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Research Article

Bessel-Riesz Operators on Lebesgue Spaces and Morrey Spaces Defined in Measure Metric Spaces

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The boundedness of Bessel–Riesz operators defined on Lebesgue spaces and Morrey spaces in measure metric spaces is discussed in this research study. The maximal operator and traditional dyadic decomposition are used to study the Bessel-Riesz operators. We investigate the interaction between the kernel and space parameters to get the results and see how this affects kernel-bound operators.

1. Introduction

The Bessel–Riesz operator is a type of singular integral operator which acts on a function by convolving it with a Bessel–Riesz kernel. These operators are defined on metric spaces and are particularly useful in the study of measure spaces. They possess many important properties such as being bounded, compact, and having a certain degree of regularity. The study of Bessel-Riesz operators is a fundamental tool in harmonic analysis and has applications in other areas of mathematics such as partial differential equations and complex analysis.

The Schrödinger's equation has been used to derive the Bessel–Riesz operators. The wave function or state function of a quantum mechanical system is described by Schrödinger's equation has been used to derive the Bessel–Riesz operators. Schrödinger's equation is a linear partial differential equation. In quantum physics, Schrödinger's equation is analogous to Newton's law. Kurata et al. [1] investigated Schrödinger's operator and the boundedness of integral operators on generalized Morrey spaces in 1999.

According to Garca-Cuerva and Eduardo Gatto [2], Riesz potential is limited between Lebesgue spaces in terms of Euclidean settings. In Kokilashvili [3] (see also Edmunds, [4], Chapter 6), the nondoubling measure is covered in great detail, including conclusions addressing the boundedness of the fractional integral operator on Lebesgue spaces with the nondoubling measure. Theorems of the Sobolev and Adams types for fractional integrals built on quasimetric measure spaces were used to show the result for potentials in Euclidean settings [3]. Garcia-Cuerva and Jos Mara [5] demonstrate that fractional integrals on measure metric spaces are bounded. Adams [6] and Peetre [7] studied the Riesz potential in Euclidean situations using Morrey spaces.

According to Eridani's proof [8], fractional integral operators are constrained in weighted Morrey spaces with quasimetric measure spaces. In [9, 10], it was proven that maximum and fractional integral operators in classical Morrey spaces are bounded. The boundedness of these operators in Morrey spaces was recently established by Idris et al. ([11], Theorem 6, pp.3). They found similar results to the Chiarenza result [12] about the boundedness of fractional integral operators on Morrey spaces. Additionally, results for weighted Morrey spaces and generalized Morrey spaces can be found in [13–17]. The examination by Eridani et al. [8] of the boundedness of fractional integral operators in Morrey spaces based on quasimetric measure spaces is novel in this study. The paper by Idris et al. [11] also includes some discoveries on Young with weight and demonstrates that these operators on Young are bound in Euclidean contexts. Euclidean spaces are the most straightforward way to represent metric spaces.

A separate approach will be used to demonstrate the boundedness of these operators on Young in measure metric spaces. We shall also discover that the norm of the kernels constrains the norm of the operators. In [18], Young inequality was utilized to demonstrate that Bessel-Riesz operators are bounded. With measure metric spaces, we will look into the boundedness of these integral operators on Lebesgue and Morrey spaces. The Young inequality cannot be used indefinitely to produce the same results. To demonstrate that Bessel–Riesz operators are bound on Young, we will employ the maximal operator. The boundedness of Bessel-Riesz operators on quasimetric spaces, a recent scientific innovation, is demonstrated in this article as an extension of the findings [19] from earlier work on Morrey spaces. The kernels of the operators hold some parameters. The usual dyadic decomposition and the Hardy–Littlewood maximal operator will all be applied in the proofs as before. We will find that the norm of the kernels dominates the norm of these integral operators. This paper will discuss an integral operator's boundedness in measure metric spaces.

The following Table 1 contains a list of notations used in this article.

2. Preliminaries

2.1. Measurable Spaces

Definition 1. A distinguished collection Σ of subsets of X is called sigma algebra if it satisfies the following axioms:

(1) If $A \in \Sigma$, then $A^c \in \Sigma$,

(2) If $A_1, A_2, \ldots$ are countable family of sets in Σ then $\cup_{i=1}^{\infty} A_i \in \Sigma$,

(3) The intersection of countable sets also belongs to Σ i.e., $\cap_{i=1}^{\infty} A_i \in \Sigma$.

Definition 2. A measure μ defined on sigma-algebraΣ, is a function $\mu\colon \Sigma \longrightarrow \mathbb{R}_+$ such that

(1) $\mu(\varnothing) = 0$, where $\varnothing$ is an empty set.

(2) If $A_1, A_2, \ldots$ is a sequence of disjoint sets in Σ, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{n=1}^{\infty} \mu(A_i). \tag{1}$$

If Σ is σ-algebra of subsets of X, then the pair (X, Σ) is said to be a measurable space and members of Σ are called Σ -measurable sets. Moreover, if (X, Σ) is a measurable space, and if μ is a measure on Σ, then the triplet (X, Σ, μ) is said to be a measure space.

