Hindawi
International Journal of Differential Equations
Volume 2022, Article ID 6468278, 9 pages
https://doi.org/10.1155/2022/6468278

Research Article

Existence of Solution for a Conformable Fractional Cauchy Problem with Nonlocal Condition

Khalid Hilal, Ahmed Kajouni, and Najat Chefnaj

Laboratoire de Mathématiques Appliquées & Calcul Scientifique, Université Sultan Moulay Slimane, BP 523, Beni Mellal 23000, Morocco

Correspondence should be addressed to Najat Chefnaj; najatchefnajj@gmail.com

Received 15 December 2021; Accepted 9 February 2022; Published 24 March 2022

Academic Editor: Gaston Mandata N gu r kata

In this work, we prove the existence and uniqueness of mild solution of the fractional conformable Cauchy problem with nonlocal condition. We obtained these results by applying the fixed point theorems precisely to the fixed point theorem of Krasnoselskii and Banach's fixed point theorem. At the end, we provide application.

1. Introduction

Many dynamic processes in physics, biology, economics, and other fields of application can be governed by differential evolution equations of neutral type of the following form:

$$\frac{d}{dt}\left(x(t) - F\left(t, x\left(h_1(t)\right)\right)\right) = -A\left(x(t) - F\left(t, x\left(h_1(t)\right)\right)\right) + G\left(t, x\left(h_2(t)\right)\right). \tag{1}$$

We replace the partial derivative by a fractional derivative [1, 2] because fractional derivatives have been proven to be a very good way to model many phenomena with memory in various fields of science and engineering [1–9]. Consequently, several researchers are working to form the best definition of the fractional derivative. A new definition called conformal fractional derivative is introduced in [7]. This new fractional derivative becomes the subject of many contributions in several areas of science [9]. Motivated by the better effect of the fractional derivative and simple properties of the conformable fractional derivative, we consider model (1) in this work; we are going to study Cauchy problem with fractional derivative. Precisely, we consider fractional Cauchy problem of the following form:

$$\begin{cases} \dfrac{d^\alpha}{dt^\alpha}\left(x(t) - F\left(t, x\left(h_1(t)\right)\right)\right) = -A\left(x(t) - F\left(t, x\left(h_1(t)\right)\right)\right) + G\left(t, x\left(h_2(t)\right)\right). (1.1), \\ x(0) + g(x) = x_0 \in X = D\left(A^\alpha\right), \end{cases} \tag{2}$$

where $\mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$ is the conformable fractional derivative of $\alpha \in (0,1)$. A is a sectorial operator which generates a strongly analytic semigroup $(T(t))$ on a Banach space X. For more details about semigroup theory, we refer to [7]. We denote by $\mathscr{C}([0,a], D(A^{\alpha})$ the banach space of continuous function from $[0,a]$ into $D(A^{\alpha}$ with the norm $\|u\|_{\alpha} = \sup_{s\in[0,a]}\|u(s)\|_{\alpha}$. For our study, the functions F: $[0,a]\times D(A^{\alpha}) \longrightarrow D(A^{\alpha})$ and G: $[0,a]\times D(A^{\alpha}) \longrightarrow D(A^{\alpha})$ and g: $\mathscr{C}([0,a], D(A^{\alpha})) \longrightarrow D(A^{\alpha})$ and also $h_1, h_2 \in \mathscr{C}([0,a], D(A^{\alpha}))$.

$x(0) + g(x) = x_0$ is nonlocal condition; this notion has been a hot topic in recent years. Their association to classical problems has brought a lot of improvement at the level of modeling, thus making it more realistic. The nonlocal condition joined the main equation instead of the classical initial condition which is necessary to model well and write mathematically, physical phenomena like in electronics, in mechanics of materials, or in biomathematics in the way closest to the reality of many phenomena in multiple disciplines. The nonlocal condition means that the initial condition depends on some future times.

In this paper, we prove the existence of mild solution of conformable fractional differential equations with nonlocal condition. The main results are based on semigroup theory combined with the Krasnoselskii fixed point theorem.

The content of this paper is organized as follows. In Section 2, we recall some preliminary facts on the conformable fractional calculus and Section 3 is devoted to prove the main result.

In this section, we recall some concepts on conformable fractional calculus.

2. Preliminaries

Definition 1. The conformable fractional derivative of x of order α at $t > 0$ is defined as

$$\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = \lim_{h\longrightarrow 0}\frac{x\left(t + he^{(\alpha-1)t}\right) - x(t)}{h}. \tag{3}$$

When the limit exists, we say that x is (α)-differentiable at t.

