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Research Article

On Hilfer-Type Fractional Impulsive Differential Equations

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Using the Schauder fixed point theorem, we prove the existence of impulsive fractional differential equations using Hilfer fractional derivative and nearly sectorial operators in this paper. We've gone over the two scenarios where the related semigroup is compact and noncompact for this purpose. We also go over an example to back up the main points.

1. Introduction

We consider the following impulsive fractional differential equations involving Hilfer fractional derivative and almost sectorial operators

$$\begin{aligned} &\mathfrak{D}^{I,\varkappa}\pounds(t) + \mathfrak{A}\pounds(t) = \mathfrak{Y}(t,\pounds(t)), \quad t \in (0,T] = \mathcal{J}, \\ &\Delta\pounds|_{t=t_k} = \mathfrak{I}_k(\pounds(t_k^-)), \quad k = 1,2,3,\ldots,m, \\ &I_{0+}^{(1-\lambda)(1-\varkappa)}\pounds(0) = \pounds_0, \end{aligned} \tag{1}$$

where $\mathfrak{D}_{0+}^{I,\varkappa}$ Hilfer fractional derivative of order $I \in (0,1)$ and type $\varkappa \in [0,1]$ and $\mathfrak{A}$ is an almost sectorial operator in $\mathcal{Y}$ having norm $\|\cdot\|$ and $\Delta\pounds|_{t=t_k}$ denotes the jump of $\pounds(t)$ at $t = t_k$, i.e., $\Delta\pounds|_{t=t_k} = \pounds(t_k^+) - \pounds(t_k^-)$, where $\pounds(t_k^+)$ and $\pounds(t_k^-)$. $\mathfrak{I}_k$: $\mathcal{U} \longrightarrow \mathcal{D}(\sqsupset)$ and $\pounds_0 \in \mathcal{D}(\sqsupset)$. $\mathfrak{Y}$: $\mathcal{J} \times \mathcal{Y} \longrightarrow \mathcal{Y}$ is a function which is defined later.

In fact, rapid changes in the dynamics of evolutionary processes might occur due to shocks, harvesting, or natural disasters, among other things. These short-term disturbances are frequently handled as impulses. Hilfer derivative fractional differential equations have recently gained a lot of attention in the literature. [1–12] appears to be the case.

In [3], Anjali Jaiswal and Bahuguna studied the equations of Hilfer fractional deritaive with almost sectorial operator in the abstract sense.

$$\begin{aligned} &\mathfrak{D}^{I,\varkappa}(t) + \mathfrak{A}\pounds(t) = \mathfrak{Y}(t,\pounds(t)), \quad t \in (0,T], \\ &I_{0+}^{(1-I)(1-\varkappa)}\pounds(0) = \pounds_0. \end{aligned} \tag{2}$$

We also refer to Hamdy M. Ahmed et.al studied in [13], which looked at the existence of nonlinear Hilfer fractional derivative differential equations with control. Sufficient circumstances have been found where the Hilfer derivative is the time fractional derivative. In [14], Yong Zhoy et.al studied the factional cauchy problems with almost sectorial operators of the form

$$\begin{aligned} &\mathfrak{D}^{I}\pounds(t) = \mathfrak{A}\pounds(t) + \mathcal{F}(t,\pounds(t)), \quad t \in (0,T], \\ &I_{0+}^{(1-I)}\pounds(0) = \pounds_0, \end{aligned} \tag{3}$$

where $\mathfrak{D}^I$ is Riemann–Liouville derivative of order §, $I^{(1-I)}$ is Riemann–Liouville integral of order $1-I$, $0 < I < 1$. $\mathfrak{A}$ is an almost sectorial operator on a complex Banach space, and $\mathcal{F}$ is a given function.

The paper is structured as follows: we have presented some information in section 2 about Hilfer derivative, almost sectorial operators, measure of non-compactness, mild solutions of equations (1.1) – (1.3) along with some basic definitions, results and lemmas. We discuss the main results for mild solutions for the equations (1.1) – (1.3) in section 3. In section 4 5, we have discussed the two cases if associated semigroup is compact and noncompact respectively. Finally, an example is discussed to illustrate the main result.

The following sections describes the supporting results of the given problem and also generalizes the results in [14–16].

2. Preliminaries

Definition 1 (see [17]). For $I > 0$, the fractional integral of order § of a function $f(t)$ is defined by

$$I_{0+}^{I} f(t) = \frac{1}{\varkappa(I)} \int_0^t f(r)(t-r)^{I-1} \mathrm{d}r. \tag{4}$$

Definition 2 (see [17]). For $0 < I < 1$, the Riemann-Liouville (R-L) fractional derivative with order § of a function $f(t)$ is defined by

$$\mathfrak{D}^{I} f(t) = \frac{1}{\varkappa(1-I)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(r)}{(t-r)^{I}} \mathrm{d}r. \tag{5}$$

Definition 3 (see [17]). For $0 < I < 1$, the Caputo fractional derivative with order § of a function $f(t)$ is defined by

$${}^{c}\mathfrak{D}^{I} f(t) = \frac{1}{\varkappa(1-I)} \int_0^t \frac{f\prime(r)}{(t-r)^{I}} \mathrm{d}r. \tag{6}$$

Definition 4 (see [11]). Let $0 < I < 1$ and $0 \le \nu \le 1$. The Hilfer fractional derivative derivative of order § and type ν is defined by

$$\mathfrak{D}_{0+}^{I,\varkappa} £(t) = I_{0+}^{\nu(1-I)} \frac{\mathrm{d}}{\mathrm{d}t} I_{0+}^{1-\varkappa(1-I)} £(t). \tag{7}$$

2.1. Measure of Non-compactness. Let $\sqsupset \subset \mathscr{Y}$ also bounded. The Hausdorff measure of non-compactness is considered as follows

$$\Xi(\sqsupset) = \inf\left\{\zeta > 0 \text{ that is } \sqsupset \subset \bigcup_{j=1}^{m} B_{\zeta}\left(x_j\right) \text{where } x_j \hat{\imath} \mathrm{Y}, \mathrm{m} \in \mathrm{N}\right\}. \tag{8}$$

The Kurtawoski measure of noncompactness Φ on a bounded set $\mathscr{B} \subset \mathscr{Y}$ is considered as follows

$$\Phi(\sqsupset) = \inf\left\{\epsilon > 0 \text{ implies } \sqsupset \subset \bigcup_{\mathrm{j}=1}^{\mathrm{m}} M_j \text{also diam}\left(M_j\right) \le \epsilon\right\}, \tag{9}$$

with the following properties

1. $\sqsupset_1 \subset \sqsupset_2$ gives $\Xi(\sqsupset_1) \le \Xi(\sqsupset_2)$ where $\sqsupset_1, \sqsupset_2$ are bounded subsets of $\mathscr{Y}$
2. $\Xi(\sqsupset) = 0$ iff $\sqsupset$ is relatively compact in $\mathscr{Y}$
3. $\Xi(\{z\} \bigcup \sqsupset) = \Xi(\sqsupset)$ for all $z \in \mathscr{Y}$ $\sqsupset \subseteq \mathscr{Y}$
4. $\Xi(\sqsupset_1 \bigcup \sqsupset_2) \le \max\{\Xi(\sqsupset_1), \Xi(\sqsupset_2)\}$
5. $\Xi(\sqsupset_1 + \sqsupset_2) \le \Xi(\sqsupset_1) + \Xi(\sqsupset_2)$
6. $\Xi(r\sqsupset) \le |r| \Xi(\sqsupset)$ for $r \in R$.