Definition 3 (see [8]). We assume that $X \coloneqq (X, \delta, \mu)$ is a topological space, endowed with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a function (quasimetric) $\delta\colon X \times X \longrightarrow [0, \infty)$ satisfying the conditions:

(i) $\delta(x, y) > 0$ for all $x \neq y$, and $\delta(x, x) = 0$ for all $x \in X$;

(ii) There exists a constant $a_0 \geq 1$, such that $\delta(x, y) \leq a_0 \delta(y, x)$ for all $x, y \in X$;

(iii) There exists a constant $a_1 \leq 1$, such that $\delta(x, y) \leq a_1 (\delta(x, z) + \delta(z, y))$ for all $x, y, z \in X$.

We assume that the balls $B(a, r) \coloneqq x \in X\colon \delta(a, x) < r$ are μ-measurable and $0 < \mu(B(a, r)) < \infty$ for $a \in X, r > 0$. For every neighborhood V of $x \in X$, there exists $r > 0$ such that $B(x, r) \subset V$. We also assume that $\mu(X) = \infty, \mu(a) = 0$ and $B(a, r_2)/B(a, r_1) \neq \varnothing$ for all $a \in X, 0 < r_1 < r_2 < \infty$. The triple (X, δ, μ) will be called quasimetric measure space.

Definition 4 (see [20]). In measure metric spaces (X, δ, μ), we will have the properties $\mu(B) \sim r^{\beta}$ for every balls $B \coloneqq B(a, r)$. For $0 < \alpha < \beta, \gamma > 0$, we define

$$G_{\alpha,\gamma}\colon = \frac{\delta(x, y)^{\alpha-\beta}}{(1 + \delta(x, y))^{\gamma}}. \tag{2}$$

For any x in X, the function $G_{\alpha,\gamma}$ may be regarded as the product of two kernels, J_{γ} be a Bessel kernel, and G_{α} be a Riesz kernel. We define $L^p(\mu)$ as any measurable functions f such that $1 \leq p < \infty$.

$$\|f\colon L^p(\mu)\| = \left(\int_X |f(t)|^p \mathrm{d}\mu(t)\right)^{1/p} < \infty. \tag{3}$$

If $\gamma = 0$, then we have G_{α} which also referred to as Riesz Potentials or fractional integrals. Since the 1920s, research on Riesz potentials has been conducted. These operators' boundedness has been studied by Hardy [21, 22] and Sobolev [23]. See [1], for example, of using the aforementioned operators in settings involving Euclidean spaces. We have evidence that Eridani [8] citation of the G_{α} boundedness results is correct. Spanne [2] has demonstrated the boundedness of I_{α} on Morrey spaces. Additionally, Adams [6] and Chiarenza [12] derived a stronger conclusion about the boundedness of Riesz operators. Using the multiplication operator W, Kurata et al. [1] demonstrated that the $G_{\alpha,\gamma} f(x)$ function is confined on generalized Morrey spaces. The boundedness of $G_{\alpha,\gamma} f(x)$ on Young in Euclidean settings was discussed by Idris et al. [11] in the cited article. In this section, we will talk about Bessel–Riesz operators with boundedness in measure metric spaces. For every $a \in X$, and $r > 0$, then

Table 1: The list of notations used in this paper.

$\mathbb{R}$	The real numbers
$\mathbb{R}^+$	The positive real numbers
$\mathbb{R}^\mathbf{n}$	The Euclidean spaces
Σ	Sigma-algebra
μ	A measure defined on sigma-algebra
δ	Quasimetric function
$\mathbf{L^p}(\mu)$	The class of Lebesgue measurable set $(1 \le p < \infty)$
ϕ	Function $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$
$\mathbf{L^p_\phi}(\mu)$	(Classical) Morrey space $(1 \le p \le q)$
$\mathbf{X}$	Quasimetric measure space with a complete measure μ and quasimetric function δ
$\mathbf{B}$	$B(a, r)$ Centered at $a \in X$ with radius $r > 0$
$\widetilde{\mathbf{B}}$	$B(a, 2r)$ Centered at $a \in X$ with radius $2r > 0$
$\mathbf{G}_{\alpha,\gamma}$	Bessel–Riesz kernel $0 < \alpha < \gamma, \gamma \ge 0$
$\mathbf{G}_{\alpha,\gamma} f(x)$	Bessel–Riesz operators $0 < \alpha < \gamma, \gamma \ge 0$
$\mathbf{M}_q f(x)$	Hardy–Littlewood maximal operator

$$\int_X \left|G_{\alpha,\gamma}\right| \mathrm{d}\mu(x) = \sum_{k\in\mathbb{Z}} \int_{2^k r \le \delta(a,x) < 2^{k+1} r} \left|G_{\alpha,\gamma}\right| \mathrm{d}\mu(x) \sim \sum_{k\in\mathbb{Z}} \frac{\left(2^k r\right)^\alpha}{\left(1 + 2^k r\right)^\gamma}. \tag{4}$$

We know that $G_{\alpha,\gamma} \in L^1(\mu)$, provided that $0 < \alpha < \gamma$. Now, we introduce the following operator:

$$\left(G_{\alpha,\gamma} f(x)\right) := \int_X \frac{\delta(x,y)^{\alpha-1} f(y)}{(1 + \delta(x,y))^\gamma} \mathrm{d}\mu(y), \quad x \in X. \tag{5}$$

2.2. Hardy–Littlewood Maximal Operator. The M_q operator which includes Hardy–Littlewood maximal operator defined as follows [11]:

$$M_q f(x) := \left(\sup_{x\in B} \frac{1}{\mu(B)} \int_B |f(y)|^q \mathrm{d}\mu(y)\right)^{1/q}. \tag{6}$$

A typical outcome for M_q is that it is constrained by $L^p(\mu)$ for $1 \le q < p \le \infty$.