If x is (α)-differentiable and $\lim_{t\longrightarrow 0^{+}}\mathrm{d}^{\alpha}x(t)/\mathrm{d}t^{\alpha}$ exists, then define

$$\frac{\mathrm{d}^{\alpha}x(0)}{\mathrm{d}t^{\alpha}} = \lim_{t\longrightarrow 0^{+}}\frac{d^{\alpha}x(t)}{dt^{\alpha}}, \tag{4}$$

The (α)-fractional integral of a function x is defined by

$$I^{\alpha}(f)(t) = \int_0^t s^{\alpha-1}f(s)ds\ . \tag{5}$$

Theorem 1. *If x is a continuous function in the domain of I^{α}, then*

$$\frac{\mathrm{d}^{\alpha}(I^{\alpha}(x(t)))}{\mathrm{d}t^{\alpha}} = x(t). \tag{6}$$

Defintion 2. The fractional Laplace transform of order α of x is defined by

$$\mathscr{L}_{\alpha}(x(t))(\lambda) = \int_0^{+\infty} t^{\alpha-1}e^{-\lambda\frac{t^{\alpha}}{\alpha}}x(t)dt\ . \tag{7}$$

The fractional Laplace transform of conformable fractional derivative is given by the following proposition.

Proposition 1. *If $x(t)$ is differentiable, then*

$$\mathscr{L}_{\alpha}\left(\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}}\right)(\lambda) = \lambda\mathscr{L}_{\alpha}(x(t))(\lambda) - x(0). \tag{8}$$

Fractional powers of an operator.

Definition 3. Let A be a sectorial operator defined on a Banach space X, such that $\mathscr{R}e\sigma(A) > 0$; for $\alpha > 0$, we note by $A^{-\alpha}$ the operator defined by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} t^{\alpha-1}T(t)dt\ . \tag{9}$$

Definition 4. Let A be a sectorial operator defined on a Banach space X, such that $\mathscr{R}e\sigma(A) > 0$. We define the family of operators $(A^{\alpha})_{\alpha\geq 0}$ as follows: $A^0 = I_X$, and for $\alpha > 0$,

$$A^{\alpha} = \left(A^{-\alpha}\right)^{-1}, D\left(A^{\alpha}\right) = \operatorname{Im}\left(A^{-\alpha}\right). \tag{10}$$

Theorem 2. *If $(-A)$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ and if $0 \in \rho(A)$, then*

(1) $D(A^{\alpha})$ is a Banach space with the norm $\|x\|_{\alpha} = \|A^{\alpha}x\|$ for every $x \in D(A^{\alpha})$.

(2) $T(t)$: $X \longrightarrow D(A^{\alpha})$ for all $t > 0$ and $\alpha \geq 0$.

(3) For every $x \in D(A^{\alpha})$, we have $T(t)A^{\alpha}x = A^{\alpha}T(t)x$.

We assume that M is a closed bounded convex subset of a Banach space E:

(i) $Ax + By \in M$ for each $x, y \in M$.

(ii) A is continuous and compact.

(iii) B is contraction.

Then, there exists $y \in M$ such that $y = Ay + By$.

We end these preliminaries with the notion of sectorial operator.

Definition 3. A: $D(A) \subset X \longrightarrow X$ is said to be sectorial operator of type (M, ω, Θ) if there exists $M > 0$, $\omega \in \mathbb{R}$, and $0 < \Theta < \pi/2$ as follows:

(1) A is closed and linear operator.

(2) $\forall\lambda \notin w + S_{\theta}$, the resolvent $(\lambda I - A)^{-1}$ of A exists.

(3) $\forall\lambda \notin w + S_{\theta}$, $|(\lambda I - A)^{-1}| \leq M/\lambda - w$,

where $w + S_{\theta}$: $= \{w + \lambda/\lambda \in \mathbb{C}avec|Arg(-\lambda)| < \theta\}$.

Theorem 2. *A densely sectorial operator generates a strongly analytic semigroup $(T(t))_{t\geq 0}$. Moreover,*

$$T(t) = \frac{1}{2\pi i}\int_{\gamma} e^{\lambda t}(\lambda I - A)^{-1} d\lambda \ , \tag{11}$$

with gamma being a suitable path $\lambda \notin w + S_{\theta}$.

Now, we give the main contribution results.

