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Research Article

Boundary Value Problem for the Langevin Equation and Inclusion with the Hilfer Fractional Derivative

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In this work, we discuss the existence and uniqueness of solution for a boundary value problem for the Langevin equation and inclusion with the Hilfer fractional derivative. First of all, we give some definitions, theorems, and lemmas that are necessary for the understanding of the manuscript. Second of all, we give our first existence result, based on Krasnoselskii's fixed point, and to deal with the uniqueness result, we use Banach's contraction principle. Third of all, in the inclusion case, to obtain the existence result, we use the Leray–Schauder alternative. Last but not least, we give an illustrative example.

1. Introduction

Fractional derivatives give an excellent description of memory and hereditary properties of different processes. Properties of the fractional derivatives make the fractional-order models more useful and practical than the classical integral-order models. Several researchers in the recent years have employed the fractional calculus as a way of describing natural phenomena in different fields such as physics, biology, finance, economics, and bioengineering (for more details see [1–10] and many other references).

With the recent outstanding development in fractional differential equations, the Langevin equation has been considered a part of fractional calculus, and thus, important results have been elaborated [11–15].

An equation of the form $md^2x/dt^2 = \lambda dx/dt + \eta(t)$ is called Langevin equation, introduced by Paul Langevin in 1908. The Langevin equation is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [8]. For some new developments on the fractional Langevin equation, see, for example, [16–18].

In this study, we investigate the existence and uniqueness criteria for the solutions of the following nonlocal boundary value problem:

$$\left\{ {}^{H}D^{\alpha_1,\beta_1}\left({}^{H}D^{\alpha_2,\beta_2} + \lambda\right)x(t) = f(t, x(t)), \quad a \le t \le b, x(a) = 0, x(b) = \sum_{i=1}^{n} \mu_i \left(I^{\nu_i}(x)\right)(\eta), \quad a < \eta < b, \right. \tag{1}$$

where ${}^{H}D^{\alpha_i,\beta_i}, i = 1, 2$ is the Hilfer fractional derivative of order α_i, $0 < \alpha_i < 1$, and parameter β_i, $0 \le \beta_i \le 1$, $i = 1, 2$, $1 < \alpha_1 + \alpha_2 \le 2$, $\lambda \in \mathbb{R}$, $a \ge 0$, I^{ν_i} is the Riemann–Liouville fractional integral of order $\nu_i > 0$, $\mu_i \in \mathbb{R}$, $i = 1, 2$, and $f\colon [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

In order to study problem (1), we transform it into a fixed-point problem and then use Krasnoselskii's fixed-point theorem to prove the existence results.

As a second problem, we study the multivalued case of (1) by considering the inclusion problem:

$$\left\{ {}^{H}D^{\alpha_1,\beta_1}\left({}^{H}D^{\alpha_2,\beta_2}+\lambda\right)x(t)\in F(t,x(t)),\quad a\le t\le b, x(a)=0, x(b)=\sum_{i=1}^{n}\mu_i\left(I^{\gamma_i}(x)\right)(\eta),\quad a<\eta<b, \right. \tag{2}$$

where $F\colon [a,b]\times\mathbb{R}\longrightarrow\mathscr{A}(\mathbb{R})$ is a multivalued map ($\mathscr{A}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

We prove the existence of solution for problem (2) by applying the nonlinear alternative of the Leray–Schauder [19]. Finally, in the last part, we give an example to support our study.

2. Preliminaries

Let us recall some basic definitions and notations of fractional calculus and multivalued analysis which are needed throughout this study.

2.1. Fractional Calculus

Definition 1 (see [4]). The Riemann–Liouville fractional integral of order $\alpha>0$ for a continuous function $f\colon [a,\infty)\longrightarrow\mathbb{R}$ can be defined as

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\mathrm{d}s, \tag{3}$$

provided that the right-hand side exists on (a,∞).

Definition 2 (see [4]). The Riemann–Liouville fractional derivative of order $\alpha>0$ of a continuous function f is defined by

$${}^{RL}D^{\alpha}f(t)\coloneqq D^{n}I^{n-\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{\mathrm{d}t}\right)^{n}\int_0^t (t-s)^{n-\alpha-1}f(s)\mathrm{d}s, \tag{4}$$

where $n-1<\alpha<n,\quad n=[\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α .

Definition 3 (see [4]). The Caputo fractional derivative of order $\alpha>0$ of a continuous function f is defined by

$${}^{C}D^{\alpha}f(t)\coloneqq I^{n-\alpha}D^{n}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\left(\frac{d}{\mathrm{d}t}\right)^{n}f(s)\mathrm{d}s, \tag{5}$$

where $n-1<\alpha<n,\quad n=[\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α.

Definition 4. Hilfer fractional derivative [3].

The Hilfer fractional derivative of order α and parameter β of a function (also known as the generalized Riemann–Liouville fractional derivative) is defined by

$${}^{H}D^{\alpha,\beta}f(t)=I^{\beta(n-\alpha)}D^{n}I^{(1-\beta)(n-\alpha)}f(t), \tag{6}$$

where $n-1<\alpha<n$, $0\le\beta\le1$, $t>a$, and $D=(d/\mathrm{d}t)$.

Remark 1. When $\beta=0$, the Hilfer fractional derivative corresponds to the Riemann–Liouville fractional derivative:

$${}^{H}D^{\alpha,0}f(t)=D^{n}I^{(n-\alpha)}f(t). \tag{7}$$

When $\beta=1$, the Hilfer fractional derivative corresponds to the Caputo fractional derivative:

$${}^{H}D^{\alpha,1}f(t)=I^{(n-\alpha)}D^{n}f(t). \tag{8}$$

The following lemma plays a fundamental role in establishing the existence results for the given problem.