Definition 5. Morrey Spaces [8] For $1 \le p < \infty$ and an appropriate $\phi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. In order to define the Morrey space, we use the formula $L^p_\phi(\mu) = L^p_\phi(X,\mu)$ is the set of all functions, where $f \in L^p_{\mathrm{loc}}(X)$ for

$$\left\|f: L^p_\phi(\mu)\right\| := \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(y)|^p \mathrm{d}\mu(y)\right)^{1/p} < \infty. \tag{7}$$

Here, we define $\phi(B) := \phi(\mu(B))$.

3. Main Results

Theorem 1. *For* $1 < p < \infty, 0 < \alpha < \min\{1, \gamma\}$, *there exists* $C > 0$ *such that*

$$\left\|G_{\alpha,\gamma} f(x): L^p(\mu)\right\| \le C \left\|G_{\alpha,\gamma}: L^1(\mu)\right\| \cdot \left\|f: L^p(\mu)\right\|. \tag{8}$$

Proof. Suppose $f \ge 0$, and for every $r > 0$, $x \in X$, we have

$$\left|G_{\alpha,\gamma} f(x)\right| = \left(\int_{\delta(x,y)<r} + \int_{\delta(x,y)\ge r}\right) \frac{\delta(x,y)^{\alpha-1} f(y)}{(1 + \delta(x,y))^\gamma} \quad d\mu(y) =: I_1(x,r) + I_2(x,r). \tag{9}$$

It is easy to see that the following estimates is holds. Then we have

$$\begin{aligned} I_1(x,r) &= \int_{\delta(x,y)<r} \frac{\delta(x,y)^{\alpha-1} f(y)}{(1 + \delta(x,y))^\gamma} \mathrm{d}\mu(y) \\ &= \sum_{k=-\infty}^{-1} \int_{2^k r \le \delta(x,y) < 2^{k+1} r} \frac{\delta(x,y)^{\alpha-1} f(y)}{(1 + \delta(x,y))^\gamma} \mathrm{d}\mu(y) \end{aligned}$$

$$\begin{aligned}
&\sim \sum_{k=-\infty}^{-1} \frac{\left(2^k r\right)^{\alpha-1}}{\left(1+2^k r\right)^{\gamma}} \int_{B\left(x,2^{k+1} r\right)} f(y) \mathrm{d}\mu(y) \\
&\leq C M_1 f(x) \cdot \left\|G_{\alpha,\gamma}: L^1(\mu)\right\|, \\
I_2(x,r) &= \sum_{k=0}^{\infty} \int_{2^k r \leq \delta(x,y)<2^{k+1} r} \frac{\delta(x,y)^{\alpha-1} f(y)}{(1+\delta(x,y))^{\gamma}} \mathrm{d}\mu(y) \\
&\sim \sum_{k=0}^{\infty} \frac{\left(2^k r\right)^{\alpha-1}}{\left(1+2^k r\right)^{\gamma}} \int_{B\left(x,2^{k+1} r\right)} f(y) \mathrm{d}\mu(y) \\
&\leq C\left\|f: L^p\right\| \sum_{k=0}^{\infty} \frac{\left(2^k r\right)^{\alpha}\left(2^{k+1} r\right)^{-1/p}}{\left(1+2^k r\right)^{\gamma}} \\
&\leq C r^{-1/p}\left\|G_{\alpha,\gamma}: L^1(\mu)\right\| \cdot \left\|f: L^p(\mu)\right\|.
\end{aligned} \tag{10}$$

So, for every $x \in X$, and $r > 0$, we have

$$\left|\left(G_{\alpha,\gamma} f(x)\right)\right| \leq C\left(M_1 f(x) + r^{-1/p}\left\|f: L^p(\mu)\right\|\right)\left\|G_{\alpha,\gamma}: L^1(\mu)\right\|. \tag{11}$$

If we choose $r > 0$ such that $M_1 f(x) = r^{-1/p}\|f: L^p(\mu)\|$, then

$$\left|\left(G_{\alpha,\gamma} f(x)\right)\right| \leq C M_1 f(x) \cdot \left\|G_{\alpha,\gamma}: L^1(\mu)\right\|, \tag{12}$$

and this finish the proof. (QED).

