Hindawi
International Journal of Differential Equations
Volume 2022, Article ID 3352789, 6 pages
https://doi.org/10.1155/2022/3352789

Research Article

Oscillatory Behavior of Even-Order Half-Linear Neutral Differential Equations

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Received 8 December 2021; Accepted 25 January 2022; Published 25 May 2022

Academic Editor: Elena Braverman

This paper discusses some sufficient conditions for oscillatory behavior of even-order half-linear neutral differential equation. An example is given to illustrate the main result.

1. Introduction

In this paper, we investigate the oscillatory behavior of even-order half-linear neutral differential equation

$$\left[p(\gamma)\left(y^{(n-1)}(\gamma)\right)^{\alpha}\right]' + q(\gamma)\left(y^{(n-1)}(\gamma)\right)^{\alpha} + r(\gamma)x^{\alpha}(\sigma(\gamma)) = 0, \tag{1}$$

where $n \geq 2$ is an even integer and $y(\gamma) = x(\gamma) + h(\gamma)x(\zeta(\gamma))$ with α is the ratio of two odd positive integers.

Throughout this paper, we assume that the following conditions are satisfied:

(K_1) p, q, r $\epsilon([\gamma_0,\infty),\mathbb{R})$, $p(\gamma) \geq 0, q(\gamma) \geq 0, r(\gamma) \geq 0$ and

$$E(\gamma) = \int_{\gamma_0}^{\gamma}\left[\frac{1}{p(s)}\exp\left(-\int_{\gamma_0}^{s}\frac{q(\iota)}{p(\iota)}d\iota\right)\right]^{1/\alpha}ds = \infty, \quad \text{as } \gamma \longrightarrow \infty. \tag{2}$$

(K_2) h: $([\gamma_0,\infty),\mathbb{R}), 0 \leq h(\gamma) < 1$ for all $\gamma \geq \gamma_0$, (K_3) ζ, σ: $((\gamma_0,\infty],\mathbb{R})$ with $\zeta(\gamma) \leq \gamma, \sigma(\gamma) \leq \gamma$ and $\lim_{\gamma\longrightarrow\infty}\zeta(\gamma) = \lim_{\gamma\longrightarrow\infty}\sigma(\gamma) = \infty$.

By a solution x of (1), we mean a function $x\,\epsilon\,\mathbb{C}([\gamma_x,\infty],\mathbb{R})$ for some $\gamma_x \geq \gamma_0$ which has a property $y\,\epsilon\,\mathbb{C}^2([\gamma_x,\infty],\mathbb{R})$ and satisfies (1) on $[\gamma_x,\infty]$.We consider the solution x of (1) on some half-line $[\gamma_x,\infty]$ and satisfy the condition as follows:

$$\sup\{|x(\gamma)|\colon\ \Gamma \leq \gamma \leq \infty\} > 0 \text{ for any } \Gamma \geq \gamma_x. \tag{3}$$

Such solutions can be called as oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is non-oscillatory.

The oscillation problem for neutral differential (1) has been studied by many authors with various techniques. Most of the results are obtained for the case $0 \leq h(\iota) < \infty$, see [1–6], $h(\iota) \geq 1$, see [7, 8], and $\alpha \geq 1$, see [2, 5, 6, 8–12]. The purpose of this paper is to extend the main results for $\alpha \leq 1$ using the Philos function method. Furthermore, the oscillation conditions are obtained in order to make it easy to apply. Also, it is our hope that the present paper will be a good contribution to study the oscillatory behavior of solutions of the even-order neutral differential equations. An example is provided to illustrate the main result.