Lemma 1 (see [3]). *Let $f\in L(a,b)$, $n-1<\alpha\le n$, $n\in\mathbb{N}$, $0\le\beta\le1$, and $I^{(1-\beta)(n-\alpha)}f\in AC^{k}[a,b]$; then,*

$$\left(I^{\alpha H}D^{\alpha,\beta}f\right)(t)=f(t)-\sum_{k=1}^{n-1}\frac{(t-a)^{k-(n-\alpha)(1-\beta)}}{\Gamma(k-(n-\alpha)(1-\beta)+1)}\cdot\lim_{t\longrightarrow +a}\frac{d^{k}}{\mathrm{d}t^{k}}\left(I^{(1-\beta)(n-\alpha)}f\right)(t). \tag{9}$$

The following lemma deals with a linear variant of the boundary value problem (1).

Lemma 2. *Let $a\ge0$, $0<\alpha_i<1$, $\gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i$, $i=1,2$, $1<\alpha_1+\alpha_2\le2$, and $h\in C([a,b],\mathbb{R})$. Then, the function x is a solution of the boundary value problem:*

$$\left\{ {}^{H}D^{\alpha_1,\beta_1}\left({}^{H}D^{\alpha_2,\beta_2}+\lambda\right)x(t)=h(t),\quad a\le t\le b, x(a)=0, x(b)=\sum_{i=1}^{n}\mu_i\left(I^{\gamma_i}(x)\right)(\eta)\mathrm{d}s,\quad a<\eta<b, \right. \tag{10}$$

if and only if

$$x(t)=I^{\alpha_1+\alpha_2}h(t)-\lambda I^{\alpha_2}x(t)+\frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\left[I^{\alpha_1+\alpha_2}h(b)-\lambda I^{\alpha_2}x(b)-\sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i}h(\eta)+\lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i}x(\eta)\right], \tag{11}$$

where

$$\Lambda=\frac{\sum_{i=1}^{n}\mu_i(\eta-a)^{\gamma_1+\alpha_2+\nu_i-1}}{\Gamma(\gamma_1+\alpha_2+\nu_i)}-\frac{(b-a)^{\gamma_1+\alpha_2-1}}{\Gamma(\gamma_1+\alpha_2)}\neq 0. \tag{12}$$

Proof. Applying the Riemann–Liouville fractional integral of order α_1 to both sides of (10), we obtain by using Lemma 1

$${}^{H}D^{\alpha_2,\beta_2}x(t)+\lambda x(t)=I^{\alpha_1}h(t)+\frac{c_0}{\Gamma(\gamma_1)}(t-a)^{\gamma_1-1}, \tag{13}$$

where c_0 is the constant and $\gamma_1=\alpha_1+\beta_1-\alpha_1\beta_1$. Applying the Riemann–Liouville fractional integral of order α_2 to both sides of (13), we obtain by using Lemma 1

$$x(t)=I^{\alpha_1+\alpha_2}h(t)-\lambda I^{\alpha_2}x(t)+\frac{c_0}{\Gamma(\gamma_1+\alpha_2)}(t-a)^{\gamma_1+\alpha_2-1}+\frac{c_1}{\Gamma(\gamma_2)}(t-a)^{\gamma_2-1}. \tag{14}$$

From using the boundary condition $x(a)=0$ in (14), we obtain that $c_1=0$. Then, we get

$$x(t)=I^{\alpha_1+\alpha_2}h(t)-\lambda I^{\alpha_2}x(t)+\frac{c_0}{\Gamma(\gamma_1+\alpha_2)}(t-a)^{\gamma_1+\alpha_2-1}. \tag{15}$$

From using the boundary condition $x(b)=\sum_{i=1}^{n}\mu_i(I^{\nu_i}(x))(\eta)$, in (15), we find

$$c_0=\frac{1}{\Lambda}\left[I^{\alpha_1+\alpha_2}h(b)-\lambda I^{\alpha_2}x(b)-\sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i}h(\eta)+\lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i}x(\eta)\right]. \tag{16}$$

Substituting the value of (c_0) in (13), we obtain the solution (11).

Conversely, suppose that x is the solution of the fractional integral (11). By applying fractional derivetive ${}^{H}D^{\alpha_2,\beta_2}$ on both sides of (11) and then applying fractional derivative, we obtain that

$${}^{H}D^{\alpha_1,\beta_1}\left({}^{H}D^{\alpha_2,\beta_2}\right)x(t)=h(t)-\lambda^{H}D^{\alpha_1,\beta_1}x(t). \tag{17}$$

It follows that

$${}^{H}D^{\alpha_1,\beta_1}\left({}^{H}D^{\alpha_2,\beta_2}+\lambda\right)x(t)=h(t),\quad a\le t\le b. \tag{18}$$

Now, we will prove that x satisfies the boundary conditions; for that, we have $x(a)=0$, and from (11), we have

$$\begin{aligned}\sum_{i=1}^{n}\mu_i(I^{\nu_i}(x))(\eta)&=\sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i}h(\eta)-\lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i}x(\eta)+\sum_{i=1}^{n}\mu_i\frac{(\eta-a)^{\gamma_1+\alpha_2+\nu_i-1}}{\Gamma(\gamma_1+\alpha_2+\nu_i)}\\&\quad\times\frac{1}{\Lambda}\left[I^{\alpha_1+\alpha_2}h(b)-\lambda I^{\alpha_2}x(b)-\sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i}h(\eta)+\lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i}x(\eta)\right].\end{aligned} \tag{19}$$