3. Main Results

Before presenting our main results, we introduce the following assumptions:

(H1) F: $[0,a] \times D(A^{\alpha}) \longrightarrow D(A^{\alpha})$ is continuous and there exists a constant $L_1 > 0$ such that $\|F(t,x_1) - F(t,x_2)\|_{\alpha} \le L_1\|x_1 - x_2\|_{\alpha}$ for all $x_1, x_2 \in D(A^{\alpha})$ and for $0 \le t \le a$.

(H2) G: $[0,a] \times D(A^{\alpha}) \longrightarrow D(A^{\alpha})$ is continuous and for all $r > 0$, there exists a function $\mu_r \in \mathscr{L}^{\infty}([0,a], \mathbb{R}^{+})$ such that $sup_{\|x\|\le r}\|G(t,x)\|_{\alpha} \le \mu_r(t)$.

(H3) g: $\mathscr{C}([0,a], D(A^{\alpha})) \longrightarrow D(A^{\alpha})$; there exists a constant $L_3 > 0$ such that $\|g(u_1) - g(u_2)\|_{\alpha} \le L_3\|u_1 - u_2\|_{\alpha}$, for all $u_1, u_2 \in \mathscr{C}([0,a], D(A^{\alpha}))$ with $\|u\|_{\alpha} = \sup_{s\in[0,a]} \|u(s)\|_{\alpha}$ for $u \in \mathscr{C}([0,a], D(A^{\alpha}))$.

(H4) $h_1, h_2 \in \mathscr{C}([0,a], D(A^{\alpha}))$.

H(5) There exists a constant $L_2 > 0$ such that

$$\left\|G(t,x_1) - G(t,x_2)\right\|_{\alpha} \le L_2\left\|x_1 - x_2\right\|_{\alpha}, \tag{12}$$

for all $x_1, x_2 \in D(A^{\alpha})$ and for $0 \le t \le a$.

Existence of mild solution:

Applying the Laplace transform to equation (2), we obtain

$$\begin{aligned}
&\mathscr{L}_{\alpha}\left(\frac{d^{\alpha}}{dt^{\alpha}}(x(t) - F(t,x(h_1(t))))\right)(\lambda) = -\mathscr{L}_{\alpha}(A(x(t) - F(t,x(h_1(t)))) + \mathscr{L}_{\alpha}(G(t,x(h_2(t)))))(\lambda)\\
&\lambda\mathscr{L}_{\alpha}((x(t) - F(t,x(h_1(t)))))(\lambda) - (x(0) - F(0,x(h_1(0)))\\
&= -A\mathscr{L}_{\alpha}(x(t) - F(t,x(h_1(t)))(\lambda) + \mathscr{L}_{\alpha}(G(t,x(h_2(t)))(\lambda))(\lambda + A)\mathscr{L}_{\alpha}\\
&((x(t) - F(t,x(h_1(t)))))(\lambda)\\
&= x(0) - F(0,x(h_1(0))) + \mathscr{L}_{\alpha}(G(t,x(h_2(t))))(\lambda).
\end{aligned} \tag{13}$$

Then,

$$\begin{aligned}
\mathscr{L}_{\alpha}((x(t) - F(t,x(h_1(t)))))(\lambda) &= (\lambda + A)^{-1}(x(0) - F(0,x(h_1(0))))\\
&+ (\lambda + A)^{-1}\mathscr{L}_{\alpha}(G(t,x(h_2(t))))(\lambda).
\end{aligned} \tag{14}$$

Hence,

$$\begin{aligned}
\mathscr{L}_{\alpha}((x(t) - F(t,x(h_1(t)))))(\lambda) &= (\lambda + A)^{-1}(x_0 - g(x) - F(0,x(h_1(0))))\\
&+ (\lambda + A)^{-1}\mathscr{L}_{\alpha}(G(t,x(h_2(t))))(\lambda).
\end{aligned} \tag{15}$$

Then,

$$\begin{aligned}
&(\lambda + A)^{-1}(x(0) - F(0,x(h_1(0))))\\
&= \mathscr{L}_{\alpha}(T\left(\frac{t^{\alpha}}{\alpha}\right)(\lambda)(x_0 - g(x) - F(0,x(h_1(0)))).
\end{aligned} \tag{16}$$