2. Main Results

To prove our main results, we use the following notations:

$$\psi(\gamma) = (1 - h(\gamma)). \tag{4}$$

We introduce a function class P by following Philos [13]. Let $G_0 = \{(\gamma, \iota) \epsilon R^2 : \gamma > \iota \geq \gamma_0$ and $G = \{(\gamma, \iota) \epsilon R^2 : \gamma \geq \iota \geq \gamma_0\}$. We say that the function $H \epsilon \mathbb{C}(G, \mathbb{R})$ belongs to the class P, denoted by $H \epsilon P$ if

(i) $H(\gamma, \gamma) = 0$ for $\gamma \geq \gamma_0$, and $H(\gamma, \iota) > 0$ on $(\gamma, \iota) \epsilon G_0$

(ii) H has a continuous and nonpositive partial derivative on G_0 with respect to the second variable

Theorem 1. *Let the conditions $(K_1) - (K_3)$ hold. Let h, $H: G \longrightarrow \mathbb{R}$ be continuous functions such that H belongs to the class P and*

$$-\frac{\partial H}{\partial \iota}(\gamma, \iota) = h(\gamma, \iota)\sqrt{H(\gamma, \iota)} \text{ for all } (\gamma, \iota) \epsilon G_0. \tag{5}$$

If there exists a positive function $\rho \epsilon \mathbb{C}^1([\gamma_0, \infty), \mathbb{R})$ such that for some $\beta \geq 1$,

$$\lim_{\gamma \longrightarrow \infty} \sup \frac{1}{H(\gamma, \Gamma)} \int_\Gamma^\gamma \left[H(\gamma, \iota)\Psi(\iota) - \frac{\beta^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\Phi^{\alpha+1}(\gamma, \iota)}{H^\alpha(\gamma, \iota) F^\alpha(\iota)} \right] d\iota = \infty, \tag{6}$$

for sufficiently large $\gamma \geq \Gamma$, where

$$\beta(\gamma) = \left[\frac{\rho'(\gamma)}{\rho(\gamma)} - \frac{q(\gamma)}{p(\gamma)} \right], F(\gamma) = \frac{\mu \gamma^{n-2}}{\rho^{1/\alpha}(\gamma) p^{1/\alpha}(\gamma)(n-2)},$$

$$\Psi(\gamma) = \rho(\gamma) r(\gamma) \psi^\alpha(\sigma(\gamma)) \left(\frac{\sigma(\gamma)}{\gamma} \right)^{\alpha(n-1)}, \tag{7}$$

$$\Phi(\gamma, \iota) = (-h(\gamma, \iota)\sqrt{H(\gamma, \iota)} + \beta(\iota) H(\gamma, \iota)).$$

for some $\mu \epsilon (0, 1)$, then every solution of (1) is oscillatory.

Proof. Let $x(\gamma)$ be a positive solution of (1). Then, $y(\gamma)$ is positive for all $\gamma \geq \gamma_0$. From equation (1), we get that

$$\left[p(\gamma) \left(y^{(n-1)}(\gamma) \right)^\alpha \right]' + q(\gamma) \left(y^{(n-1)}(\gamma) \right)^\alpha < 0. \tag{8}$$

Let $u(\gamma) = p(\gamma)(y^{(n-1)}(\gamma))^\alpha$; then, from (8), we get that

$$u'(\gamma) + \frac{q(\gamma)}{p(\gamma)} u(\gamma) < 0,$$
$$\left(\exp\left(\int_{\gamma_0}^\gamma \frac{q(\iota)}{p(\iota)} d\iota \right) u(\gamma) \right)' < 0, \tag{9}$$

which says that $\exp(\int_{\gamma_0}^\gamma (q(\iota)/p(\iota)) d\iota) u(\gamma)$ is decreasing and does not change its sign which leads to two cases:

(I) $y^{(n-1)}(\gamma) < 0$

(II) $y^{(n-1)}(\gamma) > 0$

Now, we consider that Case (I) holds. If $\gamma \geq \Gamma$, then we obtain

$$\begin{aligned} \exp\left(\int_{\gamma_0}^\gamma \frac{q(\iota)}{p(\iota)} d\iota \right) p(\gamma) \left[y^{(n-1)}(\gamma) \right]^\alpha &\leq \exp\left(\int_{\gamma_0}^\gamma \frac{q(\iota)}{p(\iota)} d\iota \right) p(\Gamma) \left[y^{(n-1)}(\Gamma) \right]^\alpha \\ &\equiv -M^\alpha \exp\left(\int_{\gamma_0}^\gamma \frac{q(\iota)}{p(\iota)} d\iota \right), \end{aligned} \tag{10}$$

where $M = p^{1/\alpha}(\Gamma)[y^{(n-1)}(\Gamma)] > 0$. Now, we have

$$y^{(n-1)}(\gamma) \leq \left[\frac{-M^\alpha}{p(\gamma)} \exp\left(-\int_\Gamma^{\gamma_0} \frac{q(\iota)}{p(\iota)} d\iota \right) \right]^{1/\alpha}. \tag{11}$$