Next, by using Definition 5 of Morrey spaces, we have

Theorem 2. *Suppose* $1 < p < \infty, 0 < \alpha < \min\{1,\gamma\}$, *and* ϕ *is a decreasing functions. Then there exists* $C > 0$ *such that*

$$\left\|G_{\alpha,\gamma} f(x): L_\phi^p(\mu)\right\| \leq C\left\|G_{\alpha,\gamma}: L^1(\mu)\right\| \cdot \left\|f: L_\phi^p(\mu)\right\|. \tag{13}$$

Proof. Suppose for $B := B(a,r)$, and $\widetilde{B} := B(a,2r)$, we define

$$f = f_1 + f_2 := f\chi_{\widetilde{B}} + f\chi_{\widetilde{B}^c},\ f \in L_\phi^p(\mu). \tag{14}$$

Since $\|f_1: L^p(\mu)\| \leq \phi(\widetilde{B})\mu(\widetilde{B})^{1/p}\|f: L_\phi^p(\mu)\| < \infty$, then $f_1 \in L^p(\mu)$. By the previous result, for every B, we have

$$\begin{aligned}
&\left(\int_B \left|\left(G_{\alpha,\gamma} f_1(x)\right)(y)\right|^p \mathrm{d}\mu(y)\right)^{1/p} \\
&\leq \left\|G_{\alpha,\gamma} f_1(x): L^p(\mu)\right\| \leq C\left\|G_{\alpha,\gamma}: L^1(\mu)\right\| \cdot \left\|f: L^p(\mu)\right\| \\
&\leq C\phi(B)\mu(B)^{1/p}\left\|G_{\alpha,\gamma}: L^1(\mu)\right\| \cdot \left\|f: L_\phi^p(\mu)\right\|.
\end{aligned} \tag{15}$$

It is easy to see that for every $x \in B(a,r)$, we have the following estimate:

$$\begin{aligned}
\left|\left(G_{\alpha,\gamma} f_2(x)\right)\right| &\leq \sum_{k=1}^{\infty} \int_{2^k r \leq \delta(x,y)<2^{k+1} r} \frac{\delta(x,y)^{\alpha-1}|f(y)|}{(1+\delta(x,y))^{\gamma}} \mathrm{d}\mu(y) \\
&\sim \sum_{k=1}^{\infty} \frac{\left(2^k r\right)^{\alpha-1}}{\left(1+2^k r\right)^{\gamma}} \int_{B\left(x,2^{k+1} r\right)} |f(y)| \mathrm{d}\mu(y) \\
&\leq C\left\|f: L_\phi^p(\mu)\right\| \sum_{k=1}^{\infty} \frac{\left(2^k r\right)^{\alpha}\phi\left(2^{k+1} r\right)}{\left(1+2^k r\right)^{\gamma}} \\
&\leq C\phi(r)\left\|f: L_\phi^p(\mu)\right\| \cdot \left\|G_{\alpha,\gamma}: L^1(\mu)\right\|,
\end{aligned} \tag{16}$$

and this complete the proof. (QED).

Suppose $0 < \alpha < 1$, and $1 < p < \infty$, for every $r > 0$, we have

$$\int_X \left|G_{\alpha,\gamma}(x)\right|^p \mathrm{d}\mu(x) \sim \sum_{k\in\mathbb{Z}} \frac{\left(2^k r\right)^{1+p(\alpha-1)}}{\left(1+2^k r\right)^{p\gamma}}, \tag{17}$$

and $G_{\alpha,\gamma} \in L^p(\mu)$, provided that $1 < p < 1/(1-\alpha)$.

Theorem 3. *For* $1 < p < \infty, 0 < \alpha < \min\{1,\gamma\}$, *there exists* $C > 0$ *such that*

$$\left\|G_{\alpha,\gamma} f(x): L^p(\mu)\right\| \leq C\left\|G_{\alpha,\gamma}: L^p(\mu)\right\| \cdot \left\|f: L^p(\mu)\right\|. \tag{18}$$

Proof. Suppose $f \geq 0$, and for every $r > 0, x \in X$, $|G_{\alpha,\gamma}| \in L^p(\mu)$ we have

$$\left|G_{\alpha,\gamma}\right| f(x) = \left(\int_{\delta(x,y)<r} + \int_{\delta(x,y)\geq r}\right) \frac{\delta(x,y)^{1+p(\alpha-1)} f(y)}{(1+\delta(x,y))^{\gamma p}} \quad d\mu(y) =: I_1(x,r) + I_2(x,r). \tag{19}$$