Let $\urcorner \subset C(I, \mathscr{Y})$ and $\urcorner(r) = \{v(r) \in \mathscr{Y} \mid v \in \urcorner\}$. We define

$$\int_0^t \urcorner(r) \mathrm{d}r = \left\{\int_0^t v(r) \mathrm{d}r \mid v \in \urcorner\right\}, \quad t \in \mathscr{J}. \tag{10}$$

2.2. Almost Sectorial Operators. Let $0 < \mu < \pi$ and $-1 < \beta < 0$. We define $S_{\mu}^{0} = \{v \in C, \{0\}, \text{that is}, |\arg v| < \mu\}$ and its closure by S_{μ}, that is $S_{\mu} = \{v \in C, \{0\}\colon |\arg v| < \mu\} \bigcup \{0\}$.

Definition 5 (see [18]). For $-1 < \beta < 0$, $0 < \omega < \pi/2$, we define $\left\{\odot_{\omega}^{\beta}\right\}$ as a family of all closed and linear operators $\mathfrak{A}\colon D(\mathfrak{A}) \subset \mathscr{Y} \longrightarrow \mathscr{Y}$ this implies

1. $\sigma(\mathfrak{A})$ is contained in S_{ω}.
2. For all $\mu \in (\omega, \pi)$, there exists M_{μ} implies

$$\|\mathscr{R}(z, \mathfrak{A})\|_{L(X)} \le M_{\mu} |z|^{\beta}, \tag{11}$$

where $\mathscr{R}(z, \mathfrak{A}) = (zI - \mathfrak{A})^{-1}$ is the resolvent operator and $\mathfrak{A} \in \odot_{\omega}^{\beta}$ is said to be an almost sectorial operator on $\mathscr{Y}$.

We assume the following Wright-type function [17]

$$\mathfrak{M}_I(\theta) = \sum_{n \in \mathbb{N}} \frac{(-\theta)^{n-1}}{\varkappa(1-In)(n-1)!}, \quad \theta \in \mathbb{C}. \tag{12}$$

For $-1 < \sigma < \infty$, $r > 0$,

(A1) $\mathfrak{M}_I(\theta) \ge 0$, $t > 0$;
(A2) $\int_0^{\infty} \theta^{\sigma} \mathfrak{M}_I \mathrm{d}\theta = \varkappa(1+\sigma)/\varkappa(1+I\sigma)$;
(A3) $\int_0^{\infty} I/\theta^{I+1} e^{-r\theta} \mathfrak{M}_I(1/\theta^{I}) \mathrm{d}\theta = e^{-r^{I}}$.

We have define $\{\mathfrak{S}_I(t)\}|_{t \in S_{\pi/2-w}^{0}}$, $\{\mathfrak{Q}_I(t)\}|_{t \in S_{\pi/2-w}^{0}}$ by

$$\begin{aligned} \mathfrak{S}_I(t) &= \int_0^{\infty} \mathfrak{M}_I(\zeta) \varrho\left(t^{I}\zeta\right) \mathrm{d}\zeta, \\ \mathfrak{Q}_I(t) &= \int_0^{\infty} I\zeta \mathfrak{M}_I(\zeta) \varrho\left(t^{I}\zeta\right) \mathrm{d}\zeta. \end{aligned} \tag{13}$$

Theorem 1 (Theorem 4.6.1 in [17]). *For each fixed $t \in S_{\pi/2-\omega}^{0}$, $\mathfrak{S}_I(t)$ and $\mathfrak{Q}_I(t)$ are bounded and linear operators on $\mathscr{Y}$. Also*

$$\left\|\mathfrak{S}_I(t)\right\| \le \mathfrak{C}_s t^{-I(1+\beta)}, \quad \left\|\mathfrak{Q}_I(t)\right\| \le \mathfrak{C}_p t^{-I(1+\beta)}, \ t > 0, \tag{14}$$

where $\mathfrak{C}_s$ and $\mathfrak{C}_p$ are constants.

We introduce the following hypotheses to obtain our main results.

(H1) For $t \in \mathcal{J}'$, $\mathfrak{Y}(t,.)\colon \mathcal{Y} \longrightarrow \mathcal{Y}$ is continuous function and for each $\pounds \in \mathfrak{C}(\mathcal{J}\prime, \mathcal{Y})$, $\mathfrak{Y}(., \pounds, \mathcal{B}\pounds)$: $\mathcal{J}\prime \longrightarrow \mathcal{Y}$ is strongly measurable.

(H2) $\exists$ a function $k \in L^1(\mathcal{J}\prime, \mathbb{R}^+)$ satisfying

$$I_{0+}^{-I\beta} k \in \mathfrak{C}(\mathcal{J}\prime, \mathbb{R}), \quad \lim t^{(1+I\beta)(1-\varkappa)} I_{0+}^{-I\beta} k(t) = 0. \tag{15}$$

(H3)

$$\sup_{[0,T]}\left(t^{(1+I\beta)(1-\varkappa)}\left\|\mathfrak{S}_{I,\varkappa(t)}u_0\right\| + t^{(1+I\beta)(1-\varkappa)}\int_0^t (t-r)^{-I\beta-1}k(r)\mathrm{d}r\right) \le r, \tag{16}$$

for $r > 0$ and $u_0 \in D(\mathfrak{A}^\theta)$, $\theta > 1+\beta$, where $\mathfrak{S}_{I,\varkappa(t)} = I_{0+}^{\varkappa(1-I)} t^{-1}\mathfrak{Q}_I(t)$.

(H4) $\exists$ constants ω_k such that $\|\mathfrak{I}_k(\pounds)\| \le \omega_k$, $k = 1, 2, \ldots, m$ for each $\pounds \in \mathcal{Y}$.

Lemma 1. *The fractional Cauchy problem* (1.1) − (1.3) *is equivalent to an integral equation given by*

$$\pounds(t) = \frac{\pounds_0}{\varkappa(\nu(1-I)+I)} t^{(1-I)(\nu-1)} + \frac{1}{\varkappa(I)}\int_0^t (t-r)^{I-1}[-\mathfrak{A}\pounds(r) + \mathfrak{Y}(r, \pounds(r))]\mathrm{d}r + \sum_{0<t_k<t} \mathfrak{S}_{I,\varkappa}(t-t_k)\mathfrak{I}_k(\pounds(t_k^-)), \quad t \in \mathcal{J}. \tag{17}$$

Lemma 2. *Let $\pounds$ is a solution of the integral equation given in (2.3), then $\pounds$ satisfies*

$$\pounds(t) = \mathfrak{S}_{I,\varkappa}(t)\pounds_0 + \int_0^t \mathfrak{K}_I(t-r)\mathfrak{Y}(r, \pounds(r))\mathrm{d}r + \sum_{0<t_k<t} \mathfrak{S}_{I,\varkappa}(t-t_k)\mathfrak{I}_k(\pounds(t_k^-)), \tag{18}$$

where $\mathfrak{S}_{I,\varkappa}(t) = I_{0+}^{\varkappa(1-I)}\mathfrak{K}_I(t)$, $\mathfrak{K}_I = t^{I-1}\mathfrak{Q}_I(t)$.