Integrating the last inequality with respect to γ, we get

$$\int_\Gamma^\gamma y^{(n-1)}(t) dt \leq -M \int_\Gamma^\gamma \left[\frac{1}{p(\iota)} \exp\left(-\int_\Gamma^\iota \frac{q(u)}{p(u)} du \right) \right]^{1/\alpha} d\iota. \tag{12}$$

As $\gamma \longrightarrow \infty$, we have $\lim_{\gamma \longrightarrow \infty} y^{(n-2)}(\gamma) = -\infty$, and consequently, we see that $\lim_{\gamma \longrightarrow \infty} y(\gamma) = -\infty$ which is a contradiction.

Now, let us take Case (II).

Since $y^{(n-1)} > 0$, from Lemma 1 of [5], we have $y' > 0$. From the definition of $y(\gamma)$, we see that

$$x(\gamma) = y(\gamma) - h(\gamma) x(\zeta(\gamma)) \geq (1 - h(\gamma)) y(\gamma), \tag{13}$$

$$x(\sigma(\gamma)) \geq \psi(\sigma(\gamma)) y(\sigma(\gamma)). \tag{14}$$

Then, (1) becomes

$$\begin{aligned} &\left[p(\gamma) \left(y^{(n-1)}(\gamma) \right)^\alpha \right]' + q(\gamma) \left(y^{(n-1)}(\gamma) \right)^\alpha \\ &+ r(\gamma) \psi^\alpha(\sigma(\gamma)) y^\alpha(\sigma(\gamma)) \leq 0. \end{aligned} \tag{15}$$

We consider the Riccati transformation

$$w(\gamma) = \frac{\rho(\gamma) p(\gamma) \left(y^{(n-1)}(\gamma) \right)^\alpha}{y^\alpha(\gamma)}. \tag{16}$$

We see that $w(\gamma) > 0$, and from (16), we get

$$w'(\gamma) \leq \left[\frac{\rho'(\gamma)}{\rho(\gamma)} - \frac{q(\gamma)}{p(\gamma)}\right] w(\gamma) - \frac{r(\gamma)\rho(\gamma)\psi^{\alpha}(\sigma(\gamma))y^{\alpha}(\sigma(\gamma))}{y^{\alpha}(\gamma)} - \frac{\alpha y'(\gamma)p(\gamma)\rho(\gamma)\left(y^{(n-1)}(\gamma)\right)^{\alpha}}{y^{\alpha+1}(\gamma)}. \tag{17}$$

From (18), we get

$$w'(\gamma) \leq \left[\frac{\rho'(\gamma)}{\rho(\gamma)} - \frac{q(\gamma)}{p(\gamma)}\right] w(\gamma) - \Psi(\gamma) - \frac{\alpha w(\gamma)y'(\gamma)}{y(\gamma)}. \tag{18}$$

The above inequality can be written as

$$w'(\gamma) \leq \beta(\gamma)w(\gamma) - \Psi(\gamma) - \alpha F(\gamma)w^{1+1/\alpha}(\gamma). \tag{19}$$

Multiplying (19) by $H(\gamma, \iota)$ and integrating from Γ *to* γ, we have for some $\beta \geq 1$ and for all $\gamma \leq \Gamma \leq \gamma_0$

$$\begin{aligned}\int_{\Gamma}^{\gamma} H(\gamma,\iota)\Psi(\iota)d\iota \leq & -\int_{\Gamma}^{\gamma} H(\gamma,\iota)w'(\iota)d\iota + \int_{\Gamma}^{\gamma} H(\gamma,\iota)\beta(\iota)w(\iota)d\iota \\ & - \frac{\alpha}{\beta}\int_{\Gamma}^{\gamma} H(\gamma,\iota)w^{1+1/\alpha}(\iota)F(\iota)d\iota \\ & - \frac{\alpha(\beta-1)}{\beta}\int_{\Gamma}^{\gamma} H(\gamma,\iota)w^{1+1/\alpha}(\iota)F(\iota)d\iota.\end{aligned} \tag{20}$$