By (12), we get

$$\begin{aligned}\sum_{i=1}^{n}\mu_i(I^{\nu_i}(x))(\eta)&=\sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i}h(\eta)\\&\quad-\lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i}x(\eta)+\left(1+\frac{(b-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\right)\\&\quad\times\left[I^{\alpha_1+\alpha_2}h(b)-\lambda I^{\alpha_2}x(b)-\sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i}h(\eta)+\lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i}x(\eta)\right]\\&=I^{\alpha_1+\alpha_2}h(b)-\lambda I^{\alpha_2}x(b)\\&\quad+\frac{(b-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\left[I^{\alpha_1+\alpha_2}h(b)-\lambda I^{\alpha_2}x(b)-\sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i}h(\eta)+\lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i}x(\eta)\right]\\&=x(b).\end{aligned} \tag{20}$$

This completes the proof. □

2.2. Multivalued Analysis. For a normed space $(X, \|.\|)$, we define

$$\mathscr{A}(X) = \{Y \subset X\colon Y \neq \varnothing\},$$
$$\mathscr{A}_{c,cp}(X) = \{Y \subset X\colon Y \text{ is convex and compact}\}. \tag{21}$$

For the basic concepts of multivalued analysis, we refer to ([2, 3]).

Definition 5. A multivalued map $F\colon [a,b] \times \mathbb{R} \longrightarrow \mathscr{A}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longrightarrow F(t,x)$ is measurable for each $x \in \mathbb{R}$

(ii) $x \longrightarrow F(t,x)$ is upper semicontinuous for almost all $t \in [a,b]$

Furthermore, a Carathéodory function F is called $\mathbb{L}^1$-Carathéodory if

(iii) For each $\rho > 0$, there exists $\varphi_\rho \in \mathbb{L}^1([a,b];\mathbb{R})$, such that

$$\|F(t,x)\| = \sup\{|v|\colon v \in F(t,x)\} \leq \varphi_\rho(t), \tag{22}$$

for all $x \in \mathbb{R}$ with $\|x\| \leq \rho$ and for a.e. $t \in [a,b]$.

Fixed-point theorems play a major role in establishing the existence theory for problem (1) and problem (2). We collect here some well-known fixed-point theorems used in this study.

Theorem 1 Krasnoselskii's fixed-point theorem [19]. *Let $\mathscr{M}$ be a closed, bounded, convex, and nonempty subset of a Banach space. Let $\mathscr{A}, \mathscr{B}$ be the operators such that*

(i) $\mathscr{A}x + \mathscr{B}y \in \mathscr{M}$ whenever $x, y \in \mathscr{M}$

(ii) $\mathscr{A}$ is compact and continuous

(iii) $\mathscr{B}$ is contraction mapping

Then, there exists $z \in \mathscr{M}$, such that $z = \mathscr{A}z + \mathscr{B}z$.

Theorem 2 (Leray–Schauder nonlinear alternative [19]). *Let X be a Banach space, C a closed, convex subset of X, U an open subset of C, and $0 \in U$. Suppose that $F\colon \overline{U} \longrightarrow C$ is a continuous, compact ($F(\overline{U})$ is a relatively compact subset of C) map. Then, either*

(i) F has a fixed point in $\overline{U}$ or

(ii) There exists $x \in \partial U$ (the boundary of U in C) and $\theta \in (0,1)$ with $x = \theta F(x)$.

3. Existence and Uniqueness Results for Problem (1)

In this section, we deal with the existence and uniqueness of solution for the boundary value problem (1).

By Lemma 2, we define an operator $\mathscr{A}\colon \mathscr{C} \longrightarrow \mathscr{C}$ by

$$\begin{aligned}(\mathscr{A}x)(t) &= I^{\alpha_1+\alpha_2} f(t,x(t)) - \lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\ &\cdot \left[I^{\alpha_1+\alpha_2} f(b,x(b)) - \lambda I^{\alpha_2} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} f(\eta, x(\eta)) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x(\eta) \right],\end{aligned} \tag{23}$$

where $\mathscr{C} = C([a,b],\mathbb{R})$ denotes the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm $\|x\| \coloneqq \sup\{|x(t)|; t \in [a,b]\}$. We will show that the boundary value problem (1) has a solution if and only if the operator $\mathscr{A}$ has a fixed point.

To simplify the computations, we use the following notations:

$$\Omega_1 = \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \left[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} \right], \tag{24}$$

$$\Omega_2 = |\lambda| \left\{ \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)} \right] \right\}. \tag{25}$$

Our first result is an existence result, based on well-known Krasnoselskii's fixed-point theorem.

Theorem 3 *Assume that*

(H1) $f\colon [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, such that $(f(t,x)) \leq \varphi(t)$, $\forall (t,x) \in [a,b] \times \mathbb{R}$, with $\varphi \in C([a,b];\mathbb{R})$.

(H2) $\Omega_2 < 1$, where Ω_2 is given by (19)

Then, there exists at least one solution for the boundary value problem (1) on $[a,b]$.

Proof. We will show that the operator $\mathscr{A}$ defined by (17) satisfies the assumptions of Krasnoselskii's fixed-point theorem. We split the operator $\mathscr{A}$ into the sum of two operators $\mathscr{A}_1$ and $\mathscr{A}_2$ on the closed ball $\mathscr{B}_\rho = \{x \in \mathscr{C}; \|x\| \le \rho\}$ with $\rho \ge \|\varphi\|\Omega_1/1-\Omega_2$, $\sup_{t\in[a,b]}\varphi(t) = \|\varphi\|$, where

$$(\mathscr{A}_1 x)(t) = I^{\alpha_1+\alpha_2} f(t,x(t)) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\left[I^{\alpha_1+\alpha_2} f(b,x(b)) - \sum_{i=1}^{n}\mu_i I^{\alpha_1+\alpha_2+\nu_i} f(\eta,x(\eta))\right], \tag{26}$$

$$(\mathscr{A}_1 x)(t) = -\lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)}\left[-\lambda I^{\alpha_2} x(b) + \lambda\sum_{i=1}^{n}\mu_i I^{\alpha_2+\nu_i} x(\eta)\right]. \tag{27}$$