Definition 6. By a mild solution of the Cauchy problem (1.1) − (1.3) we mean a function $\pounds \in \mathbb{C}(\mathcal{J}\prime, X)$ that satisfies

$$\pounds(t) = \mathfrak{S}_{I,\varkappa}(t)\pounds_0 + \int_0^t \mathfrak{K}_I(t-r)\mathfrak{Y}(r, \pounds(r))\mathrm{d}r + \sum_{0<t_k<t} \mathfrak{S}_{I,\varkappa}(t-t_k)\mathfrak{I}_k(\pounds(t_k^-)), \quad t \in \mathcal{J}\prime. \tag{19}$$

Now, we define an operator $\mathfrak{P}\colon \mathfrak{B}_r(\mathcal{J}\prime) \longrightarrow \mathfrak{B}_r(\mathcal{J}\prime)$ as

$$(\mathfrak{B}\pounds)(t) = \mathfrak{S}_{I,\varkappa}(t)\pounds_0 + \int_0^t (t-r)^{I-1}\mathfrak{Q}_I(t-r)\mathfrak{Y}(r, \pounds(r))\mathrm{d}r + \sum_{0<t_k<t} \mathfrak{S}_{I,\varkappa}(t-t_k)\mathfrak{I}_k(\pounds(t_k^-)). \tag{20}$$

Lemma 3 (see [3]). *$\mathfrak{K}_I(t)$ and $\mathfrak{S}_{I,\varkappa}(t)$ are bounded linear operators on $\mathcal{Y}$, for every fixed $t \in S^0_{\pi/2-\omega}$. Also for $t > 0$.*

$$\left\|\mathfrak{K}_I(t)x\right\| \le \mathfrak{C}_p t^{-1-I\beta}\|x\|, \quad \left\|\mathfrak{S}_{I,\varkappa(t)}x\right\| \le \frac{\varkappa(-I\beta)}{\varkappa(\varkappa(1-I)-I\beta)}\mathfrak{C}_p t^{\varkappa(1-I)-I\beta-1}\|x\|. \tag{21}$$

Proposition 1 (see [3]). *$\mathfrak{K}_I(t)$ and $\mathfrak{S}_{I,\varkappa(t)}$ are strongly continuous, for $t>0$.*

3. Main results

Theorem 2. *Let $\mathfrak{A}\in\odot_{\omega}^{\beta}$ for $-1<\beta<0$ and $0<\omega<\pi/2$. Assuming $(H1)-(H4)$ are satisfied, the operators $\{\mathfrak{F}y:\ y\in\mathfrak{B}_r(\mathcal{J})\}$ is equicontinuous provided $\pounds_0\in\mathcal{D}(\mathfrak{A}^{\theta})$ with $\theta>1+\beta$.*

Proof. For $y\in\mathfrak{B}_r(\mathcal{J})$ and $t_1=0<t_2\le T$, we gave

$$\begin{aligned}\|\mathfrak{F}y(t_2)-\mathfrak{F}y(0)\| &= \left\| t_2^{(1+I\mu)(1-\varkappa)}\left(\mathfrak{S}_{I,\varkappa(t_2)}u_0+\int_0^{t_2}(t_2-r)^{I-1}\mathfrak{Q}_I(t_2-r)\times\mathfrak{Y}(r,\pounds(r))\mathrm{d}r+\sum_{0<t_k<t_2}\mathfrak{S}_{I,\varkappa}(t_2-t_k)\mathfrak{I}_k(\pounds(t_k^-))\right)\right\| \\ &\le \left\| t_2^{(1+I\mu)(1-\varkappa)}\mathfrak{S}_{I,\varkappa(t_2)}u_0\right\| \\ &\quad+\left\| t_2^{(1+I\mu)(1-\varkappa)}\int_0^{t_2}(t_2-r)^{I-1}\mathfrak{Q}_I(t_2-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r\right\|+\left\| t_2^{(1+I\mu)(1-\varkappa)}\sum_{0<t_k<t_2}\mathfrak{S}_{I,\varkappa}(t_2-t_k)\mathfrak{I}_k(\pounds(t_k^-))\right\|.\end{aligned} \tag{22}$$

$\longrightarrow 0$, as $t_2\longrightarrow 0$.

Now, let $0<t_1<t_2\le T$,

$$\begin{aligned}\|\mathfrak{F}y(t_2)-\mathfrak{F}y(t_1)\| &\le \left\| t_2^{(1+I\mu)(1-\varkappa)}\mathfrak{S}_{I,\varkappa(t_2)}u_0-t_1^{(1+I\mu)(1-\varkappa)}\mathfrak{S}_{I,\varkappa(t_1)}u_0\right\| \\ &\quad+\left\| t_2^{(1+I\mu)(1-\varkappa)}\int_0^{t_2}(t_2-r)^{I-1}\mathfrak{Q}_I(t_2-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r\right. \\ &\quad\left.-t_1^{(1+I\mu)(1-\varkappa)}\int_0^{t_1}(t_1-r)^{I-1}\mathfrak{Q}_I(t_1-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r\right\| \\ &\quad+\left\| t_2^{(1+I\mu)(1-\varkappa)}\sum_{0<t_k<t_2}\mathfrak{S}_{I,\varkappa}(t_2-t_k)\mathfrak{I}_k(\pounds(t_k^-))-t_1^{(1+I\mu)(1-\varkappa)}\sum_{0<t_k<t_1}\mathfrak{S}_{I,\varkappa}(t_1-t_k)\mathfrak{I}_k(\pounds(t_k^-))\right\|,\end{aligned} \tag{23}$$

here, using the triangle inequality, we have

$$\begin{aligned}\|\mathfrak{F}y(t_2)-\mathfrak{F}y(t_1)\| &\le \left\| t_2^{(1+I\mu)(1-\varkappa)}\mathfrak{S}_{I,\varkappa(t_2)}u_0-t_1^{(1+I\mu)(1-\varkappa)}\mathfrak{S}_{I,\varkappa(t_1)}u_0\right\| \\ &\quad+\left\| t_2^{(1+I\mu)(1-\varkappa)}\int_{t_1}^{t_2}(t_2-r)^{I-1}\mathfrak{Q}_I(t_2-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r\right\| \\ &\quad+\left\| t_2^{(1+I\mu)(1-\varkappa)}\int_0^{t_1}(t_2-r)^{I-1}\mathfrak{Q}_I(t_2-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r-t_1^{(1+I\mu)(1-\varkappa)}\int_0^{t_1}(t_1-r)^{I-1}\mathfrak{Q}_I(t_2-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r\right\| \\ &\quad+\left\| t_1^{(1+I\mu)(1-\varkappa)}\int_0^{t_1}(t_1-r)^{I-1}\mathfrak{Q}_I(t_2-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r-t_1^{(1+I\mu)(1-\varkappa)}\int_0^{t_1}(t_1-r)^{I-1}\mathfrak{Q}_I(t_1-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r\right\|, \\ &\quad+\left\| t_2^{(1+I\mu)(1-\varkappa)}\sum_{0<t_k<t_2}\mathfrak{S}_{I,\varkappa}(t_2-t_k)\mathfrak{I}_k(\pounds(t_k^-))-t_1^{(1+I\mu)(1-\varkappa)}\sum_{0<t_k<t_1}\mathfrak{S}_{I,\varkappa}(t_1-t_k)\mathfrak{I}_k(\pounds(t_k^-))\right\| \\ &= \mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3+\mathfrak{I}_4+\mathfrak{I}_5.\end{aligned} \tag{24}$$

Since the strong continuous of $\mathfrak{S}_{I,\varkappa}(t)$, we get $\mathfrak{I}_1 \longrightarrow 0$ as $t_2 \longrightarrow t_1$. Also

$$\begin{aligned}\mathfrak{I}_2 &\leq \mathfrak{C}_p t_2^{(1+I\beta)(1-\varkappa)} \int_{t_1}^{t_2} (t_2 - r)^{-I\beta-1} \kappa(r)\mathrm{d}r \\ &\leq \mathfrak{C}_p \left| t_2^{(1+I\beta)(1-\varkappa)} \int_0^{t_2} (t_2 - r)^{-I\beta-1} \kappa(r)\mathrm{d}r - t_2^{(1+I\beta)(1-\varkappa)} \times \int_0^{t_1} (t_1 - r)^{-I\beta-1} \kappa(r)\mathrm{d}r \right| \\ &\leq \mathfrak{C}_p \int_0^{t_1} \left| t_1^{(1+I\beta)(1-\varkappa)} (t_1 - r)^{-I\beta-1} - t_2^{(1+I\beta)(1-\varkappa)} (t_2 - r)^{-I\beta-1} \right| \kappa(r)\mathrm{d}r.\end{aligned} \tag{25}$$