Integrating by parts yields

$$\int_{\Gamma}^{\gamma} H(\gamma,\iota)w'(\iota)d\iota = -H(\gamma,\Gamma)w(\Gamma) - \int_{\Gamma}^{\gamma} \frac{\partial H(\gamma,\iota)}{\partial \iota} w(\iota)d\iota. \tag{21}$$

Substituting (5) and (21) into (20), we get

$$\begin{aligned}\int_{\Gamma}^{\gamma} H(\gamma,\iota)\Psi(\iota)d\iota \leq & H(\gamma,\Gamma)w(\Gamma) + \int_{\Gamma}^{\gamma} [\Phi(\gamma,\iota)w(\iota)d\iota] \\ & - \frac{\alpha}{\beta}\int_{\Gamma}^{\gamma} H(\gamma,\iota)w^{1+1/\alpha}(\iota)F(\iota)d\iota \\ & - \frac{\alpha(\beta-1)}{\beta}\int_{\Gamma}^{\gamma} H(\gamma,\iota)w^{1+1/\alpha}(\iota)F(\iota)d\iota.\end{aligned} \tag{22}$$

Now, by applying the inequality, $Au - Bu^{1+1/\alpha} \leq \alpha^{\alpha}/(\alpha+1)^{\alpha+1}A^{\alpha+1}/B^{\alpha}$ (22), we get

$$\begin{aligned}\int_{\Gamma}^{\gamma} \left[H(\gamma,\iota)\Psi(\iota) - \frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)}\right]d\iota \\ \leq H(\gamma,\Gamma)w(\Gamma) \\ - \frac{\alpha(\beta-1)}{\beta}\int_{\Gamma}^{\gamma} H(\gamma,\iota)w^{1+1/\alpha}(\iota)F(\iota)d\iota.\end{aligned} \tag{23}$$

So, for every $\gamma \geq \Gamma$, we obtain

$$\int_{\Gamma}^{\gamma} \left[H(\gamma,\iota)\Psi(\iota) - \frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)}\right]d\iota \leq H(\gamma,\Gamma)w(\Gamma), \tag{24}$$

which is a contradiction.

Next, oscillation result follows immediately from Theorem 1. ☐

Corollary 1. *Let the assumptions of Theorem 1 be holded except that the condition (4) is replaced by*

$$\begin{aligned}&\lim_{\gamma\longrightarrow\infty}\sup \frac{1}{H(\gamma,\iota)}\int_{\Gamma}^{\gamma} H(\gamma,\iota)\Psi(\iota)d\iota = \infty, \\ &\lim_{\gamma\longrightarrow\infty}\sup \frac{1}{H(\gamma,\iota)}\int_{\Gamma}^{\gamma} \frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)}d\iota < \infty.\end{aligned} \tag{25}$$

Then, every solution of (1) is oscillatory.

Theorem 2. *Let the conditions* $(K_1)-(K_3)$ *and (2) be satisfied. Let H and h be same as in Theorem 1. We suppose that*

$$0 \leq \inf_{\iota\geq\gamma_0}\left\{\liminf_{\gamma\longrightarrow\infty} \frac{H(\gamma,\iota)}{H(\gamma,\gamma_0)}\right\} \leq \infty. \tag{26}$$