For any $x, y \in \mathscr{B}_\rho$, we have

$$\begin{aligned}
(\mathscr{A}_1 x)(t) + (\mathscr{A}_2 y)(t) \le & \sup_{t\in[a,b]} \\
& \cdot \Bigg\{ I^{\alpha_1+\alpha_2}|f(t,x(t))| + |\lambda| I^{\alpha_2}|y(t)| + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\
& \times\left[I^{\alpha_1+\alpha_2}|f(b,x(b))| + \sum_{i=1}^{n}|\mu_i| I^{\alpha_1+\alpha_2+\nu_i}|f(\eta,x(\eta))| + |\lambda| I^{\alpha_2}|y(b)| + |\lambda|\sum_{i=1}^{n}|\mu_i| I^{\alpha_2+\nu_i}|y(\eta)|\right]\Bigg\} \\
\le & \|\varphi\|\left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n}|\mu_i|\frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)}\right]\right\} \\
& + \|y\||\lambda|\left\{\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\mu_i|\frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)}\right]\right\}, \quad \le \|\varphi\|\Omega_1 + \rho\Omega_2, \le \rho,
\end{aligned} \tag{28}$$

and hence, $\|\mathscr{A}_1 x + \mathscr{A}_2 y\| \le \rho$, which implies that $\mathscr{A}_1 x + \mathscr{A}_2 y \in \mathscr{B}_\rho$.

Next, by using $(H2)$, we show that $\mathscr{A}_2$ is a contraction mapping. Let $x, y \in \mathscr{C}$, for $t \in [a,b]$,

$$\begin{aligned}
\left|(\mathscr{A}_2 x)(t) + (\mathscr{A}_2 y)(t)\right| \le & |\lambda| I^{\alpha_2}|x(t) - y(t)| + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\
& \times\left[|\lambda| I^{\alpha_2}|x(b) - y(b)| + |\lambda|\sum_{i=1}^{n}|\mu_i| I^{\alpha_2+\nu_i}|x(\eta) - y(\eta)|\right], \\
\le & \|x-y\||\lambda|\left\{\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\mu_i|\frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)}\right]\right\} \\
\le & \Omega_2\|x-y\|.
\end{aligned} \tag{29}$$

This shows that $\|\mathscr{A}_2 x + \mathscr{A}_2 y\| \le \Omega_2\|x-y\|$; then, by using $(H2)$, $\mathscr{A}_2$ is a contraction mapping.

The operator $\mathscr{A}_1$ is continuous, since f is continuous. It is uniformly bounded on $\mathscr{B}_\rho$ as

$$\left\|\mathscr{A}_1 x\right\| \le \Omega_1\|\varphi\|. \tag{30}$$

Now, we prove that the operator $\mathscr{A}_1$ is compact. Setting $\sup_{(t,x)\in[a,b]\times\mathscr{B}_\rho}|f(t,x)| = \overline{f} < \infty$, and let $t_1, t_2 \in [a,b]$, $t_1 < t_2$, we obtain

$$
\begin{aligned}
&\left|(\mathscr{A}_1 x)(t_2) - (\mathscr{A}_1 x)(t_1)\right| = \Bigg| I^{\alpha_1+\alpha_2} f(t_2, x(t_2)) + \frac{(t_2-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\
&\left[I^{\alpha_1+\alpha_2} f(b, x(b)) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} f(\eta, x(\eta)) \right] - I^{\alpha_1+\alpha_2} f(t_1, x(t_1)) \\
&-\frac{(t_1-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \left[I^{\alpha_1+\alpha_2} f(b, x(b)) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} f(\eta, x(\eta)) \right], \Bigg| \\
&\le \frac{\overline{f}}{\Gamma(\alpha_1+\alpha_2)} \left| \int_a^{t_1} \left((t_2-s)^{\alpha_1+\alpha_2-1} - (t_1-s)^{\alpha_1+\alpha_2-1} ds \right) + \int_{t_1}^{t_2} (t_2-s)^{\alpha_1+\alpha_2-1} ds \right| \\
&\quad + \frac{(t_2-a)^{\gamma_1+\alpha_2-1} - (t_1-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\overline{f} \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \overline{f} \sum_{i=1}^{n} |\mu_i| \frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} \right],
\end{aligned} \tag{31}
$$

where the right-hand side tends to zero as $t_2 - t_1 \longrightarrow 0$, independently of $x \in \mathscr{B}_\rho$. Then, $\mathscr{A}_1$ is equicontinuous, and hence, $\mathscr{A}_1$ is relatively compact on $\mathscr{B}_\rho$. By the Arzelá–Ascoli theorem, $\mathscr{A}_1$ is compact on $\mathscr{B}_\rho$. It follows by Krasnoselskii's fixed-point theorem that problem (1) has at least one solution on $[a, b]$.

To deal with the uniqueness of solution for our problem (1), we use Banach's contraction principle. □

Theorem 4. *Assume that* $|f(t,x) - f(t,y)| \le L|x-y|$, $L > 0$, *for each* $t \in [a,b]$ *and* $x, y \in \mathbb{R}$.

If $L\Omega_1 + \Omega_2 < 1$, where Ω_1, Ω_2 are, respectively, given by (24) and (25), then problem (1) has a unique solution on $[a, b]$.

Proof. Consider the operator $\mathscr{A}$ defined in (17). The problem (1) is then transformed into a fixed-point problem $x = \mathscr{A}x$. By using Banach contraction principle, we will show that $\mathscr{A}$ has a unique fixed point.