Hence,

$$\begin{aligned}
(\lambda + A)^{-1}\mathscr{L}_{\alpha}(G(t,x(h_2(t))))(\lambda) &= \mathscr{L}_{\alpha}\left(\int_0^{t^{\alpha}/\alpha} T\left(\frac{t^{\alpha}}{\alpha} - s\right)G\left((\alpha s)^{1/\alpha}, x\left(h_2\left((\alpha s)^{1/\alpha}\right)\right)\right)ds\right)(\lambda)\\
&= \mathscr{L}_{\alpha}\left(\int_0^{t} s^{\alpha-1}T\left(\frac{t^{\alpha}}{\alpha} - \frac{s^{\alpha}}{\alpha}\right)G(s,x(h_2(s)))ds\right)(\lambda).
\end{aligned} \tag{17}$$

Therefore,

$$\mathscr{L}_\alpha((x(t)-F(t,x(h_1(t)))))(\lambda)=\mathscr{L}_\alpha(T\left(\frac{t^\alpha}{\alpha}\right)(x_0-g(x)-F(0,x(h_1(0)))(\lambda) + \mathscr{L}_\alpha\left(\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s\right)(\lambda). \tag{18}$$

According to inverse fractional Laplace transform, we find the formula

$$x(t)-F(t,x(h_1(t))=T\left(\frac{t^\alpha}{\alpha}\right)(x_0-g(x)-F(0,x(h_1(0))) + \mathscr{L}_\alpha\left(\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s\right)(\lambda). \tag{19}$$

Then, we obtain

$$x(t)=F(t,x(h_1(t))+T\left(\frac{t^\alpha}{\alpha}\right)(x_0-g(x)-F(0,x(h_1(0))) + \int_0^t s^{\alpha-1}T\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s. \tag{20}$$

Defintion 1. We say that $x\in\mathscr{C}([0,a],D(A^\alpha))$ is a mild solution of equation (2) if the following assertion is true:

$$x(t)=F(t,x(h_1(t))+T\left(\frac{t^\alpha}{\alpha}\right)(x_0-g(x)-F(0,x(h_1(0))) + \int_0^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)G(h_1 s,x(v))\mathrm{d}s. \tag{21}$$

Theorem 3. *If $(T(t))_{t>0}$ is compact and $(H1)-(H4)$ are satisfied, then problem (2) has at least one mild solution, provided that*

$$L_1+\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|(L_3+L_1)<1. \tag{22}$$

Proof. Choosing

$$r\geq\frac{\|F(t,x(h_1(t)))\|_\alpha+\sup_{t\in[0,a]}\|T(t^\alpha/\alpha)\|\left((a^\alpha/\alpha)|\mu_r|_{\mathscr{L}^\infty}+\|x_0\|_\alpha+\|g(x)\|_\alpha+\|F(t,x(h_1(t)))\|_\alpha\right)}{1-\left(L_1+\sup_{t\in[0,a]}\|T(t^\alpha/\alpha)\|(L_3+L_1)\right)}, \tag{23}$$

let $B_r=\{x\in X\|x\|_\alpha\leq r\}$, for $x\in B_r$, and $t\in[0,a]$ define the operators Γ_1 and Γ_2 by

$$\Gamma_1(x)(t)=F(t,x(h_1(t))+T\left(\frac{t^\alpha}{\alpha}\right)(x_0-g(x)-F(0,x(h_1(0))),$$
$$\Gamma_2(x)(t)=\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s. \tag{24}$$

□

Claim 1. We prove that Γ_1 is contraction on B_r. We have

$$\begin{aligned}\left\|\Gamma_1(x)(t)-\Gamma_1(y)(t)\right\|_\alpha &\leq \left\|A^\alpha\left(F\left(t,x\left(h_1(t)\right)-F\left(t,y\left(h_1(t)\right)\right)\right)\right\| \\ &\quad+\left\|T\left(\frac{t^\alpha}{\alpha}\right)A^\alpha\left(g(y)-g(x)\right)\right\| \\ &\quad+\left\|T\left(\frac{t^\alpha}{\alpha}\right)A^\alpha\left(F\left(0,y\left(h_1(0)\right)\right)-F\left(0,y\left(h_1(0)\right)\right)\right)\right\| \\ &\leq L_1\sup_{0\leq s\leq a}\|x(s)-y(s)\|_\alpha+\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\left(L_3+L_1\right)\sup_{0\leq s\leq a}\|x(s)-y(s)\|_\alpha \\ &\leq\left(L_1+\sup_{t\in[0,ta]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\left(L_3+L_1\right)\right)\sup_{0\leq s\leq a}\|x(s)-y(s)\|_\alpha.\end{aligned} \tag{25}$$

Then,

$$\left\|\Gamma_1(x)-\Gamma_1(y)\right\|_\alpha\leq\left(L_1+\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\left(L_3+L_1\right)\right)\|x-y\|_\alpha. \tag{26}$$

Since $L_1+\sup_{t\in[0,a]}\|T(t^\alpha/\alpha)\|(L_3+L_1)<1$, so Γ_1 is a contraction on B_r.