If there exist functions $\phi\epsilon C([\gamma_0,\infty),\mathbb{R})$ and $\rho\epsilon\mathbb{C}^1([\gamma_0,\infty),\mathbb{R})$ such that for some $\beta \geq 1$

$$\limsup_{\gamma\longrightarrow\infty} \frac{1}{H(\gamma,\Gamma)}\int_{\Gamma}^{\gamma}\left[\mathrm{H}(\gamma,\iota)\Psi(\iota) - \frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)\mathrm{F}^{\alpha}(\iota)}\right]\mathrm{d}\iota \geq \Phi(\Gamma), \tag{27}$$

$$\int_{\Gamma}^{\infty} F(\iota)\Phi_{+}^{1+1/\alpha}(\iota)d\iota = \infty, \tag{28}$$

for all sufficiently large $\gamma_2 \epsilon [\gamma_1, \infty) \subseteq [\gamma_0, \infty)$ and all $\Gamma \geq \gamma_2$, where $\psi(\gamma, \iota)$ and $F(\gamma)$ are as in Theorem 1 and $\Phi_{+}(\gamma) = \max\{\phi(\gamma), 0\}$, then every solution of (1) is oscillatory.

Proof. Let $x(\gamma)$ be a non-oscillatory solution of (1), say $x(\gamma) > 0, x(\sigma(\gamma)) > 0$ and $x(\zeta(\gamma)) > 0$ for $\gamma \geq \gamma_1$ for some $\gamma_1 \geq \gamma_0$. Proceeding as in the proof of Theorem 1, we again have two cases:

(I) $y^{(n-1)}(\gamma) < 0$

(II) $y^{(n-1)}(\gamma) > 0$

For $\gamma \geq \gamma_2$, if Case (I) holds proceeding as in the proof of Theorem 1, we obtain a contradiction to the positivity of y.

Next, we assume case (II) holds, proceeding as in the proof of Theorem 1; we again arrive at (23), which can be written as $\gamma > \Gamma > \gamma_0$.

$$\begin{aligned} &\frac{1}{H(\gamma,\Gamma)} \int_{\Gamma}^{\gamma} \left[H(\gamma,\iota)\Psi(\iota) - \frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)} \right] d\iota \leq w(\Gamma) \\ &- \frac{1}{H(\gamma,\Gamma)} \int_{\Gamma}^{\gamma} \frac{\alpha(\beta-1)}{\beta} H(\gamma,\iota) w^{1+1/\alpha}(\iota) F(\iota) d\iota. \end{aligned} \tag{29}$$

From (29), we have

$$\begin{aligned} &\lim_{\gamma\longrightarrow\infty} \sup \frac{1}{H(\gamma,\Gamma)} \int_{\Gamma}^{\gamma} \left[H(\gamma,\iota)\Psi(\iota) - \frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)} \right] d\iota \leq w(\Gamma) \\ &- \lim_{\gamma\longrightarrow\infty} \inf \frac{1}{H(\gamma,\Gamma)} \int_{\Gamma}^{\gamma} \frac{\alpha(\beta-1)}{\beta} H(\gamma,\iota) w^{1+1/\alpha}(\iota) F(\iota) d\iota. \end{aligned} \tag{30}$$

In view of (27), it follows from (30) that

$$\begin{aligned} w(\Gamma) &\geq \Phi(\Gamma) + \lim_{\gamma\longrightarrow\infty} \inf \frac{1}{H(\gamma,\Gamma)} \\ &\cdot \int_{\Gamma}^{\gamma} \frac{\alpha(\beta-1)}{\beta} H(\gamma,\iota) w^{1+1/\alpha}(\iota) F(\iota) d\iota, \end{aligned} \tag{31}$$

for all $\gamma \geq \gamma_3$ for any $\beta > 1$. Then, for all $\Gamma \geq \gamma_3$,

$$W(\Gamma) \geq \phi(\Gamma), \tag{32}$$

$$\begin{aligned} &\lim_{\gamma\longrightarrow\infty} \inf \frac{1}{H(\gamma,\Gamma)} \int_{\gamma_4}^{\gamma} H(\gamma,\iota) w^{1+1/\alpha}(\iota) F(\iota) d\iota \\ &\leq \frac{\beta(w(\gamma_3) - \phi(\gamma_3))}{\alpha(\beta-1)} < \infty. \end{aligned} \tag{33}$$