It is easy to see that the following estimates is holds. That is, we have

$$\begin{aligned}
I_1(x,r) &= \int_{\delta(x,y)<r} \frac{\delta(x,y)^{1+p(\alpha-1)} f(y)}{(1+\delta(x,y))^{\gamma p}} \mathrm{d}\mu(y) \\
&= \sum_{k=-\infty}^{-1} \int_{2^k r \le \delta(x,y) < 2^{k+1} r} \frac{\delta(x,y)^{1+p(\alpha-1)} f(y)}{(1+\delta(x,y))^{\gamma p}} \mathrm{d}\mu(y) \\
&\sim \sum_{k=-\infty}^{-1} \frac{\left(2^k r\right)^{1+p(\alpha-1)}}{\left(1+2^k r\right)^{p\gamma}} \int_{B(x,2^{k+1}r)} f(y) \mathrm{d}\mu(y) \\
&\le C M_1 f(x) \cdot \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\|, \\
I_2(x,r) &= \sum_{k=0}^{\infty} \int_{2^k r \le \delta(x,y) < 2^{k+1} r} \frac{\delta(x,y)^{1+p(\alpha-1)} f(y)}{(1+\delta(x,y))^{\gamma p}} \mathrm{d}\mu(y) \\
&\sim \sum_{k=0}^{\infty} \frac{\left(2^k r\right)^{1+p(\alpha-1)}}{\left(1+2^k r\right)^{p\gamma}} \int_{B(x,2^{k+1}r)} f(y) \mathrm{d}\mu(y) \\
&\le C \left\| f \colon L^p \right\| \sum_{k=0}^{\infty} \frac{\left(2^k r\right)^{\alpha(1+p)} \left(2^{k+1} r\right)^{-1/p}}{\left(1+2^k r\right)^{p\gamma}} \\
&\le C r^{-1/p} \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\| \cdot \left\| f \colon L^p(\mu) \right\|.
\end{aligned} \tag{20}$$

So, for every $x \in X$, and $r > 0$, we have

$$\left| G_{\alpha,\gamma} \right| f(x) \le C\left(M_1 f(x) + r^{-1/p} \left\| f \colon L^p(\mu) \right\| \right) \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\|. \tag{21}$$

If we choose $r > 0$ such that $M_1 f(x) = r^{-1/p} \| f \colon L^p(\mu) \|$, then

$$\left| G_{\alpha,\gamma} \right| f(x) \le C M_1 f(x) \cdot \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\|, \tag{22}$$

and this finish the proof. (QED).

Next, by using Definition 5, the following theorem exists:

Theorem 4. *Suppose* $1 < p < \infty, 0 < \alpha < \min\{1, \gamma\}$, *and* ϕ *is a decreasing functions. Then there exists* $C > 0$ *such that*

$$\left\| G_{\alpha,\gamma} f(x) \colon L^p_\phi(\mu) \right\| \le C \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\| \cdot \left\| f \colon L^p_\phi(\mu) \right\|. \tag{23}$$

Proof. Suppose for $B := B(a,r)$, and $\widetilde{B} := B(a,2r)$, we define

$$f = f_1 + f_2 := f\chi_{\widetilde{B}} + f\chi_{\widetilde{B}^c}, \quad f \in L^p_\phi(\mu). \tag{24}$$

Since $\| f_1 \colon L^p(\mu) \| \le \phi(\widetilde{B}) \mu(\widetilde{B})^{1/p} \| f \colon L^p_\phi(\mu) \| < \infty$, then $f_1 \in L^p(\mu)$ and $G_{\alpha,\gamma} \in L^p(\mu)$ By the previous result, for every B, we have

$$\begin{aligned}
&\left(\int_B \left| G_{\alpha,\gamma} f_1(x) \right|^p \mathrm{d}\mu(y) \right)^{1/p} \\
&\le \left\| G_{\alpha,\gamma} f_1(x) \colon L^p(\mu) \right\| \\
&\le C \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\| \cdot \left\| f_1 \colon L^p(\mu) \right\| \\
&\le C \phi(B) \mu(B)^{1/p} \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\| \cdot \left\| f \colon L^p_\phi(\mu) \right\|.
\end{aligned} \tag{25}$$

It is easy to see that for every $x \in B(a,r)$, we have the following estimate:

$$\begin{aligned}
\left| G_{\alpha,\gamma} f_2(x) \right| &\le \sum_{k=1}^{\infty} \int_{2^k r \le \delta(x,y) < 2^{k+1} r} \frac{\delta(x,y)^{1+p(\alpha-1)} |f(y)|}{(1+\delta(x,y))^{\gamma p}} \mathrm{d}\mu(y) \\
&\sim \sum_{k=1}^{\infty} \frac{\left(2^k r\right)^{1+p(\alpha-1)}}{\left(1+2^k r\right)^{p\gamma}} \int_{B(x,2^{k+1}r)} |f(y)| \mathrm{d}\mu(y) \\
&\le C \left\| f \colon L^p_\phi(\mu) \right\| \sum_{k=1}^{\infty} \frac{\left(2^k r\right)^{\alpha(1+p)} \phi\left(2^{k+1} r\right)}{\left(1+2^k r\right)^{p\gamma}} \\
&\le C \phi(r) \left\| f \colon L^p_\phi(\mu) \right\| \cdot \left\| G_{\alpha,\gamma} \colon L^p(\mu) \right\|,
\end{aligned} \tag{26}$$

and this complete the proof (QED).

At infinity, the Bessel–Riesz kernel disappears more quickly than the fractional integral operator does. By accomplishing this, we can demonstrate that the Bessel–Riesz kernel is a member of some Lebesgue spaces. First, the following lemma, which is helpful to demonstrate that $G_{\alpha,\gamma}$ belongs to some Lebesgue spaces.

