}(y, i) \,di = \infty.
\]

Then, every solution of (1) is oscillatory.

**Theorem 2.** Let the conditions \((K_1) \sim (K_3)\) and (2) be satisfied. Let \( H \) and \( h \) be same as in Theorem 1. We suppose that

\[
0 \leq \inf_{y \geq y_0} \left\{ \liminf_{y \to \infty} \frac{H(y, \Gamma)}{H(y, y_0)} \right\} \leq \infty.
\]

If there exist functions \( \Phi \in \mathcal{C}([y_0, \infty), \mathbb{R}) \) and \( \rho \in \mathcal{C}^1([y_0, \infty), \mathbb{R}) \) such that for some \( \beta \geq 1 \)

\[
\limsup_{y \to \infty} \frac{1}{H(y, \Gamma)} \int_{\Gamma}^{y} H(y, i)\Psi(i) \leq \frac{\beta^{\alpha}}{(\alpha + 1)^{\alpha+1}} \frac{\Phi^{\alpha+1}(y, i)}{H^\alpha(y, i)F^\alpha(i)} \,di \geq \Phi(\Gamma).
\]
\[
\int_{\gamma}^{\infty} F(i)\Phi_{+}^{\alpha+1}(i)di = \infty, 
\]

for all sufficiently large \( \gamma \in \{\gamma_1, \infty\} \subset [\gamma_0, \infty) \) and all \( \Gamma \geq \gamma_2 \), where \( \psi(y, i) \) and \( F(y) \) are as in Theorem 1 and \( \Phi_+, (\gamma) = \max\{\psi(\gamma, 0), 0\} \), then every solution of (1) is oscillatory.

**Proof.** Let \( x(y) \) be a non-oscillatory solution of (1), say \( x(y) > 0, x(\sigma(y)) > 0 \) and \( x(\zeta(y)) > 0 \) for \( \gamma \geq \gamma_1 \) for some \( y \geq \gamma_0 \). Proceeding as in the proof of Theorem 1, we again have two cases:

1. \( y^{(n-1)}(y) < 0 \)
2. \( y^{(n-1)}(y) > 0 \)

For \( y \geq \gamma_2 \), if Case (I) holds proceeding as in the proof of Theorem 1, we obtain a contradiction to the positivity of \( y \).

Next, we assume Case (II) holds, proceeding as in the proof of Theorem 1; we again arrive at (23), which can be written as \( y > \Gamma > \gamma_0 \).

\[
\frac{1}{H(\gamma, \Gamma)} \int_{\gamma}^{\Gamma} H(\gamma, i)\Psi(i) - \frac{\beta^\alpha}{(\alpha + 1)^{\alpha+1}} \Phi(i) \frac{\Phi_{+}^{\alpha+1}(y, i)}{H^\alpha(y, i)F^\alpha(i)} \] 
\[
- \frac{1}{H(\gamma, \Gamma)} \int_{\gamma}^{\Gamma} \frac{\alpha(\beta - 1)}{\beta} H(\gamma, i)w^{1+\alpha}(i)F(i)di.
\]

From (29), we have

\[
\lim_{\gamma \to \infty} \sup_{\gamma \in \gamma_0} \frac{1}{H(\gamma, \Gamma)} \int_{\gamma}^{\Gamma} H(\gamma, i)\Psi(i) - \frac{\beta^\alpha}{(\alpha + 1)^{\alpha+1}} \Phi(i) \frac{\Phi_{+}^{\alpha+1}(y, i)}{H^\alpha(y, i)F^\alpha(i)} di \leq w(\Gamma)
\]

\[
- \lim_{\gamma \to \infty} \inf_{\gamma \in \gamma_0} \frac{1}{H(\gamma, \Gamma)} \int_{\gamma}^{\Gamma} \frac{\alpha(\beta - 1)}{\beta} H(\gamma, i)w^{1+\alpha}(i)F(i)di.
\]