We set $\sup_{t\in[a,b]} = |f(t,0)| = M < \infty$ and choose $r > 0$, such that

$$
r \ge \frac{M\Omega_1}{1 - L\Omega_2 - \Omega 1}. \tag{32}
$$

Now, we show that $\mathscr{A}\mathscr{B}_r \subset \mathscr{B}_r$, where $\mathscr{B}_r = \{x \in \mathscr{C}([a, b], \mathbb{R}); \|x\| \le r\}$. For any $x \in \mathscr{B}_r$, we have

$$
\begin{aligned}
|(\mathscr{A}x)(t)| \le& \sup_{t\in[a,b]} \\
&\left\{ I^{\alpha_1+\alpha_2}|f(t,x(t))| + |\lambda| |I^{\alpha_2}|x(t)| + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[I^{\alpha_1+\alpha_2}|f(b,x(b))| + \sum_{i=1}^{n} |\mu_i| I^{\alpha_1+\alpha_2+\nu_i}|f(\eta,x(\eta))| + |\lambda||x(b)| + |\lambda| \sum_{i=1}^{n} |\mu_i| I^{\alpha_2+\nu_i}|x(\eta)| \right] \right\} \\
&\le (L\|x\| + M) \left\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} \right] \right\} \\
&\quad + \|x\||\lambda| \left\{ \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)} \right] \right\} \\
&\le (L\|x\| + M)\Omega_1 + \|x\|\Omega_2, \quad \le (Lr + M)\Omega_1 + r\Omega_2,
\end{aligned} \tag{33}
$$

which implies that $\mathscr{A}\mathscr{B}_r \subset \mathscr{B}_r$.

Next, let $x, y \in \mathscr{C}([a,b],\mathbb{R})$. Then, for $t \in [a,b]$, we have

$$
\begin{aligned}
&|(\mathscr{A}x)(t) - (\mathscr{A}y)(t)| \\
&\le \left\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} \right] \right\} L\|x-y\| + |\lambda| \\
&\cdot \left\{ \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n} |\mu_i| \frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)} \right] \right\} \\
&\cdot \|x-y\|, \quad \le (L\Omega_1 + \Omega_2)\|x-y\|,
\end{aligned}
\tag{34}
$$

which implies $\|(\mathscr{A}x)(t) - (\mathscr{A}y)(t)\| \le (L\Omega_1 + \Omega_2)\|x-y\|$. As $L\Omega_1 + \Omega_2 < 1$, $\mathscr{A}$ is a contraction. Therefore, by Banach's fixed-point theorem, the operator $\mathscr{A}$ has a fixed point which is indeed a unique solution of problem (1). □

4. Existence Results for Problem (2)

Definition 6. A continuous function x is said to be a solution of problem (2) if $x(a) = 0, x(b) = \sum_{i=1}^{n} \mu_i (I^{\nu_i}(x))(\eta)$, and there exists a function $v \in \mathbb{L}^1([a,b],\mathbb{R})$ with $v \in F(t, x(t))$, a.e., on $[a,b]$, such that

$$
\begin{aligned}
x(t) = I^{\alpha_1+\alpha_2} v(t) - \lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\
\cdot \left[I^{\alpha_1+\alpha_2} v(b) - \lambda I^{\alpha_2} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} v(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x(\eta) \right].
\end{aligned}
\tag{35}
$$

For each $x \in \mathscr{C}[a,b], \mathbb{R}$, define the set of selections of F by

$$
\mathcal{S}_{F,x} := \left\{ v \in \mathbb{L}^1([a,b],\mathbb{R}) \colon v \in F(t,x(t)) \text{ on } [a,b] \right\}. \tag{36}
$$

Lemma 3 (see [5]). *Let X be a Banach space, and $F\colon [a,b] \times \mathbb{R} \longrightarrow \mathscr{A}_{c,cp}$ be a $\mathbb{L}^1$-Carathéodory multivalued map. Let Υ be a linear continuous mapping from $L^1([a,b],X)$ to $\mathscr{C}([a,b],X)$. Then, the operator is*

$\Upsilon \circ \mathcal{S}_F \colon \mathscr{C}([a,b],X) \longrightarrow \mathscr{A}_{c,cp}(\mathscr{C}([a,b],X)); x \longrightarrow (\Upsilon \circ \mathcal{S}_F)(x) = \Upsilon(\mathcal{S}_{F,x})$ is a closed graph operator in $\mathscr{C}([a,b],X) \times \mathscr{C}([a,b],X)$.

Our second existence result is based on the nonlinear alternative of the Leray–Schauder for multivalued maps [19].

Theorem 5. *Suppose that (H2) and the following assumptions hold:*

(H3) $F\colon [a,b] \times \mathbb{R} \longrightarrow \mathscr{A}_{c,cp}(\mathbb{R})$ is $\mathbb{L}^1$-Carathéodory and has nonempty compact and convex values, and for each fixed $x \in \mathscr{C}([a,b],\mathbb{R})$, the set $\mathcal{S}_{F,x} = \{v \in \mathbb{L}^1([a,b],X) \colon v(t) \in F(t,x(t)); t \in [a,b]\}$ is nonempty.

(H4) $\|F(t,x)\| := \sup\{|v| \colon v \in F(t,x)\} \le p(t)\Psi(\|x\|)$ for all $t \in [a,b]$ and all $x \in \mathscr{C}([a,b],X)$, where $p \in \mathbb{L}^1([a,b],\mathbb{R}^+)$, and $\Psi\colon \mathbb{R}^+ \longrightarrow [0,+\infty)$ is a continuous and nondecreasing function.

(H5) There exists a constant $M > 0$, such that

$$
\frac{(1-\Omega_2)M}{\|p\|\Psi(M)\Omega_1} > 1, \tag{37}
$$

where Ω_1, Ω_2 are, respectively, given by (24) and (25).

Then, there exists at least one solution for problem (2) on $[a,b]$.