Lemma 1. *If* $\mu(B(a,r)) \sim r^\beta$, *then*

$$\int_X G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y) < \infty, \tag{27}$$

where

$$G_{\alpha,\gamma}(x,y) = \frac{\delta(x,y)^{\alpha-\beta}}{(1+\delta(x,y))^{\gamma}}. \tag{28}$$

Let

$$\int_X G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y) = \int_{\delta(x,y)<r} G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y) + \int_{\delta(x,y)\geq R} G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y), \tag{29}$$

by simplifying

$$\begin{aligned}\int_{\delta(x,y)<r} G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y) &= \sum_{k=-\infty}^{-1}\int_{2^kR\leq\delta(x,y)<2^{k+1}R} G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y)\\ &\sim \sum_{k=-\infty}^{-1}\left(2^kR\right)^{\alpha-\beta}\int_{\delta(x,y)<2^{k+1}R} G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y)\\ &\leq C_1\sum_{k=-\infty}^{-1}\left(2^kR\right)^{\alpha-\beta}\mu\left(\beta_{2^{k+1}R}(x)\right)\\ &= C_1\sum_{k=-\infty}^{-1}\left(2^kR\right)^{\alpha}\leq C_1R^{\alpha}<\infty,\\ \int_{\delta(x,y)\geq R} G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y) &= \sum_{k=0}^{\infty}\int_{2^kR\leq\delta(x,y)<2^{k+1}R} G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y)\\ &= C_2\sum_{k=0}^{\infty}\frac{\left(2^{k+1}R\right)^{\alpha-\beta}}{\left(1+2^{k+1}R\right)^{\gamma}}\leq C_2\sum_{k=0}^{\infty}\frac{\left(2^kR\right)^{\alpha-\beta}}{\left(2^kR\right)^{\gamma}}\\ &= C_2\sum_{k=0}^{\infty}\left(2^kR\right)^{\alpha-\beta-\gamma}\leq C_2R^{\alpha-\beta-\gamma}<\infty.\end{aligned} \tag{30}$$

For every $x \in X$, and $R > 0$, (3) implies

$$\int_X G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y) = C\left(R^{\alpha}+R^{\alpha-\beta-\gamma}\right). \tag{31}$$

Hence, we obtain

$$\int_X G_{\alpha,\gamma}(x,y)\mathrm{d}\mu(y) < \infty \quad \forall x \in X. \tag{32}$$

To prove that $G_{\alpha,\gamma}$ is the member of some Lebesgue spaces, we have the following theorem:

Theorem 5. *If $0 < \alpha < \beta$ and $\gamma \in (0,\infty)$, then we have for every $R > 0$,*

$$\left\|G_{\alpha,\gamma}\colon L^p\right\|^p \sim \sum_{k\in Z}\frac{\left(2^kR\right)^{(\alpha-\beta)p+\beta}}{\left(2^kR\right)^{\gamma p}}, \quad 1\leq\frac{\beta}{\beta+\gamma-\alpha}<p<\frac{\beta}{\beta-\alpha}. \tag{33}$$

Proof. For every $R > 0$, we have

$$\begin{aligned}\int_X \left|G_{\alpha,\gamma}(x,y)\right|^p d\mu(y) &= \int_{0\le\delta(x,y)} \left|G_{\alpha,\gamma}(x,y)\right|^p d\mu(y)\\ &= \sum_{k\in Z}\int_{2^kR\le\delta(x,y)<2^{k+1}R} \frac{\delta(x,y)^{(\alpha-\beta)p}}{(1+\delta(x,y))^{\gamma p}} d\mu(y)\\ &\sim \sum_{k\in Z}\frac{\left(2^kR\right)^{(\alpha-\beta)p}}{\left(1+\left(2^kR\right)\right)^{\gamma p}}\int_{2^kR\le\delta(x,y)<2^{k+1}R} d\mu(y)\\ &\sim \sum_{k\in Z}\frac{\left(2^kR\right)^{(\alpha-\beta)p+\beta}}{\left(1+\left(2^kR\right)\right)^{\gamma p}}.\end{aligned} \quad (34)$$

Using Lemma 1, for $1\le p<\infty$, we define $G_{\alpha,\gamma}\in L^p(\mu)$, if and only if

$$\left\|G_{\alpha,\gamma}\colon L^p\right\| = \left(\int_{\mathbb{R}^n}\left|G_{\alpha,\gamma}(x,y)\right|^p d\mu(y)\right)^{1/p} < \infty. \quad (35)$$

From the above definition, it easy to see that

$$\left\|G_{\alpha,\gamma}\colon L^p\right\| < \infty \Leftrightarrow 1 \le \frac{\beta}{\beta+\gamma-\alpha} < p < \frac{\beta}{\beta-\alpha}. \quad (36)$$

With the result of the membership of $G_{\alpha,\gamma}$ on Lebesgue spaces, we can prove the boundedness of $G_{\alpha,\gamma}f(x)$ on Lebesgue spaces by using Young inequality.

Theorem 6 (see [18]). *Assume $p,q,r\in[1,\infty]$ with*

$$\frac{1}{r} = \frac{1}{p}+\frac{1}{q}-1. \quad (37)$$

If $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$, then $f*g$ exists and belongs to $L^r(\mathbb{R}^n)$. Moreover,

$$\|f*g\|_r \le \|f\|_p\|g\|_q. \quad (38)$$

3.1. Boundedness of the Bessel–Riesz Operators on Morrey Spaces. This section will cover Bessel–Riesz operators on Morrey spaces in measure metric spaces and their boundedness. In the previous section, we have $G_{\alpha,\gamma}\in L^p(X)$, where $\beta/\beta+\gamma-\alpha<p<\beta/\beta-\alpha$ and the inclusion property of Morrey spaces, so $G_{\alpha,\gamma}\in L^p_\phi(\mu)$. Accordingly, we have the following theorem.