In view of (27), it follows from (30) that

\[
\lim_{\gamma \to \infty} \sup_{\gamma \in \gamma_0} \frac{1}{H(\gamma, \Gamma)} \int_{\gamma}^{\Gamma} H(\gamma, i)\Psi(i) - \frac{\beta^\alpha}{(\alpha + 1)^{\alpha+1}} \Phi(i) \frac{\Phi_{+}^{\alpha+1}(y, i)}{H^\alpha(y, i)F^\alpha(i)} di \leq w(\Gamma)
\]

\[
\lim_{\gamma \to \infty} \inf_{\gamma \in \gamma_0} \frac{1}{H(\gamma, \Gamma)} \int_{\gamma}^{\Gamma} \frac{\alpha(\beta - 1)}{\beta} H(\gamma, i)w^{1+\alpha}(i)F(i)di.
\]

for all \( y \geq \gamma_3 \) for any \( \beta > 1 \). Then, for all \( \Gamma \geq \gamma_3 \),

\[
W(\Gamma) \geq \phi(\Gamma),
\]

\[
\lim_{\gamma \to \infty} \inf_{\gamma \in \gamma_0} \frac{1}{H(\gamma, \Gamma)} \int_{\gamma}^{\Gamma} H(\gamma, i)w^{1+\alpha}(i)F(i)di \leq \frac{\beta(w(\gamma_3) - \phi(\gamma_3))}{\alpha(\beta - 1)} < \infty.
\]

Now, we claim that

\[
\int_{\gamma_3}^{\infty} w^{1+\alpha}(i)F(i)di < \infty.
\]

If not, then

\[
\int_{\gamma_3}^{\infty} w^{1+\alpha}(i)F(i)di = \infty.
\]

By (26), there exists a constant \( \epsilon > 0 \) such that

\[
y_1 \geq \gamma_0.
\]

(33) \[
\frac{1}{H(\gamma, \gamma_4)} \int_{\gamma_4}^{\gamma} H(\gamma, i)w^{1+\alpha}(i)F(i)di
\]

\[
= \frac{1}{H(\gamma, \gamma_4)} \int_{\gamma_4}^{\gamma} H(\gamma, i) d \left[ \int_{\gamma_4}^{\gamma} w^{1+\alpha}(s)F(s)ds \right]
\]

\[
\geq \frac{1}{\epsilon H(\gamma, \gamma_4)} \int_{\gamma_4}^{\gamma} \left[ \frac{-\partial H(y, i)}{\partial t} \right] di
\]

\[
\geq \frac{\delta}{\epsilon H(\gamma, \gamma_4)} \int_{\gamma_4}^{\gamma} H(y, i) \phi(\gamma_3) \frac{\partial \phi(\gamma_3)}{\partial t} di
\]

(38)
It follows from (36) that
\[
\lim_{\gamma \to -\infty} \inf_{\gamma} \frac{H(y,\gamma)}{H(y,\gamma_0)} \geq \epsilon > 0,
\]
and hence, there exists \( \gamma_6 \geq \gamma_5 \) such that
\[
\frac{H(y,\gamma_5)}{H(y,\gamma_6)} \geq \epsilon \text{ for } \gamma \geq \gamma_6.
\]

From the latter inequality and (32), we see that
\[
\frac{1}{H(y,\gamma_1)} \int_{\gamma_1}^{\gamma} H(y,\gamma) w^{1+\alpha}(\gamma) F(\gamma) d\gamma \geq \delta, \text{ for } \gamma \geq \gamma_6.
\]

Since \( \delta \) is an arbitrary positive constant, we obtain
\[
\lim_{\gamma \to -\infty} \inf_{\gamma} \frac{1}{H(y,\gamma)} \int_{\gamma}^{\gamma'} H(y,\gamma) \Psi(\gamma) - \frac{\beta^a}{(\alpha+1)^{a+1}} \frac{\Phi^{a+1}(y,\gamma)}{H^a(y,\gamma) F^a(\gamma)} d\gamma \geq \Phi(\Gamma),
\]
then every solution of (1) is oscillatory.