Proof. Let us introduce the operator $\mathscr{A}\colon \mathscr{C}([a,b],\mathbb{R}) \longrightarrow \mathscr{A}_{c,cp}([a,b],\mathbb{R})$, in order to transform problem (2) into a fixed-point problem:

$$\mathscr{A}(x) := \Bigg\{ h \in \mathscr{C}([a,b],\mathbb{R}) \colon h(t) = \Bigg\{ I^{\alpha_1+\alpha_2} v(t) - \lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda \Gamma(\gamma_1+\alpha_2)} \times \Bigg[I^{\alpha_1+\alpha_2} v(b) - \lambda I^{\alpha_2} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} v(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x(\eta), t \in [a,b], v \in \mathscr{S}_{F,x} \Bigg] \Bigg\}. \tag{38}$$

We will show that the operator $\mathscr{A}$ satisfies all conditions of the Leray–Schauder nonlinear alternative [19], and we give the proof in several steps: □

Step 1. $\mathscr{A}(x)$ is convex for each $x \in \mathscr{C}([a,b],\mathbb{R})$.

Indeed, if h_1, h_2 belong to $\mathscr{A}(x)$, then there exist $v_1, v_2 \in \mathscr{S}_{F,x}$, such that for each $t \in [a,b]$, we have

$$h_j(t) = I^{\alpha_1+\alpha_2} v_j(t) - \lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda \Gamma(\gamma_1+\alpha_2)} \left[I^{\alpha_1+\alpha_2} v_j(b) - \lambda I^{\alpha_2} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} v_j(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x(\eta) \right], \tag{39}$$

for $j = 1, 2$. Let $0 \le k \le 1$; then, for each $t \in [0,1]$, we have

$$k h_1(t) + (1-k) h_2(t) = I^{\alpha_1+\alpha_2} [k v_1(s) + (1-k) v_2(s)] - \lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda \Gamma(\gamma_1+\alpha_2)} \times \left[I^{\alpha_1+\alpha_2} [k v_1(s) + (1-k) v_2(s)] - \lambda I^{\alpha_2} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} [k v_1(\eta) + (1-k) v_2(\eta)] + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x(\eta) \right]. \tag{40}$$

Thus, $k v_1 + (1-k) v_2 \in \mathscr{A}(x)$ (because $S_{F,x}$ is convex); then, $\mathscr{A}(x)$ is convex for each $x \in \mathscr{C}([a,b],\mathbb{R})$.

Step 2. $\mathscr{A}(x)$ maps bounded set into bounded set in $\mathscr{C}([a,b],\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant l, such that for each $h \in \mathscr{A}(x)$, $x \in \mathscr{B}_\rho = \{x \in \mathscr{C}[a,b],\mathbb{R} \colon |x| \le \rho\}$, and we have $\|h\| \le l$.

If $h \in \mathscr{A}(x)$, then there exist $v \in \mathscr{S}_{F,x}$, such that

$$h(t) = I^{\alpha_1+\alpha_2} v(t) - \lambda I^{\alpha_2} x(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda \Gamma(\gamma_1+\alpha_2)} \cdot \left[I^{\alpha_1+\alpha_2} v(b) - \lambda I^{\alpha_2} x(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} v(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x(\eta) \right]. \tag{41}$$

Then, for every $t \in [a,b]$, we have

$$\begin{aligned}
|h(x)(t)| \leq & \sup_{t\in[a,b]} \\
& \left\{ I^{\alpha_1+\alpha_2}|v(t)| + |\lambda| I^{\alpha_2}|x(t)| + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[I^{\alpha_1+\alpha_2}|v(b)| + \sum_{i=1}^{n}|\mu_i| I^{\alpha_1+\alpha_2+\nu_i}|v(\eta)| + |\lambda| I^{\alpha_2}|x(b)| + |\lambda| \sum_{i=1}^{n}|\mu_i| I^{\alpha_2+\nu_i}|x(\eta)| \right] \right\} \\
& \leq \|p\|\Psi(\|x\|) \left\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \sum_{i=1}^{n}|\mu_i| \frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} \right] \right\} \\
& + \|x\||\lambda| \left\{ \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{(b-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \times \left[\frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \sum_{i=1}^{n}|\mu_i| \frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)} \right] \right\} \\
& \leq \|p\|\Psi(\|x\|)\Omega_1 + \|x\|\Omega_2, \quad \leq \|p\|\Psi(\rho)\Omega_1 + \rho\Omega_2.
\end{aligned} \tag{42}$$

Then,

$$\|h\| \leq \|p\|\Psi(\rho)\Omega_1 + \rho\Omega_2 := l, \tag{43}$$

where Ω_1, Ω_2 are, respectively, given by (24) and (25).

Step 3. $\mathscr{A}$ maps bounded set into equicontinuous sets of $\mathscr{C}([a,b],\mathbb{R})$.

Let $t_1, t_2 \in [a,b]; t_1 < t_2$, and $x \in \mathscr{B}_\rho$, where $\mathscr{B}_\rho$, as above, is a bounded set of $\mathscr{C}([a,b],\mathbb{R})$; for each $x \in \mathscr{B}_\rho$ and $h \in \mathscr{A}(x)$, there exist $v \in \mathscr{S}_{F,x}$; then, we obtain