Then $\mathfrak{I}_2 \longrightarrow 0$ as $t_2 \longrightarrow t_1$, by using $(H2)$ and the dominated convergence theorem. Since

$$\begin{aligned}\mathfrak{I}_3 &\leq \mathfrak{C}_p \int_0^{t_1} (t_2 - r)^{-I-I\beta} \times \left| t_2^{(1+I\beta)(1-\varkappa)} (t_2 - r)^{I-1} - t_1^{(1+I\beta)(1-\varkappa)} (t_1 - r)^{I-1} \right| \kappa(r)\mathrm{d}r \\ &\quad \cdot (t_2 - r)^{-I-I\beta} \left| t_2^{(1+I\beta)(1-\varkappa)} (t_2 - r)^{I-1} - t_1^{(1+I\beta)(1-\varkappa)} (t_1 - r)^{I-1} \right| \kappa(r) \\ &\leq t_2^{(1+I\beta)(1-\varkappa)} (t_2 - r)^{I-1} \kappa(r) + t_1^{(1+I\beta)(1-\varkappa)} (t_1 - r)^{I-1} \kappa(r) \\ &\leq 2t_1^{(1+I\beta)(1-\varkappa)} (t_1 - r)^{I-1} \kappa(r),\end{aligned} \tag{26}$$

and $\int_0^{t_1} 2t_1^{(1+I\beta)(1-\varkappa)} (t_1 - r)^{I-1} \kappa(r)$ exists, i.e, $I_3 \longrightarrow 0$ as $t_2 \longrightarrow t_1$.

For $\varepsilon > 0$, we have

$$\begin{aligned}\mathfrak{I}_4 &= \left\| \int_0^{t_1} t_1^{(1+I\beta)(1-\varkappa)} [\mathfrak{Q}_I(t_2 - r) - \mathfrak{Q}_I(t_1 - r)] (t_1 - r)^{I-1} \mathfrak{Y}(r, \pounds(r))\mathrm{d}r \right\| \\ &\leq \int_0^{t_1-\epsilon} t_1^{(1+I\beta)(1-\varkappa)} \|\mathfrak{Q}_I(t_2 - r) - \mathfrak{Q}_I(t_1 - r)\|_{L(X)} (t_1 - r)^{I-1} \kappa(r)\mathrm{d}r \\ &\quad + \int_{t_1}^{t_1} t_1^{(1+I\beta)(1-\varkappa)} \|\mathfrak{Q}_I(t_2 - r) - \mathfrak{Q}_I(t_1 - r)\|_{L(X)} (t_1 - r)^{I-1} \kappa(r)\mathrm{d}r \\ &\leq t_1^{(1+I\beta)(1-\varkappa)} \int_0^{t_1} (t_1 - r)^{I-1} \kappa(r)\mathrm{d}r. \sup_{s\in[0,t_1-\epsilon]} \|\mathfrak{Q}_I(t_2 - r) - \mathfrak{Q}_I(t_1 - r)\|_{L(X)} \\ &\quad + \mathfrak{C}_p \int_{t_1}^{t_1} t_1^{(1+I\beta)(1-\varkappa)} \left((t_2 - r)^{-I-I\beta} + (t_1 - r)^{-I-I\beta} \right) (t_1 - r)^{I-1} \kappa(r)\mathrm{d}r \\ &\leq t_1^{(1+I\beta)(1-\varkappa)+I(1+\beta)} \int_0^{t_1} (t_1 - r)^{-I\beta-1} \kappa(r)\mathrm{d}r. \sup_{s\in[0,t_1-\epsilon]} \|\mathfrak{Q}_I(t_2 - r) - \mathfrak{Q}_I(t_1 - r)\|_{L(X)} \\ &\quad + 2\mathfrak{C}_p \int_{t_1-\epsilon}^{t_1} t_1^{(1+I\beta)(1-\varkappa)} (t_1 - r)^{-I\beta-1} \kappa(r)\mathrm{d}r.\end{aligned} \tag{27}$$

Since $\mathfrak{Q}_I(t)$ is uniformly continuous and $\lim_{t_2 \longrightarrow t_1} \mathfrak{I}_2 = 0$, then $\mathfrak{I}_4 \longrightarrow 0$ as $t_2 \longrightarrow t_1$ i.e. independent of $y \in \mathfrak{B}_r(\mathcal{F})$.

Clearly, since the strongly continuous of of $\mathfrak{S}_{I,\varkappa}(t)$, we get

$$\mathfrak{I}_5 = \left\| t_2^{(1+I\mu)(1-\varkappa)} \sum_{0<t_k<t_2} \mathfrak{S}_{I,\varkappa}(t_2 - t_k) \mathfrak{I}_k(\pounds(t_k^-)) - t_1^{(1+I\mu)(1-\varkappa)} \sum_{0<t_k<t_1} \mathfrak{S}_{I,\varkappa}((t_1 - t_k) \mathfrak{I}_k(\pounds(t_k^-))) \right\| \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1. \tag{28}$$

Hence $\|\mathfrak{F}y(t_2) - \mathfrak{F}y(t_1)\| \longrightarrow 0$ independently of $y \in \mathfrak{B}_r(\mathcal{J})$ as $t_2 \longrightarrow t_1$ therefore $\{\mathfrak{F}y: y \in \mathfrak{B}_r(\mathcal{J})\}$ is equicontinuous. □

Theorem 3. *Let $-1 < \beta < 0$ and $0 < \omega < \pi/2$ and $\mathfrak{A} \in \odot_{\omega}^{\beta}$. Then under consideration $(H1) - (H3)$ the operator $\{\mathfrak{F}y: y \in \mathfrak{B}_r(\mathcal{J})\}$ is continuous and bounded provided $\pounds_0 \in \mathcal{D}(\mathfrak{A}^{\theta})$ with $\theta > 1 + \beta$.*

Proof. We verify that $\mathfrak{F}$ maps $\mathfrak{B}_r(\mathcal{J})$. Taking $y \in \mathfrak{B}_r(\mathcal{J})$ and define $\pounds(t) = t^{-(1+I\beta)(1-\varkappa)}y(t)$, we have $\pounds \in \mathfrak{B}_r^{\mathfrak{Y}}(\mathcal{J}\prime)$. Let $t \in [0, T]$,

$$\|\mathfrak{F}\| \leq \left\|t^{(1+I\beta)(1-\varkappa)}\mathfrak{S}_{I,\varkappa}(t)u_0\right\| + t^{(1+I\beta)(1-\varkappa)} \times \left\|\int_0^t (t-r)^{I-1}\mathfrak{Q}_I(t-r)\mathfrak{Y}(r, \pounds(r))\mathrm{d}r\right\|. \tag{29}$$

From $(H2) - (H3)$, we get

$$\begin{aligned}\|\mathfrak{F}y(t)\| &\leq t^{(1+I\beta)(1-\varkappa)}\left\|\mathfrak{S}_{I,\varkappa}(t)u_0\right\| + t^{(1+I\beta)(1-\varkappa)} \times \int_0^t (t-r)^{-I\beta-1}\kappa(r)\mathrm{d}r \\ &\leq \sup_{[0,T]} t^{(1+I\beta)(1-\varkappa)} \int_0^t (t-r)^{-I\beta-1}\kappa(r)\mathrm{d}r \leq r.\end{aligned} \tag{30}$$

Hence $\|\mathfrak{F}y\| \leq r$, for any $y \in \mathscr{B}_r(I)$.

Now, to verify $\mathfrak{F}$ is continuous in $\mathscr{B}_r(I)$, let y_n, $y \in \mathscr{B}(I)$, $n = 1, 2, \ldots$, with $\lim_{n\longrightarrow\infty} y_n = y$. That is $\lim_{n\longrightarrow\infty} y_n(t) = y(t)$ and $\lim_{n\longrightarrow\infty} t^{-(1+I\beta)1-\varkappa}y_n(t) = t^{-(1+I\beta)1-\varkappa}y(t)$ and $\lim_{n\longrightarrow\infty} t^{-(1+I\beta)(1-\varkappa)}y_n(t) = t^{-(1+I\beta)(1-\varkappa)}y(t)$, on $\mathcal{J}\prime$.