Hence the proof is complete. \( \square \)

3. Example

In this section, we provide an example to illustrate the main result.

Example 1. We consider the fourth-order neutral delay differential equation
\[
\left[ y^{(\alpha)}(y)^{1/3} \right] + \frac{1}{2^{\alpha}} (y^{(\alpha)}(y))^{1/3} + y^2 x^{1/3} (y - 1) = 0, \quad y \geq 2,
\]
where \( y(y) = x(y) + 1/2 x(y - 2) \).

Hence, all conditions of Theorem 1 are satisfied, and therefore, every solution of (46) is oscillatory.

It is noted that the conditions of Corollary 1 are not satisfied, and hence, Theorem 1 improves Corollary 1.

\[
\lim_{\gamma \to -\infty} \inf_{\gamma} \frac{1}{H(y,\gamma)} \int_{\gamma}^{\gamma'} H(y,\gamma) w^{1+\alpha}(\gamma) F(\gamma) d\gamma = \infty,
\]
which contradicts (33). Thus, (34) should hold and so by (32), we have
\[
\int_{\gamma}^{\infty} F(\gamma) w^{1+\alpha}(\gamma) d\gamma \leq \int_{\gamma}^{\infty} w^{1+\alpha}(\gamma) F(\gamma) d\gamma < \infty,
\]
which contradicts (28). This completes the proof of the theorem. \( \square \)

Theorem 3. Let all the conditions of Theorem 2 be satisfied except the condition (21) be replaced with
\[
\Phi(\Gamma) \leq \lim_{\gamma \to -\infty} \sup_{\gamma} \frac{1}{H(y,\gamma)} \int_{\gamma}^{\gamma'} H(y,\gamma) \Psi(\gamma) - \frac{\beta^a}{(\alpha+1)^{a+1}} \frac{\Phi^{a+1}(y,\gamma)}{H^a(y,\gamma) F^a(\gamma)} d\gamma
\]
\[
\leq \lim_{\gamma \to -\infty} \inf_{\gamma} \frac{1}{H(y,\gamma)} \int_{\gamma}^{\gamma'} H(y,\gamma) \Psi(\gamma) - \frac{\beta^a}{(\alpha+1)^{a+1}} \frac{\Phi^{a+1}(y,\gamma)}{H^a(y,\gamma) F^a(\gamma)} d\gamma.
\]

Hence, we have \( n = 4, \alpha = 1/3, \zeta(y) = y - 2, \zeta^{-1}(y) = y + 1, \zeta^{-1}(\zeta^{-1}(y)) = y + 2, \sigma(y) = y - 1, \rho(y) = 1, q(y) = 1/2y, r(y) = y^2 \).

Simple calculation shows that by taking \( \rho(y) = y \), that \( B(y) = 1/2y, F(y) = \mu/2y, \Phi(y) = 1/8y^3(1-1/y) \).

Letting \( H(y,\gamma) = (y-i)^2 \), we see that \( HeF, h(y,i) = 2 \) and
\[
\Phi(y,i) = \frac{y^2}{2i} - 3i + \frac{5i}{2}
\]
We choose \( \Gamma = 2 \), and the condition (46) becomes
\[
\lim_{\gamma \to -\infty} \sup_{\gamma} \frac{1}{(y-2)^2} \int_{\gamma}^{\gamma'} \left[ (y-2)^2 \right]^{3/4} \frac{(y-1)}{y} \frac{1}{i} = \infty.
\]

4. Conclusion

In this paper, we have obtained criteria for the oscillation of all solutions of (1) using the Philos type method. An example
is provided to illustrate the main result. It will be of interest to study the equation under the condition [14].

$$\int_{y_0}^{\infty} \left[ \frac{1}{r(y)} \exp\left(-\int_{y_0}^{y} \frac{p(t)}{r(t)} \, dt\right) \right]^{1/\alpha} \, dr < \infty. \quad (49)$$

It is also interesting to extend the results of this paper to stochastic type neutral differential equations and differential equations with impulses, see [15, 16].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**References**