Theorem 7. *Let $0<\alpha<\beta$, $0<\gamma$, then we have*

$$\left\|G_{\alpha,\gamma}f(x)\right\|_{L^{p_2,q_2}} \le C_{p_1,q_1}\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}\|f\|_{L^{p_1,q_1}}, \quad (39)$$

for every $f\in L^{p_1,q_1}(X)$, where $1<p_1<q_1<p$, $1\le s\le p$, $\beta/\beta+\gamma-\alpha<p<\beta/\beta-\alpha$, $1/p_2=1/p_1-q_1/p_1p$ and $1/q_2=1/q_1-1/p$.

Suppose $0<\alpha<\beta$, $0<\gamma$, and take $\beta/\beta+\gamma-\alpha<p<\beta/\beta-\alpha, 1\le s\le p$. Let $f\in L^{p_1,q_1}(X)$, $1<p_1<q_1<p$. For every $x\in X$, $G_{\alpha,\gamma}f(x):=I_1(x)+I_2(x)$, where $I_1(x):=\int_{\delta(x,y)<r}\delta(x,y)^{\alpha-\beta}f(y)/(1+\delta(x,y))^\gamma d\mu(y)$ and $I_2(x):=\int_{\delta(x,y)\ge R}\delta(x,y)^{\alpha-\beta}f(y)/(1+\delta(x,y))^\gamma d\mu(y)$. To estimate I_1 and I_2, we use dyadic decomposition. Now, estimate I_1:

$$\begin{aligned}\left|I_1(x)\right| &\le C_1\sum_{k=-1}^{-\infty}\frac{\left(2^kR\right)^{\alpha-\beta}}{\left(1+2^kR\right)^\gamma}\int_{2^kR\le\delta(x,y)<2^{k+1}R}|f(y)|d\mu(y)\\ &\le C_2M_qf(x)\sum_{k=-1}^{-\infty}\frac{\left(2^kR\right)^{\alpha-\beta+\beta/s}\left(2^kR\right)^{\beta/s'}}{\left(1+2^kR\right)^\gamma}.\end{aligned} \quad (40)$$

By using Hölder's inequality, we get

$$\begin{aligned}\left|I_1(x)\right| &\le C_3M_qf(x)\left(\sum_{k=-1}^{-\infty}\frac{\left(2^kR\right)^{(\alpha-\beta)s+\beta}}{\left(1+2^kR\right)^{\gamma s}}\right)^{1/s}\left(\sum_{k=-1}^{-\infty}\left(2^kR\right)^\beta\right)^{1/s'}\\ &\le C_4M_qf(x)\frac{\left(\int_{\delta(x,y)<r}G^s_{\alpha,\gamma}\delta(x,y)d\mu(y)\right)^{1/s}}{R^{\beta(1/s-1/t)}} \le C_4\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}M_qf(x)R^{\beta/p'}.\end{aligned} \quad (41)$$

Hölder's inequality is used again to estimate I_2:

$$\begin{aligned}\left|I_2(x)\right| &\le C_5\sum_{k=0}^{\infty}\frac{\left(2^kR\right)^{\alpha-\beta}}{\left(1+2^kR\right)^{\gamma}}\int_{2^kR\le\delta(x,y)<2^{k+1}R}|f(y)|\mathrm{d}\mu(y)\\ &\le C_5\sum_{k=0}^{\infty}\frac{\left(2^kR\right)^{\alpha-\beta}}{\left(1+2^kR\right)^{\gamma}}\left(\int_{2^kR\le\delta(x,y)<2^{k+1}R}|f(y)|^{p_1}\mathrm{d}\mu(y)\right)^{1/p_1}\left(2^kR\right)^{\beta/p_1'}.\end{aligned} \tag{42}$$

Next, we write

$$\begin{aligned}\left|I_2(x)\right| &\le C_6\|f\|_{L^{p_1,q_1}}\sum_{k=0}^{\infty}\frac{\left(2^kR\right)^{\alpha-\beta+\beta-\beta/q_1}}{\left(1+2^kR\right)^{\gamma}}\frac{\left(\int_{2^kR\le\delta(x,y)<2^{k+1}R}\mathrm{d}\mu(y)\right)^{1/s}}{\left(2^kR\right)^{\beta/s}}\\ &\le C_6\|f\|_{L^{p_1,q_1}}\sum_{k=0}^{\infty}\frac{\left(\int_{2^kR\le\delta(x,y)<2^{k+1}R}\delta(x,y)^{\alpha-\beta}f(y)/(1+\delta(x,y))^{\gamma}\mathrm{d}\mu(y)\right)^{1/s}}{\left(2^kR\right)^{\beta/q_1-\beta}\left(2^kR\right)^{\beta/s}},\end{aligned} \tag{43}$$

and we obtain

$$\left|I_2(x)\right| \le C_6\|f\|_{L^{p_1,q_1}}\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}\sum_{k=0}^{\infty}\left(2^kR\right)^{\beta/p'-\beta/q_1} \le C_7\|f\|_{L^{p_1,q_1}}\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}R^{\beta\left(1/p'-1/q_1\right)}. \tag{44}$$