$(H1)$ implies

$$\mathfrak{Y}(t, \pounds_n(t)) = \mathfrak{Y}\left(t, t^{-(1+I\beta)(1-\varkappa)}y_n(t), t^{-(1+I\beta)(1-\varkappa)}\right)) \longrightarrow \mathfrak{Y}\left(t, t^{-(1+I\beta)(1-\varkappa)}y(t), t^{-(1+I\beta)(1-\varkappa)}\right), \tag{31}$$

as $n \longrightarrow \infty$.

From $(H2)$ to obtain the inequality $(t-r)^{-I\beta-1}|\mathfrak{Y}(r, \pounds_n(r))| \leq 2(t-r)^{-(I\beta)(1-\varkappa)}\kappa(r)$, i.e.,

$$\int_0^t (t-r)^{-I\beta-1}\left\|\mathfrak{Y}(r, \pounds_n(r)) - \mathfrak{Y}(r, \pounds(r))\right\|\mathrm{d}r \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{32}$$

Let $t \in [0, T]$. Now,

$$\left\|\mathfrak{F}y_n(t) - \mathfrak{F}y(t)\right\| \leq t^{(1+I\beta)(1-\varkappa)} \times \left\|\int_0^t (t-r)^{I-1}\mathfrak{Q}_I(t-r)\mathfrak{Y}(r, \pounds_n(r)) - \mathfrak{Y}(r, \pounds(r))\mathrm{d}r\right\|. \tag{33}$$

Applying Theorem 1, we have

$$\left\|\mathfrak{F}y_n(t) - \mathfrak{F}y(t)\right\| \leq \mathfrak{C}_p t^{(1+I\beta)(1-\varkappa)} \times \int_0^t (t-r)^{-I\beta-1}\left\|\mathfrak{Y}(r, \pounds_n(r)) - \mathfrak{Y}(r, \pounds(r))\right\|\mathrm{d}r. \tag{34}$$

$\longrightarrow 0$ as $n \longrightarrow \infty$, i.e, $\mathfrak{F} y_n \longrightarrow \mathfrak{F} y$ pointwise on $\mathcal{J}$. Also Theorem 2 implies that $\mathfrak{F} y_n \longrightarrow \mathfrak{F} y$ uniformly on $\mathcal{J}$ as $n \longrightarrow \infty$. That is $\mathfrak{F}$ is continuous. □

4. $\mathfrak{T}t)$ is Compact

Theorem 4. *Let $-1<\beta<0$, $0<\omega<\pi/2$ and $\mathfrak{A} \in \Xi_{\omega}^{\beta}$. If $\mathfrak{T}(t)(t>0)$ is compact and $(H1)-(H4)$ hold then $\exists$ a mild solution of $(1.1)-(1.3)$ in $\mathcal{B}_r^{\mathcal{Y}}(I\imath)$ for every $\pounds_0 \in D(\mathfrak{A}^{\theta})$ with $\theta > 1+\beta$.*

Proof. Since we have assumed $\mathfrak{T}(t)$ is compact, then the equicontinuity of $\mathfrak{T}(t)(t>0)$. Moreover, by Theorem 2 and 3, $\mathfrak{F}: \mathfrak{B}_r^{\mathcal{Y}}(\mathcal{J}\imath) \longrightarrow \mathfrak{B}_r^{Y}(\mathcal{J}\imath)$ is continuous and bounded and $\varepsilon: \mathfrak{B}_r(\mathcal{J}) \longrightarrow \mathfrak{B}_r(\mathcal{J})$ is bounded, continuous and $\{\varepsilon y: y \in \mathfrak{B}_r(\mathcal{J})\}$ equicontinuous. We can write $\epsilon: \mathfrak{B}_r(\mathcal{J}) \longrightarrow \mathfrak{B}_r(\mathcal{J})$ by

$$(\epsilon y)(t) = \left(\epsilon^1 y\right)(t) + \left(\epsilon^2 y\right)(t), \tag{35}$$

where

$$\begin{aligned}
\left(\epsilon^1 y\right)(t) &= t^{(1+I\beta)(1-\varkappa)} \mathfrak{S}_{I,\varkappa}(t)\pounds_0 = t^{(1+I\beta)(1-\varkappa)} \mathcal{J}_{0+}^{\varkappa(1-I)} t^{I-1} \mathfrak{Q}_I(t)\pounds_0 \\
&= \frac{t^{(1+I\beta)(1-\varkappa)}}{\varkappa(\varkappa(1-I))} \int_0^t (t-r)^{\varkappa(1-I)-1} r^{I-1} \times \int_0^{\infty} I\theta M_I(\theta)\mathfrak{T}\left(r^I\theta\right)\pounds_0 \mathrm{d}\theta \mathrm{d}r \\
&= \frac{It^{(1+I\beta)(1-\varkappa)}}{\varkappa(\varkappa(1-I))} \int_0^t \int_0^{\infty} (t-r)^{\varkappa(1-I)-1} r^{I-1} \theta M_I(\theta)\mathfrak{T}\left(r^I\theta\right) \times \pounds_0 \mathrm{d}\theta \mathrm{d}r, \\
\left(\varepsilon^2 y\right)(t) &= t^{(1+I\beta)(1-\varkappa)} \int_0^t (t-r)^{I-1} \mathfrak{Q}_I(t-r)\mathfrak{Y}(r,\pounds(r))\mathrm{d}r + \sum_{0<t_k<t} \mathfrak{S}_{I,\varkappa}(t-t_k)\mathfrak{I}_k\left(\pounds\left(t_k^-\right)\right).
\end{aligned} \tag{36}$$

For $\sigma>0$ and $\zeta \in (0,t)$, we define an operator $\varepsilon_{\zeta,\sigma}^1$ on $\mathfrak{B}_r(\mathcal{J})$ by

$$\begin{aligned}
\left(\varepsilon_{\zeta,\sigma}^1 y\right)(t) &= \frac{t^{(1+I\beta)(1-\varkappa)}}{\varkappa(\varkappa(1-I))} \int_{\zeta}^t \int_{\sigma}^{\infty} (t-r)^{(1-I)\varkappa-1} r^{I-1} \theta M_I(\theta)\mathfrak{T}\left(r^I\theta\right) \times \pounds_0 \mathrm{d}\theta \mathrm{d}r \\
&= \frac{It^{(1+I\beta)(1-\varkappa)}}{\varkappa(\varkappa(1-I))} \mathfrak{T}\left(\zeta^I\sigma\right) \int_{\zeta}^t \int_{\sigma}^{\infty} (t-r)^{(1-I)\varkappa-1} r^{I-1} \theta M_I(\theta) \times \mathfrak{T}\left(r^I\theta - \zeta^I\sigma\right)\pounds_0 \mathrm{d}\theta \mathrm{d}r.
\end{aligned} \tag{37}$$