Now, we claim that

$$\int_{\gamma_3}^{\infty} w^{1+1/\alpha}(\iota) F(\iota) d\iota < \infty. \tag{34}$$

If not, then

$$\int_{\gamma_3}^{\infty} w^{1+1/\alpha}(\iota) F(\iota) d\iota = \infty. \tag{35}$$

By (26), there exists a constant $\epsilon > 0$ such that

$$\inf_{\iota\geq\gamma_0} \left\{ \lim_{\gamma\longrightarrow\infty} \inf \frac{H(\gamma,\iota)}{H(\gamma,\gamma_0)} \right\} > \epsilon. \tag{36}$$

On the other hand, by virtue of (35) for any positive number δ, there exists $\gamma_4 > \gamma_3$ such that

$$\int_{\gamma_4}^{\gamma} w^{1+1/\alpha}(\iota) F(\iota) d\iota \geq \frac{\delta}{\epsilon} \quad \text{for all } \gamma \geq \gamma_5. \tag{37}$$

Using integration by parts and taking (37) into account, we conclude that for all $\gamma \geq \gamma_5$,

$$\begin{aligned} &\frac{1}{H(\gamma,\gamma_4)} \int_{\gamma_4}^{\gamma} H(\gamma,\iota) \omega^{1+1/\alpha}(\iota) F(\iota) d\iota \\ &= \frac{1}{H(\gamma,\gamma_4)} \int_{\gamma_4}^{\gamma} H(\gamma,\iota) d\left[\int_{\gamma_4}^{\iota} \omega^{1+1/\alpha}(s) F(s) ds \right] \\ &= \frac{1}{H(\gamma,\gamma_4)} \int_{\gamma_4}^{\gamma} \int_{\gamma_4}^{\iota} \left[\omega^{1+1/\alpha}(s) F(s) ds \right] \left[\frac{-\partial H(\gamma,\iota)}{\partial \iota} \right] d\iota \\ &\geq \frac{\delta}{\varepsilon} \frac{1}{H(\gamma,\gamma_4)} \int_{\gamma_5}^{\gamma} \left[\frac{-\partial H(\gamma,\iota)}{\partial \iota} \right] d\iota \\ &\geq \frac{\delta}{\varepsilon} \frac{H(\gamma,\gamma_5)}{H(\gamma,\gamma_4)} \geq \frac{\delta}{\varepsilon} \frac{H(\gamma,\gamma_5)}{H(\gamma,\gamma_0)}. \end{aligned} \tag{38}$$

It follows from (36) that

$$\lim_{t\longrightarrow\infty}\inf\frac{H(\gamma,\iota)}{H(\gamma,\gamma_0)}\geq\epsilon>0, \quad (39)$$

and hence, there exists $\gamma_6\geq\gamma_5$ such that

$$\frac{H(\gamma,\gamma_5)}{H(\gamma,\gamma_0)}\geq\epsilon \text{ for } \gamma\geq\gamma_6. \quad (40)$$

From the latter inequality and (32), we see that

$$\frac{1}{H(\gamma,\gamma_4)}\int_{\gamma_4}^{\gamma}H(\gamma,\iota)w^{1+1/\alpha}(\iota)F(\iota)d\iota\geq\delta, \text{ for } \gamma\geq\gamma_6. \quad (41)$$

Since δ is an arbitrary positive constant, we obtain

$$\lim_{\gamma\longrightarrow\infty}\inf\frac{1}{H(\gamma,\gamma_4)}\int_{\gamma_4}^{\gamma}H(\gamma,\iota)w^{1+1/\alpha}(\iota)F(\iota)d\iota=\infty, \quad (42)$$

which contradicts (33). Thus, (34) should hold and so by (32), we have

$$\int_{\gamma_4}^{\infty}F(\iota)\phi_+^{1+1/\alpha}(\iota)d\iota\leq\int_{\gamma_4}^{\infty}w^{1+1/\alpha}(\iota)F(\iota)d\iota<\infty, \quad (43)$$

which contradicts (28). This completes the proof of the theorem. □

Theorem 3. *Let all the conditions of Theorem 2 be satisfied except the condition (21) be replaced with*

$$\liminf_{\gamma\longrightarrow\infty}\frac{1}{H(\gamma,\Gamma)}\int_{\Gamma}^{\gamma}\left[H(\gamma,\iota)\Psi(\iota)-\frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)}\right]d\iota\geq\Phi(\Gamma), \quad (44)$$

then every solution of (1) is oscillatory.