Claim 2. We prove that $\Gamma_1(x)+\Gamma_1(y)\in B_r$ for every $x,y\in B_r$.

$$\begin{aligned}\left\|\Gamma_1(x)(t)+\Gamma_2(y)(t)\right\|_\alpha &\leq\left\|F\left(t,x\left(h_1(t)\right)\right)\right\|_\alpha+\frac{a^\alpha}{\alpha}\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\left|\mu_r\right|_{\mathscr{L}^\infty} \\ &\quad+\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\left(\left\|x_0\right\|_\alpha+\|g(x)\|_\alpha+\left\|F\left(0,x\left(h_1(0)\right)\right)\right\|_\alpha\right) \\ &\leq\left\|F\left(t,x\left(h_1(t)\right)\right)\right\|_\alpha \\ &\quad+\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\left(\left\|x_0\right\|_\alpha+\|g(x)\|_\alpha+\left\|F\left(0,x\left(h_1(0)\right)\right)\right\|_\alpha\right. \\ &\quad\left.+\frac{a^\alpha}{\alpha}\left|\mu_r\right|_{\mathscr{L}^\infty}\right) \\ &\leq r\left(1-\left(L_1+\sup_{t\in[0,a]]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\left(L_3+L_1\right)\right)\right).\end{aligned} \tag{27}$$

Claim 3. We will prove that Γ_2 is continuous on B_r

Let $x_n\in B_r$ such that $\lim_{n\longrightarrow\infty}x_n=x$.

We will prove that $\lim_{n\longrightarrow\infty}\Gamma_2(x_n)=\Gamma_2(x)$.

We have

$$\begin{aligned}\left\|\Gamma_2(x_n)-\Gamma_2(x)\right\|_\alpha &\le \int_0^t s^{\alpha-1}\left\|T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)\right\|\left\|G(s,x_n(h_2(s)))-G(s,x(h_2(s)))\right\|_\alpha \mathrm{d}s\\ &\le \sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\int_0^t s^{\alpha-1}\left\|G(s,x_n(h_2(s)))-G(s,x(h_2(s)))\right\|_\alpha \mathrm{d}s.\end{aligned} \tag{28}$$

By $(H2)$, we have $\|s^{\alpha-1}(G(s,x_n\ (h_2(v)))\ -G(s,x(h_2(s))))\|_\alpha\le 2\mu_r s^{\alpha-1}$ and $\lim_{n\longrightarrow\infty}G(s,x_n(h_2(s)))\ = G9s,x(h_2(s))$. According to Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\longrightarrow\infty}\left\|\Gamma_2(x_n)-\Gamma_2(x)\right\|_\alpha=0. \tag{29}$$

Claim 4. We prove that Γ_2 is compact.

Step 1. We prove that $\{\Gamma_2(x)(t)|x\in B_r\}$ is relativement compact in X.

For some fixed $t\in]0,a[$, let $\varepsilon\in]0,t[$ and define the operator Γ_2^ε by

$$\begin{aligned}\Gamma_2^\varepsilon(x)(t)&=\int_0^{(t^\alpha-\varepsilon^\alpha)^{1/\alpha}} s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s\\ &=T\left(\frac{\varepsilon^\alpha}{\alpha}\right)\int_0^{(t^\alpha-\varepsilon^\alpha)^{1/\alpha}} s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha-\varepsilon^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s.\end{aligned} \tag{30}$$

According to compactness of $(T(t))_{t>0}$, the set $\{\Gamma_2^\varepsilon(x)(t)|x\in B_r\}$ is relatively compact in X.

We have

$$\left\|\Gamma_2^\varepsilon(x)(t)-\Gamma_2(x)(t)\right\|_\alpha\le\sup_{t\in[0,a]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left|\mu_r\right|_{L^\infty([0,a],\mathbb{R}^+)}\frac{\varepsilon^\alpha}{\alpha}. \tag{31}$$

Therefore, $\{\Gamma_2(x)(t)|x\in B_r\}$ is relatively compact in X. It is clear that $\{\Gamma_2(x)(0)|x\in B_r\}$ is compact.