$$\begin{aligned}
|h(t_2) - h(t_1)| \leq & \frac{1}{\Gamma(\alpha_1+\alpha_2)} \left| \int_a^{t_1} \left((t_2-s)^{\alpha_1+\alpha_2-1} - (t_1-s)^{\alpha_1+\alpha_2-1} \right) v(s) \mathrm{d}s + \int_{t_1}^{t_2} (t_2-s)^{\alpha_1+\alpha_2-1} v(s) \mathrm{d}s \right| \\
& + \frac{|\lambda|}{\Gamma(\alpha_2)} \left| \int_a^{t_1} \left((t_2-s)^{\alpha_2-1} - (t_1-s)^{\alpha_2-1} \right) x(s) \mathrm{d}s + \int_{t_1}^{t_2} (t_2-s)^{\alpha_2-1} x(s) \mathrm{d}s + \frac{(t_2-a)^{\gamma_1+\alpha_2-1} - (t_1-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \right| \\
& \times \left[\|v(s)\| \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \|v(s)\| \sum_{i=1}^{n}|\mu_i| \frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} + \|x(b)\||\lambda| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \|x(\eta)\||\lambda| \sum_{i=1}^{n}|\mu_i| \frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)} \right] \\
& \leq \frac{\|p\|\Psi(\rho)}{\Gamma(\alpha_1+\alpha_2)} \left| \int_a^{t_1} \left((t_2-s)^{\alpha_1+\alpha_2-1} - (t_2-s)^{\alpha_1+\alpha_2-1} \right) \mathrm{d}s + \int_{t_1}^{t_2} (t_2-s)^{\alpha_1+\alpha_2-1} \mathrm{d}s \right| \\
& + \frac{\rho|\lambda|}{\Gamma(\alpha_2)} \left| \int_a^{t_1} \left((t_2-s)^{\alpha_2-1} - (t_2-s)^{\alpha_2-1} \right) \mathrm{d}s + \int_{t_1}^{t_2} (t_2-s)^{\alpha_2-1} \mathrm{d}s \right| + \frac{(t_2-a)^{\gamma_1+\alpha_2-1} - (t_2-a)^{\gamma_1+\alpha_2-1}}{|\Lambda|\Gamma(\gamma_1+\alpha_2)} \\
& \times \left[\|p\|\Psi(\rho) \frac{(b-a)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \|p\|\Psi(\rho) \sum_{i=1}^{n}|\mu_i| \frac{(\eta-a)^{\alpha_1+\alpha_2+\nu_i}}{\Gamma(\alpha_1+\alpha_2+\nu_i+1)} + \rho|\lambda| \frac{(b-a)^{\alpha_2}}{\Gamma(\alpha_2+1)} + \rho|\lambda| \sum_{i=1}^{n}|\mu_i| \frac{(\eta-a)^{\alpha_2+\nu_i}}{\Gamma(\alpha_2+\nu_i+1)} \right],
\end{aligned} \tag{44}$$

as $t_2 \longrightarrow t_1$, the right-hand side of the above inequality tends to zero, implies that $\mathscr{A}(x)$ is equicontinuous. Therefore, it follows by Arzelà-Ascoli theorem that $\mathscr{A}\colon \mathscr{C}([a,b],\mathbb{R}) \longrightarrow \mathscr{A}_{c,cp}(\mathscr{C}([a,b],\mathbb{R}))$ is relatively compact; then, $\mathscr{A}$ is completely continuous.

Now, we show that the operator $\mathscr{A}$ is upper semi-continuous. To prove this, it is enough to show that $\mathscr{A}$ has a closed graph.

Step 4. $\overline{\mathscr{N}}$ has a closed graph.

Let $x_n \longrightarrow x_*$, $h_n \in \mathscr{A}(x_n)$, and $h_n \longrightarrow h_*$; we shall prove that $h_* \in \mathscr{A}(x_*)$. $h_n \in \mathscr{A}(x_n)$, and then, there exists $v_n \in \mathscr{S}_{F,x_n}$, such that for each $t \in [a,b]$,

$$
\begin{aligned}
h_n(t) &= I^{\alpha_1+\alpha_2} v_n(t) - \lambda I^{\alpha_2} x_n(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\
&\quad \cdot \left[I^{\alpha_1+\alpha_2} v_n(b) - \lambda I^{\alpha_2} x_n(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\gamma_i} v_n(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\gamma_i} x_n(\eta) \right].
\end{aligned} \tag{45}
$$

We should prove that $v_* \in \mathscr{S}_{F,x_*}$, such that for each $t \in [a,b]$,

$$
\begin{aligned}
h_*(t) &= I^{\alpha_1+\alpha_2} v_*(t) - \lambda I^{\alpha_2} x_*(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\
&\quad \cdot \left[I^{\alpha_1+\alpha_2} v_*(b) - \lambda I^{\alpha_2} x_*(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\gamma_i} v_*(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\gamma_i} x_*(\eta) \right],
\end{aligned} \tag{46}
$$

and we have that

$$
\begin{aligned}
&\Bigg\| \left(h_n(t) + \lambda I^{\alpha_2} x_n(t) - \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \left[-\lambda I^{\alpha_2} x_n(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\gamma_i} x_n(\eta) \right] \right) \\
&\quad - t\left(h_*(t) + \lambda I^{\alpha_2} x_*(t) - \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \left[-\lambda I^{\alpha_2} x_*(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\gamma_i} x_*(\eta) \right] \right) \Bigg\| \longrightarrow 0,
\end{aligned} \tag{47}
$$

as $n \longrightarrow \infty$.