Since $\mathfrak{T}(\epsilon^I\delta)$ is compact $\mathcal{V}_{\zeta,\sigma}^1(t) = \left\{\varepsilon_{\zeta,\sigma}^1 y(t), y \in \mathfrak{B}_r(\mathcal{J})\right\}$ is precompact in $\mathcal{Y}\,\forall\zeta \in (0,t)$ and $\delta>0$. Moreover, for any $y \in \mathfrak{B}_r(\mathcal{J})$

$$\begin{aligned}
&\left\|\left(\varepsilon^1 y\right)(t) - \left(\varepsilon_{\zeta,\sigma}^1 y\right)(t)\right\| \\
&\quad \le \mathcal{K}(I,\varkappa)\left\|t^{(1+I\beta)(1-\varkappa)} \int_0^t \int_0^{\sigma} (t-r)^{\varkappa(1-I)-1} r^{I-1} \theta M_I(\theta)\mathfrak{T}\left(r^I\theta\right)\pounds_0 \mathrm{d}\theta \mathrm{d}r\right\| \\
&\qquad + \mathcal{K}(I,\varkappa)\left\|t^{(1+I\beta)(1-\varkappa)} \int_0^{\zeta} \int_{\sigma}^{\infty} (t-r)^{\varkappa(1-I)-1} r^{I-1} \theta M_I(\theta)\mathfrak{T}\left(r^I\theta\right)\pounds_0 \mathrm{d}\theta \mathrm{d}r\right\| \\
&\quad \le \mathcal{K}(I,\varkappa) t^{(1+I\beta)(1-\varkappa)} \int_0^t \int_0^{\sigma} (t-r)^{\varkappa(1-I)-1} r^{I-1} \theta M_I(\theta) r^{-I\varkappa-I} \left\|\pounds_0\right\| \theta^{-\beta-1} \mathrm{d}\theta \mathrm{d}r \\
&\qquad + \mathcal{K}(I,\varkappa) t^{(1+I\beta)(1-\varkappa)} \int_0^{\zeta} \int_{\sigma}^{\infty} (t-r)^{\varkappa(1-I)-1} r^{I-1} \theta M_I(\theta) r^{-I\beta-I} \theta^{-\beta-1} \left\|\pounds_0\right\| \mathrm{d}\theta \mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= \mathscr{K}(I,\varkappa)t^{(1+I\beta)(1-\varkappa)}\int_0^t (t-r)^{\varkappa(1-I)-1}r^{-I\beta-1}\|\pounds_0\|\mathrm{d}r\int_0^\sigma \theta^{-\beta}M_I(\theta)\mathrm{d}\theta \\
&\quad + \mathscr{K}(I,\varkappa)t^{(1+I\beta)(1-\varkappa)}\int_0^\zeta (t-r)^{\varkappa(1-I)-1}r^{-I\beta-1}\|\pounds_0\|\mathrm{d}r\int_\eta^\infty \theta^{-\beta}M_I(\theta)\mathrm{d}\theta \\
&\le \mathscr{K}t^{-I\varkappa(1+\beta)}\|\pounds_0\|\int_0^\eta \theta^{-\beta}M_I(\theta)\mathrm{d}\theta \\
&\quad + \mathscr{K}t^{-I\varkappa(1+\beta)}\|\pounds_0\|\int_0^\zeta (1-s)^{\varkappa(1-I)-1}r^{-I\beta-1}\mathrm{d}r\int_\eta^\infty \theta^{-\beta}M_I(\theta)\mathrm{d}\theta, \\
&\longrightarrow 0, \text{ as } \zeta\longrightarrow 0,\ \sigma\longrightarrow 0,
\end{aligned}
\tag{38}
$$

where, $\mathscr{K}(I,\varkappa) = I/\varkappa(\varkappa(1-I))$.

Therefore, $\mathscr{V}^1_{\zeta,\sigma}(t) = \left\{\varepsilon^1_{\zeta,\sigma}y(t), y\in\mathfrak{B}_r(\mathscr{J})\right\}$ are arbitrarily close to $\mathscr{V}^1(t) = \left\{\varepsilon^1 y(t), y\in\mathfrak{B}_r(\mathscr{J})\right\}$, for $t>0$. Hence $\mathscr{V}^1(t)$, for $t>0$, is precompact in $\mathscr{Y}$.

For $\zeta\in(0,t)$, $\sigma>0$, we can present an operator $\varepsilon^2_{\zeta,\sigma}$ on $\mathfrak{B}_r(\mathscr{J})$ by

$$
\begin{aligned}
\left(\varepsilon^2_{\zeta,\sigma}y\right)(t) &= It^{(1+I\beta)(1-\varkappa)}\int_0^{t-\zeta}\int_\sigma^\infty \theta M_I(\theta)(t-r)^{I-1}\mathfrak{I}\left((t-r)^I\theta\right)\mathfrak{Y}(r,\pounds(r))\mathrm{d}\theta\mathrm{d}r \\
&\quad + \sum_{0<t_k<t}\mathfrak{S}_{I,\varkappa}(t-t_k)\mathfrak{I}_k\left(\pounds\left(t_k^-\right)\right) \\
&= It^{(1+I\beta)(1-\varkappa)}\mathfrak{T}\left(\zeta^I\sigma\right)\int_0^{t-\zeta}\int_\sigma^\infty \theta M_I(\theta)(t-r)^{I-1}\times\mathfrak{I}\left((t-r)^I\theta-\zeta^I\sigma\right)\mathfrak{Y}(r,\pounds(r))\mathrm{d}\theta\mathrm{d}r \\
&\quad + \sum_{0<t_k<t}\mathfrak{S}_{I,\varkappa}(t-t_k)\mathfrak{I}_k\left(\pounds\left(t_k^-\right)\right).
\end{aligned}
\tag{39}
$$

Hence $\mathscr{V}^2_{\zeta,\sigma}(t) = \left\{\varepsilon^2_{\zeta,\sigma}y(t), y\in\mathfrak{B}_r(\mathscr{J})\right\}$ is precompact in $\mathscr{Y}$ $\forall\zeta\in(0,t)$ and $\sigma>0$ due to the compactness of $\mathfrak{I}(\zeta^I\sigma)$. For every $y\in\mathfrak{B}_r(\mathscr{J})$, we get

$$
\begin{aligned}
&\left\|\left(\varepsilon^2 y\right)(t)-\left(\varepsilon^2_{\zeta,\sigma}y\right)(t)\right\| \\
&\le \left\|It^{(1+I\beta)(1-\varkappa)}\left(\int_0^t\int_0^\sigma \theta MI(\theta)(t-r)^{I-1}\mathfrak{I}\left((t-r)^I\theta\right)\mathfrak{Y}(r,\pounds(r))\mathrm{d}\theta\mathrm{d}r\right.\right. \\
&\quad \left.\left.+\sum_{0<t_k<t}\mathfrak{S}_{I,\varkappa}(t+\sigma-t_k)\mathfrak{I}_k\left(\pounds\left(t_k^-\right)\right)\right)\right\| \\
&\quad + \left\|It^{(1+I\beta)(1-\varkappa)}\left(\int_{t-\zeta}^t\int_\sigma^\infty (t-r)^{I-1}\theta M_I(\theta)\mathfrak{I}\left((t-r)^I\theta\right)\mathfrak{Y}(r,\pounds(r))\mathrm{d}\theta\mathrm{d}r\right.\right. \\
&\quad \left.\left.+\sum_{0<t_k<t}\mathfrak{S}_{I,\varkappa}(t+\zeta-t_k)\mathfrak{I}_k\left(\pounds\left(t_k^-\right)\right)\right)\right\| \\
&\le I\mathfrak{C}_0 t^{(1+I\beta)(1-\varkappa)}\left(\int_0^t (t-r)^{-I\beta-1}k(r)\mathrm{d}r\int_0^\sigma \theta^{-\beta}M_I(\theta)\mathrm{d}\theta+\sum_{0<t_k<\sigma}\omega_k\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ I\mathfrak{C}_0 t^{(1+I\beta)(1-\varkappa)}\left(\int_{t-\zeta}^{t}(t-r)^{-I\beta-1}k(r)\mathrm{d}r\int_0^{\infty}\theta^{-\beta}M_I(\theta)\mathrm{d}\theta+\sum_{0<t_k<\zeta}\omega_k\right)\\
&\times I\mathfrak{C}_0 t^{(1+I\beta)(1-\varkappa)}\\
&\left(\int_0^{t}(t-r)^{-I\beta-1}k(r)\mathrm{d}r\times\int_0^{\sigma}\theta^{-\beta}M_I(\theta)\mathrm{d}\theta+\sum_{0<t_k<\eta}\omega_k\right)\\
&+\frac{I\mathfrak{C}_0\varkappa(1-\beta)}{\varkappa(1-I\beta)}t^{(1+I\beta)(1-\varkappa)}\left(\int_{t-\zeta}^{t}(t-r)^{-I\beta-1}k(r)\mathrm{d}r+\sum_{0<t_k<\zeta}\omega_k\right),\\
&\longrightarrow 0\text{ as }\sigma\longrightarrow 0,\quad \sigma\longrightarrow 0.
\end{aligned}
\tag{40}
$$