Finally, $\{\Gamma_2(x)(t)|x\in B_r\}$ is relativement compact in X for all $t\in[0,a]$.

Step 2. We prove that $\Gamma_2(B_r)$ is equicontinuous.

Let $t_1,t_2\in]0,a]$ such that $t_1<t_2$. We have

$$\begin{aligned}\Gamma_2(x)(t_2)-\Gamma_2(x)(t_1)&=\int_0^{t_2} s^{\alpha-1}T\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s-\int_0^{t_1} s^{\alpha-1}T\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)G(s,x(h_2(s)))ds\\ &=\int_0^{t_1} s^{\alpha-1}\left(T\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)-T\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)\right)G(s,x(h_2(s)))\mathrm{d}s\\ &\quad+\int_{t_1}^{t_2} s^{\alpha-1}T\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s.\end{aligned} \tag{32}$$

Then,

$$\begin{aligned}\Gamma_2(x)(t_2)-\Gamma_2(x)(t_1)&=\left(T\left(\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right)-I\right)\int_0^{t_1} s^{\alpha-1}T\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s\\ &\quad+\int_{t_1}^{t_2} s^{\alpha-1}T\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s.\end{aligned} \tag{33}$$

Then,

$$\begin{aligned}\left\|\Gamma_2(x)(t_2)-\Gamma_2(x)(t_1)\right\|_\alpha &\le \left\|T\left(\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right)-I\right\|\int_0^\tau s^{\alpha-1}\left\|T\left(\frac{t_1^\alpha-s^\alpha}{\alpha}\right)\right\|\|G(s,x(h_2(s)))\|_\alpha ds\\ &\quad+\int_{t_1}^{t_2} s^{\alpha-1}\left\|T\left(\frac{t_2^\alpha-s^\alpha}{\alpha}\right)\right\|\|G(s,x(h_2(s)))\|_\alpha ds\\ &\le \sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\||\mu|_{L^\infty([0,a],\mathbb{R}^+)}\left(\left\|T\left(\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right)-I\right\|\frac{a^\alpha}{\alpha}+\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right)\\ &\le\frac{\sup_{t\in[0,a]}\|T(t^\alpha/\alpha)\||\mu|_{L^\infty([0,a],\mathbb{R}^+)}}{\alpha}\left(\left\|T\left(\frac{t_2^\alpha-t_1^\alpha}{\alpha}\right)-I\right\|a^\alpha+(t_2^\alpha-t_1^\alpha)\right).\end{aligned} \tag{34}$$

We conclude that $\Gamma_2(x)$, $x\in B_r$ are equicontinuous at $t\in[0,a]$. By using Arzela Ascoli theorem, we obtain that Γ_2 is compact. Finally, the Krasnoselskii theorem helps us to complete the proof.

Uniqueness of mild solution.

Theorem 4. *Assume that $(H1)-(H5)$ hold, then the Cauchy problem (2) has a unique solution, provided that*

$$L_1+\left(L_1+L_3+L_2\frac{a^\alpha}{\alpha}\right)\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|<1. \tag{35}$$

Proof. Define operator P: $\mathscr{C}([0,a],D(A^\alpha)\longrightarrow\mathscr{C}([0,a],D(A^\alpha)$

$$\begin{aligned}(Px)(t)&=F(t,x(h_1(t))+T\left(\frac{t^\alpha}{\alpha}\right)(x_0-g(x)-F(0,x(h_1(0)))\\ &\quad+\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)G(s,x(h_2(s)))\mathrm{d}s.\end{aligned} \tag{36}$$

Next, let $x,y\in\mathscr{C}([0,a],D(A^\alpha)$, we have

$$\begin{aligned}(Px)(t)-(Py)(t)&=F(t,x(h_1(t))-F(t,y(h_1(t))\\ &\quad+T\left(\frac{t^\alpha}{\alpha}\right)(g(y)-g(x)+F(0,y(h_1(0))-F(0,x(h_1(0)))\\ &\quad+\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)(G(s,x(h_2(s)))-(G(s,y(h_2(s))))\mathrm{d}s.\end{aligned} \tag{37}$$