Consider the linear operator:

$$
\begin{aligned}
v &\longrightarrow \Upsilon(v)(t), \\
\Upsilon\colon \mathbb{L}^1([a,b],\mathbb{R}) &\longrightarrow \mathscr{C}([a,b],\mathbb{R}),
\end{aligned} \tag{48}
$$

with

$$
\Upsilon(v)(t) = I^{\alpha_1+\alpha_2} v(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \left[I^{\alpha_1+\alpha_2} v(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\gamma_i} v(\eta) \right]. \tag{49}
$$

From Lemma 3, $\Upsilon \circ \mathscr{S}_F$ is a closed graph operator; then, we have that

$$\left(h_n(t) + \lambda I^{\alpha_2} x_n(t) - \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \left[-\lambda I^{\alpha_2} x_n(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x_n(\eta) \right] \right) \in \Upsilon\left(\mathcal{S}_{F,x_n}\right). \tag{50}$$

Since $x_n \longrightarrow x_*$ and $h_n \longrightarrow h_*$ then

$$\left(h_*(t) + \lambda I^{\alpha_2} x_*(t) - \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \left[-\lambda I^{\alpha_2} x_*(b) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x_*(\eta) \right] \right) = \Upsilon(\nu_*) \in \Upsilon\left(\mathcal{S}_{F,x_*}\right). \tag{51}$$

It follows that $\nu_* \in \mathcal{S}_{F,x_*}$, such that

$$\begin{aligned} h_*(t) &= I^{\alpha_1+\alpha_2} \nu_*(t) - \lambda I^{\alpha_2} x_*(t) + \frac{(t-a)^{\gamma_1+\alpha_2-1}}{\Lambda\Gamma(\gamma_1+\alpha_2)} \\ &\quad \cdot \left[I^{\alpha_1+\alpha_2} \nu_*(b) - \lambda I^{\alpha_2} x_*(b) - \sum_{i=1}^{n} \mu_i I^{\alpha_1+\alpha_2+\nu_i} \nu_*(\eta) + \lambda \sum_{i=1}^{n} \mu_i I^{\alpha_2+\nu_i} x_*(\eta) \right]. \end{aligned} \tag{52}$$

Step 5. $\mathcal{A}$ has a fixed point in $\mathcal{B}_\rho$.

We show that (ii) from Theorem 2 is not possible. Then, if $x \in \theta\mathcal{A}x$ for $\theta \in [0,1]$, there exist $\nu \in \mathcal{S}_{F,x}$, such that $x(t) = \theta h(t)$ implies $|x(t)| \le |h(t)|$; then,

$$\|x\| \le \|p\| \Psi(\|x\|) \Omega_1 + \|x\| \Omega_2, \tag{53}$$

and then,

$$(1-\Omega_2)\|x\| \le \|p\| \Psi(\|x\|). \tag{54}$$

If (ii) from Theorem 2 hold, then there exist $\theta \in [0,1]$ and $x \in \partial\mathcal{B}_M$ avec $x = \theta\mathcal{A}$ which means that x is solution to (2) with $\|x\| = M$; then, we have from (32) that

$$(1-\Omega_2) M \le \|p\| \Psi(M), \tag{55}$$

and then,

$$\frac{(1-\Omega_2) M}{\|p\| \Psi(M)} \le 1, \tag{56}$$

which contradicts $(H5)$. Consequently, $\mathcal{A}$ has a fixed point in $[a,b]$.

By the nonlinear alternative of Leray–Schauder, we deduce that our problem (2) has at least one solution. This completes the proof.

5. Example

Consider the following problem:

$$\begin{cases} D^{(3/7),(2/7)}\left(D^{(5/7),(4/7)} + \dfrac{1}{13}\right)x(t) \\ = \dfrac{2}{4t+31}\left(\dfrac{x^2(t) + 2|x(t)|}{1+|x(t)|}\right) + \dfrac{1}{4}, \\ \dfrac{1}{4} \le t \le \dfrac{5}{4}, \\ x\left(\dfrac{1}{4}\right) = 0, \\ x\left(\dfrac{5}{4}\right) = \dfrac{3}{8} I^{(2/3)} x\left(\dfrac{3}{4}\right) + \dfrac{5}{8} I^{(4/3)} x\left(\dfrac{3}{4}\right) + \dfrac{7}{8} I^{(5/3)} x\left(\dfrac{3}{4}\right), \end{cases} \tag{57}$$

where $\alpha_1 = 3/7$, $\alpha_2 = 5/7$, $\beta_1 = 2/7$, $\beta_2 = 4/7$, $\lambda = 1/13$, $a = 1/4$, $b = 5/4$, $n = 3$, $\mu_1 = 3/8$, $\mu_2 = 5/8$, $\mu_3 = 7/8$, $\nu_1 = 2/3$, $\nu_2 = 4/3$, $\nu_3 = 5/3$, and $\eta = 3/4$, with $\alpha_1 + \alpha_2 = 8/7 \in [1,2]$.

With these values, we get $\gamma_1 = \alpha_1 + \beta_1 - \alpha_1\beta_1 = 29/49$, $\gamma_2 = \alpha_2 + \beta_2 - \alpha_2\beta_2 = 43/49$, $|\Lambda| \simeq 0 \cdot 6572399879$, $\Omega_1 = 4 \cdot 725942806$, $\Omega_2 = 0 \cdot 3825052846 < 1$, and $L = 1/8$, while

$$|f(t,x) - f(t,y)| \le \frac{1}{8}|x-y|, \quad x, y \in \mathbb{R}. \tag{58}$$

Since $L\Omega_1 + \Omega_2 \simeq 0 \cdot 9732481354 < 1$, then the conditions of Theorem 4 are satisfied; finally, problem (1) has a unique solution on $[1/4, 5/4]$.

6. Conclusion

In the current article, we study and investigate the existence and uniqueness of solution for a boundary value problem for the Langevin equation and inclusion with the Hilfer fractional derivative. The novelty of this work is that it is more general than the works based on the derivative of Caputo and Riemann–Liouville because when $\beta = 0$, we find the Riemann–Liouville fractional derivative, and when $\beta = 1$, we find the Caputo fractional derivative. In this study, we established the existence and uniqueness results for the first problem, by using the fixed-point theorems (Banach' fixed-point theorem and Krasnoselskii's fixed-point theorem), and for the inclusion case, we use Leray–Schauder alternative to prove the existence of solution. In the end, we give an example to illustrate our results.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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