Therefore, $\mathscr{V}^2_{\zeta,\sigma}(t)=\left\{\varepsilon^2_{\zeta,\sigma}y(t), y\in\mathfrak{B}_r(\mathscr{J})\right\}$ are arbitrarily close to $\mathscr{V}^2(t)=\{\varepsilon^2 y(t), y\in\mathfrak{B}_r(\mathscr{J})\}$, $t>0$. That is, $\{\varepsilon y, y\in\mathfrak{B}_r(\mathscr{J})\}$ is relatively compact by using the Arzela-Ascoli Theorem. Then $\pounds^*$ is a mild solution of (1.1)-(1.3). □

5. $\mathfrak{T}(t)$ is Noncompact

We consider as follows, $(H5)$ $\exists$ a constant $k>0$ satisfies the following

$$\Xi(\mathfrak{Y}(t,\mathfrak{E}_1,\mathfrak{E}_2))\le k\Xi(\mathfrak{E}_1,\mathfrak{E}_2)\text{ for }a.e\,t\in[0,T], \tag{41}$$

for every bounded subset $\mathfrak{E}_1,\mathfrak{E}_2\subset\mathscr{Y}$.

Theorem 5. *Let $-1<\beta<0$, $0<\omega<\pi/2$, and $\mathfrak{A}\in\Xi\omega^{\beta}$. Suppose $(H1)-(H5)$ hold. Then, $(1.1)-(1.3)$ has a mild solution in $\mathfrak{B}_r^{\mathscr{Y}}(J\prime)$ for every $u_0\in D(\mathfrak{Y}^{\theta})$ with $\theta>1+\beta$.*

Proof. By Theorem 2 and 3, we get ε: $\mathfrak{B}_r(\mathscr{J})\longrightarrow\mathfrak{B}_r(\mathscr{J})$ is continuous, bounded, and $\{\varepsilon y\colon y\in\mathfrak{B}_r(\mathscr{J})\}$ is equicontinuous. Also, we prove that there is a subset of $\mathfrak{B}_r(\mathscr{J})$ such that ε is campact in it.

For any bounded set $\mathbb{P}_0\subset\mathfrak{B}_r(\mathscr{J})$, set

$$
\begin{aligned}
&\varepsilon^{(1)}(\mathbb{P}_0)=\varepsilon(\mathbb{P}_0),\\
&\varepsilon^{(n)}(\mathbb{P}_0)=\varepsilon\left(\overline{co}\left(\varepsilon^{(n-1)}(\mathbb{P}_0)\right)\right),\ n=2,3,\ldots.
\end{aligned}
\tag{42}
$$

For any $\epsilon>0$, we can obtain from Propositions ((2.2) – (2.4)), a subsequence $\{y_n^{(1)}\}_{n=1}^{\infty}\subset\mathbb{P}_0$ satisfying

$$
\begin{aligned}
\Xi\left(\varepsilon^{(1)}(\mathbb{P}_0(t))\right)&\le 2\Xi\left(t^{(1+I\beta)(1-\varkappa)}\int_0^t(t-r)^{I-1}\mathfrak{Q}_I(t-r)\right.\\
&\left.\mathfrak{Y}\left(r,\left\{r^{-(1+I\beta)(1-\varkappa)}\left(y_n^{(1)}(r)\right)\right\}_{n=1}^{\infty}\right)\mathrm{d}r+\sum_{0<t_k<t}\omega_k\right)\\
&\le 4\mathfrak{C}_p t^{(1+I\beta)(1-\varkappa)}\left(\int_0^t(t-r)^{-I\beta-1}\right.\\
&\left.\Xi\left(\mathfrak{Y}\left(r,\left\{r^{-(1+I\beta)(1-\varkappa)}\left(y_n^{(1)}(r)\right)\right\}_{n=1}^{\infty}\right)\right)\mathrm{d}r+\sum_{0<t_k<t}\omega_k\right)\\
&\le 4\mathfrak{C}_p k t^{(1+I\beta)(1-\varkappa)}\Xi(\mathfrak{P}_0)\left(\int_0^t(t-r)^{-I\beta-1}r^{-(1+I\beta)(1-\varkappa)}\mathrm{d}r+\sum_{0<t_k<t}\omega_k\right)\\
&=4\mathfrak{C}_p k t^{-I\beta}\Xi(\mathfrak{P}_0)\left(\frac{\varkappa(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(-2I\beta+\varkappa(1+I\beta))}+\sum_{0<t_k<t}\omega_k\right).
\end{aligned}
\tag{43}
$$

From ϵ is arbitrary,

$$\Xi\left(\varepsilon^{(1)}(\mathbb{P}_0(t))\right)\le 4\mathfrak{C}_p k t^{-I\beta}\Xi(\mathbb{P}_0)\left(\frac{\varkappa(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(-2I\beta+\varkappa(1+I\beta))}+\sum_{0<t_k<t}\omega_k\right). \tag{44}$$

Again for any $\epsilon > 0$, we can get from Propositions ((2.2) – (2.4)) a subsequence $\{y_n^{(2)}, \mathscr{B} y_n^{(2)}\}_{n=1}^{\infty} \subset \overline{co}(\varepsilon^{(1)}(\mathbb{P}_0))$ implies that

$$\begin{aligned}
\Xi\left(\varepsilon^{(2)}(\mathbb{P}_0(t))\right) &= \Xi\left(\varepsilon\left(\overline{co}\left(\varepsilon^{(1)}(\mathbb{P}_0(t))\right)\right)\right) \\
&\leq 2\Xi\Bigg(t^{(1+I\beta)(1-\varkappa)} \int_0^t (t-r)^{I-1} \mathcal{Q}_I(t-r) \\
&\quad \times \mathfrak{Y}\left(r, \left\{r^{-(1+I\beta)(1-\varkappa)}\left(y_n^{(2)}(r)\right)\right\}_{n=1}^{\infty}\right) \mathrm{d}r + \sum_{0<t_k<t} \omega_k\Bigg) \\
&\leq 4\mathfrak{C}_p t^{(1+I\beta)(1-\varkappa)}\Bigg(\int_0^t (t-r)^{-I\beta-1} \Xi \\
&\quad \times \mathfrak{Y}\left(r, \left\{r^{-(1+I\beta)(1-\varkappa)}\left(y_n^{(2)}(r)\right)\right\}_{n=1}^{\infty}\right) \mathrm{d}r + \sum_{0<t_k<t} \omega_k\Bigg) \\
&\leq 4\mathfrak{C}_p k t^{(1+I\beta)(1-\varkappa)}\Bigg(\int_0^t (t-r)^{-I\beta-1} \\
&\quad \times \Xi\left(r^{-(1+I\beta)(1-\varkappa)}\left(\left\{y_n^{(2)}(r)\right\}_{n=1}^{\infty}\right)\right) \mathrm{d}r + \sum_{0<t_k<t} \omega_k\Bigg) \\
&\leq 4\mathfrak{C}_p k t^{(1+I\beta)(1-\varkappa)}\Bigg(\int_0^t (t-r)^{-I\beta-1} r^{-(1+I\beta)(1-\varkappa)} \\
&\quad \times \Xi\left(\left\{y_n^{(2)}(r)\right\}_{n=1}^{\infty}\right) \mathrm{d}r + \sum_{0<t_k<t} \omega_k\Bigg) \\
&\leq \frac{\left(4\mathfrak{C}_p k\right)^2 t^{(1+I\beta)(1-\varkappa)} \varkappa(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(-2I\beta+\varkappa(1+I\beta))} \\
&\quad \times \Xi(\mathbb{P}_0)\Bigg(\int_0^t (t-r)^{-I\beta-1} r^{-(1+I\beta)(1-\varkappa)-I\beta} \mathrm{d}r + \sum_{0<t_k<t} \omega_k\Bigg) \\
&= \left(\frac{\left(4\mathfrak{C}_p k\right)^2 t^{-2I\beta} \varkappa^2(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(-3I\beta+\varkappa(1+I\beta))} + \sum_{0<t_k<t} \omega_k\right)\Xi(\mathbb{P}_0).
\end{aligned} \tag{45}$$