Then,

$$\begin{aligned}\|(Px)(t)-(Py)(t)\|_\alpha&\le A^\alpha(F(t,x(h_1(t))-F(t,y(h_1(t)))\\ &\quad+\left\|T\left(\frac{t^\alpha}{\alpha}\right)A^\alpha(g(y)-g(x))\|+\|T\left(\frac{t^\alpha}{\alpha}\right)A^\alpha(F(0,y(h_1(0))-F(0,x(h_1(0)))\right\|\\ &\quad+\int_0^t s^{\alpha-1}\left\|T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)A^\alpha(G(s,x(h_2(s)))-(G(s,y(h_2(s))))\right\|\mathrm{d}s.\end{aligned} \tag{38}$$

Hence,

$$\begin{aligned}\|(Px)(t)-(Py)(t)\|_\alpha \leq \left(1+\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\right)L_1\sup_{0\leq s\leq a}\|x(s)-y(s)\|_\alpha \\ +L_3\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\sup_{0\leq s\leq a}\|x(s)-y(s)\|_\alpha+\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\frac{a^\alpha}{\alpha}L_2\sup_{0\leq s\leq a}\|x(s)-y(s)\|_\alpha.\end{aligned} \tag{39}$$

Therefore,

$$\|(Px)-(Py)\|_\alpha \leq \left(L_1+\left(L_1+L_3+L_2\frac{a^\alpha}{\alpha}\right)\sup_{t\in[0,a]}\left\|T\left(\frac{t^\alpha}{\alpha}\right)\right\|\right)\|x-y\|_\alpha. \tag{40}$$

Since $L_1+(L_1+L_3+L_2a^\alpha/\alpha)\sup_{t\in[0,a]}\|T(t^\alpha/\alpha)\|<1$, as a consequence, P has a unique fixed point in $\mathscr{C}([0,a],D(A^\alpha))$. □

4. Application

We consider the conformable fractional Cauchy problem of the following form:

$$\begin{cases}\dfrac{\partial^{1/2}}{\partial t^{1/2}}\left(u(t,\xi)-\displaystyle\int_0^\pi b(t,\xi,\sigma)u(\sin t,\sigma)d\sigma\right)= \\ \dfrac{\partial^2(.)}{\partial x^2}\left(u(t,\xi)-\displaystyle\int_0^\pi b(t,\xi,\sigma)u(\sin t,\sigma)d\sigma\right)+\psi\left(t,\dfrac{\partial u(t,\xi)}{\partial\xi}\right) \\ u(t,0)=u(t,\pi)=0 \\ u(0,\xi)-\displaystyle\int_0^\pi k(\xi,\sigma)u(0,\sigma)d\sigma=u_0(\xi),\quad 0\leq\xi\leq 1.\end{cases} \tag{41}$$

With $X=\mathbb{L}^2[0,\pi]$, we define operator A by $Af=-f''$ and $D(A)=H_0^2([0,\pi])=\{f\in X\colon f',f''\in X \text{ and } f(0)=f(\pi)=0\}$ with the norm $_2$.

(i) $b\colon [0,1]\times[0,\pi]\times[0,\pi]\longrightarrow\mathbb{R}$ is class C^1 and $b(t,.,0)=b(t,.,\pi)$.

(ii) $\psi\colon [0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous and there exists a constant $a_0>0$ such that

$$|\psi(t,\xi_1)-\psi(t,\xi_2)|\leq a_0|\xi_1-\xi_2|, \xi_1,\xi_2\in\mathbb{R}, 0\leq t\leq 1. \tag{42}$$

(iii) $k\colon [0,\pi]\times[0,\pi]\longrightarrow\mathbb{R}$ is class C^1 and $k(.,0)=k(.,\pi)$.

For $\alpha=1/2, t\in[0,1], v\in D(A^{1/2})$ with $D(A^{1/2})=\{f\in X/\sum_0^\infty n(f,e_n)e_n\}$ and $w\in E=\mathscr{C}([0,1],D(A^{1/2}))$, we define $F(t,v)(.)=\int_0^\pi b(t,.,\sigma)v(\sigma)d\sigma$, $G(t,v)(.)=G(t,v')$, $g(w)(.)=\int_0^\pi k(.,\sigma)w(\sigma)d\sigma$, and $h_1(t)=h_2(t)=\sin t$ $A=\partial^2(.)/\partial x^2$.