Summing the two estimates, we get

$$\left|G_{\alpha,\gamma}f(x)\right| \le C\left|G_{\alpha,\gamma}\right|_{L^p_\phi(\mu)}\left(M_qf(x)R^{\beta/p'}+\|f\|_{L^{p_1,q_1}}R^{\beta/p'-\beta/q_1}\right), \tag{45}$$

for each $x\in X$. Assume that M_pf is finite everywhere and f is not a unique instance of 0. Make sure that $R^{\beta/q_1}=\|f\|_{L^{p_1,q_1}}/M_qf(x)$. We get

$$\left|G_{\alpha,\gamma}f(x)\right| \le C\left|G_{\alpha,\gamma}\right|_{L^p_\phi(\mu)}\|f\|_{L^{p_1,q_1}}^{q_1/p'}M_qf(x)^{1-q_1/p'}. \tag{46}$$

Define $1/p_2=1/p_1-q_1/p_1p'$ and $1/q_2=1/q_1-1/p'$. For arbitrary $r>0$, we have

$$\left(\int_{\delta(x,y)<r}\left|G_{\alpha,\gamma}f(x)\right|^{p_2}\mathrm{d}\mu(x)\right)^{1/p_2} \le C\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}\|f\|_{L^{p_1,q_1}}^{1-p_1/p_2}\left(\int_{\delta(x,y)<r}|M_qf(x)^{p_1}\mathrm{d}\mu(x)\right)^{1/p_2}. \tag{47}$$

Divide by $r^{\beta/p_2-\beta/q_2}$ and take supremum to get

$$\begin{aligned}\left\|G_{\alpha,\gamma}f(x)\right\|_{L^{p_2,q_2}} &= \sup_{r>0}\frac{\left(\int_{\delta(x,y)<r}\left|G_{\alpha,\gamma}f(x)\right|^{p_2}\mathrm{d}\mu(x)\right)^{1/p_2 s}}{r^{\beta/p_2-\beta/q_2}}\\ &\leq C\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}\|f\|^{1-p_1/p_2}_{L^{p_1,q_1}}\frac{\left(\int_{\delta(x,y)<r}|M_q f(x)^{p_1}\mathrm{d}\mu(x)\right)^{1/p_2}}{r^{\beta/p_2-\beta/q_2}}\\ &= C\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}\|f\|^{1-p_1/p_2}_{L^{p_1,q_1}}\left\|M_p f\right\|^{p_1/p_2}_{L^{p_1,q_1}}.\end{aligned} \tag{48}$$

Using the boundedness of M on Morrey spaces (Chiarenza–Frascas theorem [2]), we obtain an inequality

$$\left\|G_{\alpha,\gamma}f(x)\right\|_{L^{p_2,q_2}} \leq C\left\|G_{\alpha,\gamma}\right\|_{L^p_\phi(\mu)}\|f\|^{1-p_1/p_2}_{L^{p_1,q_1}}. \tag{49}$$

Corollary 1. *Suppose we have $p_1 = q_1, p_2 = q_2$, and $s = p\, 1 \leq \beta/\beta + \gamma - \alpha < p < \beta/\beta - \alpha$. If for some $C_1 > 0$, $\mu(B(a,r)) \leq C_1 r^\beta$, $f \in L^{p_1}(\mu)$, and $G_{\alpha,\gamma} \in L^s(\mu)$, then there exist $C_2 > 0$ such that*

$$\left\|G_{\alpha,\gamma}f(x)f\colon L^{q_1}(\mu)\right\| \leq C_2\left\|G_{\alpha,\gamma}(x,y)\colon L^s(\mu)\right\|.\left\|f\colon L^{p_1}(\mu)\right\|s. \tag{50}$$

By the above corollary, we can say that $G_{\alpha,\gamma}f(x)$ is bounded from $L^{p_1}(\mu)$ to $L^{q_1}(\mu)$. Moreover, norm of $G_{\alpha,\gamma}f(x)$ is dominated by norm of kernel Bessel-Riesz. For the boundedness of $G_{\alpha,\gamma}f(x)$.

4. Concluding Remark

This paper has investigated the existence of Morrey space for the Bessel–Riesz operators that are dominated by the Bessel–Riesz kernel. The standard dyadic decomposition and maximal operators were used to further establish the boundedness of Bessel–Riesz operators. Additionally, Saba et al. [24], have shown that the generalized Bessel–Riesz operator is bounded. Future consideration will be an ongoing research direction that is to extend the study of Bessel–Riesz operators to more general settings such as weighted spaces, variable exponent spaces, and other metric measure spaces.

Data Availability

While the results of the research are being commercialized, the data that were used to support the findings of this study are currently under embargo. After the publication of this article, requests for data will be taken into consideration by the relevant author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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