Now,

$$\Xi\left(\varepsilon^{(n)}(\mathbb{P}_0(t))\right) \leq \frac{\left(4\mathfrak{C}_p k\right)^n t^{-nI\beta} \varkappa^n(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(-(n+1)I\beta+\varkappa(1+I\beta))}\Xi(\mathbb{P}_0), \quad n \in \mathbb{N}. \tag{46}$$

Let $M = 4\mathfrak{C}_p k T^{-I\beta}\varkappa(-I\beta)$. We can find $m, k \in \mathbb{N}$ big enough such that $1/k < I\beta < 1/k - 1$ and $n + 1/k > 2$ for $n > m$. $\varkappa(-(n+1)I\beta + \varkappa(1+I\beta)) > \varkappa(n+1/k)$. That is

$$\frac{\left(4\mathfrak{C}_p k\right)^n T^{-nI\beta}\varkappa^n(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(-(n+1)I\beta+\varkappa(1+I\beta))}<\frac{\left(4\mathfrak{C}_p k\right)^n T^{-nI\beta}\varkappa^n(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(n+1/k)}. \tag{47}$$

Replace $(n+1)$ by $(j+1)k$. Then R.H.S of inequality given above becomes

$$\frac{M^{(j+1)k-1}\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(j+1)}=\frac{\left(M^k\right)^j M^{k-1}\varkappa(-I\beta+\varkappa(1+I\beta))}{j!}\longrightarrow 0 \text{ as } j\longrightarrow\infty. \tag{48}$$

Therefore, $\exists$ a constant $n_0\in\mathbb{N}$ that is

$$\frac{4\mathfrak{C}_p k^n t^{-nI\beta}\varkappa^n(-I\beta)\varkappa-I\beta+\varkappa(1+I\beta)}{\varkappa(-(n+1)I\beta+\varkappa(1+I\beta))}\leq\frac{\left(4\mathfrak{C}_p k\right)^{n_0}T^{-n_0 I\beta}\varkappa^{n_0}(-I\beta)\varkappa(-I\beta+\varkappa(1+I\beta))}{\varkappa(-(n_0+1)I\beta+\varkappa(1+I\beta))}=p<1. \tag{49}$$

Now

$$\Xi\left(\varepsilon^{(n_0)}\left(\mathbb{P}_0(t)\right)\right)\leq p\Xi\left(\mathbb{P}_0\right). \tag{50}$$

From $\varepsilon^{(n_0)}(\mathbb{P}_0(t))$ is bounded and equicontinuous, applying Proposition (2.2), we get

$$\Xi\left(\varepsilon^{(n_0)}\left(\mathbb{P}_0\right)\right)=\max_{t\in[0,T]}\Xi\left(\varepsilon^{n_0}\left(\mathbb{P}_0(t)\right)\right). \tag{51}$$

Hence,

$$\Xi\left(\varepsilon^{n_0}\left(\mathbb{P}_0\right)\right)\leq p\Xi\left(\mathbb{P}_0\right), \tag{52}$$

where $p<1$. By applying the Schauder fixed point theorem, we obtain a fixed point y^* in $\mathfrak{B}_r(\mathcal{J})$ of ε. Let $\pounds^*(t)=(\Psi(t))^{(1+I\varphi)(\varkappa-1)}y^*(t)$. Then, $\pounds^*(t)$ is a mild solution of $(1.1)-(1.3)s$. □

6. Illustrate an application for main results

We consider the following impulsive Hilfer derivative fractional system:

$$\begin{aligned}
\mathfrak{D}_{0+}^{I,\varkappa}\pounds(t,x)-\partial_x^2\pounds(t,x)&=\mathfrak{Y}(t,\pounds(t,x)),\quad t\in[0,T],\ x\in[0,a],\\
\pounds(t,0)&=\pounds(t,a)\\
&=0 \text{ on } t\in[0,T],\\
\mathcal{J}_{0+}^{(1-I)(1-\varkappa_k)}\pounds(0,x)&=\pounds_0(x),\quad x\in[0,a],\\
\Delta\pounds|_{t=1/2}&=\mathfrak{I}_1\left(\pounds\left(\frac{1^-}{2}\right)\right),
\end{aligned} \tag{53}$$

in Banach space $\mathcal{Y}=C^I([0,a])(0<I<1)$ of all Holder continuous functions, where $I=1/4$, $\varkappa=1/2$, $\mathfrak{Y}(t,\pounds,\mathcal{B}\pounds)=t^{-1/5}\cos^2\pounds$. Here, we can convert the above problem $(1.1-1.3)$ in abstract form as

$$\begin{aligned}
\mathfrak{D}^{I,\varkappa}\pounds(t)+\mathfrak{A}\pounds(t)&=\mathfrak{Y}(t,\pounds(t)),\quad t\in(0,T]=\mathcal{J},\\
\Delta\pounds|_{t=t_k}&=\mathfrak{I}_k\left(\pounds\left(t_k^-\right)\right),\quad k=1,2,3,\ldots,m,\\
I_{0+}^{(1-I)(1-\varkappa_k)}\pounds(0)&=\pounds_0.
\end{aligned} \tag{54}$$

Here $\mathfrak{A}=-\partial_x^2$ with $\mathfrak{D}(\mathfrak{A})=\left\{\pounds\in C^{2+I}([0,a])\text{ that is }\pounds(t,0)=0=\pounds(t,a)\right\}$. It follows from the work in [18] $\exists$ constants $\delta,\epsilon>0$, implies $\mathfrak{A}+\delta\in\odot_{\pi/2-\epsilon}^{\pi/2-1}(\mathcal{Y})$. To verify the compactness of semigroup $\mathfrak{I}(t)$, it is enough to prove that $\mathcal{R}I,-(\mathfrak{A}+\delta)$ is compact for every $I>0$. By $\mathfrak{D}(\mathfrak{A}\subset C^{2+I}([0,a]))$ and $C^{2+I}([0,a])$ is compactly embedded in $C^I([0,a])$, the compactness of the resolvent operator for every $I>0$. We take $l(t)=t^{-1/5}$.

$$r=\sup_{[0,T]}\left(t^{(1+I\beta)(1-\varkappa)}\left\|\mathfrak{S}_{I,\varkappa}(t)u_0\right\|\right)+\frac{T^{17/20\varkappa(-\beta/4)\varkappa(4/5)}}{\varkappa(4/5-\beta/4)}. \tag{55}$$

Then, hypotheses $(H1)-(H5)$ are satisfied. According to Theorem 4, the problem (6.1) has mild solution in $\mathfrak{B}_r^{\mathfrak{Y}}((0,T])$.

7. Conclusion

The main objective of in this paper, we discussed the mild solutions for Hilfer fractional derivative differential equations involving jump conditions and almost sectorial operator when associated semigroup is compact and noncompact using Schauder fixed point theorem. Our theorems guarantee the effectiveness of existence results It is the results of the equations concerned. we discussed an example to verify the existence results. We will find to to investigate stability of similar problem in our future research work.

Data Availability

There is no data used for this manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors have equal contribution and have finalized the manuscript.

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