The problem takes the following form:

$$\begin{cases}\dfrac{d^\alpha}{dt^\alpha}\left(x(t)-F(t,x(h_1(t)))\right)=-A\left(x(t)-F(t,x(h_1(t)))\right)+G(t,x(h_2(t))) \\ x(0)+g(x)=x_0.\end{cases} \tag{43}$$

F: $[0,1]\times D(A^{1/2}) \longrightarrow D(A^{1/2})$ and G: $[0,a]\times D(A^{\alpha}) \longrightarrow X$, on the other hand,

$$\|F(t,v)\|_{1/2}^2 = \int_0^{\pi}\left(\int_0^{\pi}\frac{\partial}{\partial\xi}b(t,\xi,\sigma)v(\sigma)\mathrm{d}\sigma\right)^2\mathrm{d}\xi \leq \left(\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial}{\partial\xi}b(t,\xi,\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}\xi\right)\int_0^{\pi}v^2(\sigma)\mathrm{d}\sigma. \tag{44}$$

From the equality of Poincare, we have

$$\|F(t,v)\|_{1/2}^2 \leq \sup_{0\leq t\leq 1}\left(\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial}{\partial\xi}b(t,\xi,\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}\xi\right)\|v\|_{1/2}^2, \tag{45}$$

which implies that hypothesis $(H1)$ is verified

Let $v_1, v_2 \in D(A^{1/2})$ and $\xi \in [0,\pi]$. We know that

$$\begin{aligned}\left\|G(t,v_1)-G(t,v_2)\right\|^2 &\leq (t,v_1)(\xi)-G(t,v_2)(\xi)|^2\mathrm{d}\xi \\ &\leq a_0^2\int_0^{\pi}\left|v_1'(\xi)-v_2'(\xi)\right|^2\mathrm{d}\xi \\ &\leq a_0^2\left\|v_1'-v_2'\right\|_{1/2}^2,\end{aligned} \tag{46}$$

and hypothesis $(H5)$ is verified.

Let $v \in\in D(A^{1/2})$; we have $\|g(v)\|_{1/2}^2 = \|A^{1/2}g(v)\|^2 = \|\partial/\partial\xi g\ (v)(\xi)\|^2$,

$$\|g(v)\|_{1/2}^2 = \grave{}\int_0^{\pi}\left(\int_0^{\pi}\frac{\partial}{\partial\xi}k(\xi,\sigma)v(\sigma)\mathrm{d}\sigma\right)^2\mathrm{d}\xi \leq \left(\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial}{\partial\xi}k(\xi,\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}\xi\right)\int_0^{\pi}v(\sigma)^2\mathrm{d}\sigma. \tag{47}$$

From the equality of Poincare, we have

$$\|g(v)\|_{1/2}^2 \leq \left(\int_0^{\pi}\int_0^{\pi}\left(\frac{\partial}{\partial\xi}k(\xi,\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}\xi\right)\|v\|_{1/2}^2. \tag{48}$$

Then, $(H3)$ is verified.

Moreover, $\sup_{0\leq t\leq 1}(\int_0^{\pi}\int_0^{\pi}(\partial/\partial\xi b\quad(t,\xi,\sigma))^2\mathrm{d}\sigma\mathrm{d}\xi) + (\int_0^{\pi}\int_0^{\pi}(\partial/\partial\xi k(\xi,\sigma))^2\mathrm{d}\sigma d\xi) + 2a_0)M \leq 1$. As a consequence, (41) has a unique fixed point in $\mathscr{C}([0,a], D(A^{\alpha})$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] K. Ezzinbi and J. H. Liu, "defined evolution equation with nonlocal conditions," *Mathematical and Computer Modelling*, vol. 36, pp. 1027–1038, 2002.

[2] A. Pazy, "Semigroups of linear operators and applications to partial diffrential equations," *Applied Mathematical Sciences*, vol. 44, Springer, New York, NY, USA, 1983.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amesterdam, Netherlands, 2006.

[4] K. S. Miller, *An Introduction to Fractional Calculus and Fractional Differential Equations*, Wiley and Sons, New York, NY, USA, 1993.

[5] K. Ezzinbi, X. Fu, and K. Hilal, "Existence and regularity in the -norm for some neutral partial differential equations with nonlocal conditions," *Nonlinear Analysis*, vol. 67, no. 5, pp. 1613–1622, 2007.

[6] M. Bouaouid, K. Hilal, and S. Melliani, "Sequential evolution conformable differential equations of second order with nonlocal condition," *Advances in Difference Equations*, vol. 2019, no. 1, p. 21, 2019.

[7] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[8] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives 7eory and Applications*, Gordon Breach Science Publishers, Amsterdam, Netherlands, 1993.

[9] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015.