Proof. The proof easily follows from the fact that

$$\begin{aligned}\Phi(\Gamma)&\leq\limsup_{\gamma\longrightarrow\infty}\frac{1}{H(\gamma,\Gamma)}\int_{\Gamma}^{\gamma}\left[H(\gamma,\iota)\Psi(\iota)-\frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)}\right]d\iota\\&\leq\liminf_{\gamma\longrightarrow\infty}\frac{1}{H(\gamma,\Gamma)}\int_{\Gamma}^{\gamma}\left[H(\gamma,\iota)\Psi(\iota)-\frac{\beta^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\Phi^{\alpha+1}(\gamma,\iota)}{H^{\alpha}(\gamma,\iota)F^{\alpha}(\iota)}\right]d\iota.\end{aligned} \quad (45)$$

Hence the proof is complete. □

3. Example

In this section, we provide an example to illustrate the main result.

Example 1. We consider the fourth-order neutral delay differential equation

$$\left[y'''(\gamma)^{1/3}\right]'+\frac{1}{2\gamma}(y'''(\gamma))^{1/3}+\gamma^2x^{1/3}(\gamma-1)=0,\quad \gamma\geq2, \quad (46)$$

where $y(\gamma)=x(\gamma)+1/2x(\gamma-2)$.

Hence, we have $n=4,\alpha=1/3,\zeta(\gamma)=\gamma-2,\zeta^{-1}(\gamma)=\gamma+1,\zeta^{-1}(\zeta^{-1}(\gamma))=\gamma+2,\quad\sigma(\gamma)=\gamma-1,p(\gamma)=1,q(\gamma)=1/2\gamma,r(\gamma)=\gamma^2$.

Simple calculation shows that by taking $\rho(\gamma)=\gamma$, that $B(\gamma)=1/2\gamma,F(\gamma)=\mu/2\gamma,\Phi(\gamma)=1/8\gamma^3(\gamma-1/\gamma)$.

Letting $H(\gamma,\iota)=(\gamma-\iota)^2$, we see that $H\epsilon P$, $h(\gamma,\iota)=2$ and

$$\Phi(\gamma,\iota)=\frac{\gamma^2}{2\iota}-3\iota+\frac{5\iota}{2}. \quad (47)$$

We choose $\Gamma=2$, and the condition (46) becomes

$$\lim_{\gamma\longrightarrow\infty}\sup\frac{1}{(\gamma-2)^2}\int_2^{\gamma}\left[\frac{(\gamma-2)^2\iota^3}{8}\left(\frac{\iota-1}{\iota}\right)-2\left(\frac{3}{4}\right)^{4/3}\frac{\left(\gamma^2/2\iota-3\gamma+5\iota/2\right)^{4/3}}{(\gamma-\iota)^{2/3}\mu}\iota\right]=\infty. \quad (48)$$

Hence, all conditions of Theorem 1 are satisfied, and therefore, every solution of (46) is oscillatory.

It is noted that the conditions of Corollary 1 are not satisfied, and hence, Theorem 1 improves Corollary 1.

4. Conclusion

In this paper, we have obtained criteria for the oscillation of all solutions of (1) using the Philos type method. An example is provided to illustrate the main result. It will be of interest to study the equation under the condition [14].

$$\int_{\gamma_0}^{\infty}\left[\frac{1}{r(\gamma)}\exp\left(-\int_{\gamma_0}^{\gamma}\frac{p(\iota)}{r(\iota)}dt\right)\right]^{1/\alpha}dr<\infty. \quad (49)$$

It is also interesting to extend the results of this paper to stochastic type neutral differential equations and differential equations with impulses, see [15, 16].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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