Research Article

Solving the Fractional Schrödinger Equation with Singular Initial Data in the Extended Colombeau Algebra of Generalized Functions

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This manuscript aims to highlight the existence and uniqueness results for the following Schrödinger problem in the extended Colombeau algebra of generalized functions.

\[
\frac{1}{i} \frac{\partial u}{\partial t}(t, x) - \Delta u(t, x) + v(x)u(t, x) \leq 0, \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n,
\]

where \(\delta\) is the Dirac distribution. The proofs of our main results are based on the Gronwall inequality and regularization method. We conclude our article by establishing the association concept of solutions.

1. Introduction

In the theory of distributions, it is well known that the multiplication of two distributions is not always well defined; for more details, see [1]. However, until 1980, this operation was not used in a rigorous way. Following, Schwarz’s renowned impossibilities relating to the build of a commutative, associative, and differential algebra in which we can inject the space of distributions \(\mathcal{D}'\). In [2], Colombeau was able to answer to this question by replacing continuous functions with indefinitely differentiable functions. The thing that has allowed us to investigate nonlinear differential equations formed from natural occurrences with parameters in the sense of distributions in a rational manner; for that, see [1]. We are talking about an algebra that is defined as the quotient of the algebra of moderate functions by the algebra of negligible functions. So, it should come as no surprise that the following presentations’ main topic will be to raise componentwise definitions to the level of equivalence classes. The derivative operator is one of the most essential operators, and its description in this context allows us to investigate differential equations in terms of distributions. Unfortunately, according to the definition, it does not pose a difficulty for generalized functions, which is not the case for the fractional derivative. In [3], Mirjana presents a method for dealing with fractional derivatives including singularities based on Colombeau’s idea of algebras of generalized functions. The Colombeau algebra of generalized functions is extended to fractional derivatives by the same author. It is used to solve ODEs and PDEs with entire and fractional derivatives in terms of temporal and spacial variables; see [4–8]. The study of fractional order integral and derivative operators over real and complex domains, as well as their applications, is the subject of fractional calculus. The features of fractional derivatives make them ideal for modeling complicated systems. The ordinary derivative is a local operator, whereas fractional operators are nonlocal. Fractional derivatives exhibit nonlocal dynamics as a result of this, i.e. the process dynamics have some memory; for more information, see [9–12]. When the nonlinear term is a \(L^\infty\text{loc}\)-function that does not satisfy the Lipschitz condition, the nonlinear Schrödinger equation with singular potential and initial data is studied in [13]. Recently, in [14], Benmerrous et al. established the existence and uniqueness of solutions for the Schrödinger equation with singular potential and initial data in the Colombeau algebra. The
reader can consult articles as well [15, 16] and the references therein for more details on the fractional Schrödinger equation.

Motivated by the above works precisely by [14], we study the existence and uniqueness results for the Schrödinger equation with singular initial data in the extended Colombeau algebra of generalized functions. Also, we give the association of solutions.

The present article is organized as follows: In Section 2, we recall some fundamental properties of the generalized function theory. Section 3 is consecrated for the proofs of the existence and uniqueness of solutions for the Schrödinger problem (12) in the Colombeau algebra and the extended Colombeau algebra. We conclude this article by proving the association of solutions.

2. Preliminaries

To define the full algebra of Colombeau, for \( r \in \mathbb{N}^* \) we define

\[
\sum_{r} = \left\{ \mu \in \mathcal{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} y^{\beta} \mu(y) dy = 0 \text{ for } 1 \leq |\beta| \leq r \right\},
\]

\( r = 1, 2,..., \)

We denote by

\[
\xi^\varepsilon_m(\mathbb{R}^n) = \left\{ \left( u, \varepsilon \right) \subset (0, 1) \subset \mathcal{D}(\mathbb{R}^n) : \forall \mu \in \mathcal{D}(\mathbb{R}^n), \exists m \in \mathbb{N} \sup_{x \in \mathcal{X}} |D^\beta u_{\varepsilon}(x)| = O(\varepsilon^m) \right\},
\]

\[
\Xi^\varepsilon(\mathbb{R}^n) = \left\{ \left( u, \varepsilon \right) \subset (0, 1) \subset \mathcal{D}(\mathbb{R}^n) : \forall \mu \in \mathcal{D}(\mathbb{R}^n), \forall \beta \in \mathbb{N} \sup_{x \in \mathcal{X}} |D^\beta u_{\varepsilon}(x)| = O(\varepsilon^m) \right\},
\]

where

\[
\begin{align*}
\mu_{\varepsilon}(x) &= \frac{1}{\varepsilon^n} \mu\left( \frac{x}{\varepsilon} \right), \forall \mu \in \mathcal{D}(\mathbb{R}^n), \\
u_{\varepsilon}(x) &= u(\mu_{\varepsilon}, x) \forall \mu \in \Sigma_1,
\end{align*}
\]

\[
\xi(\mathbb{R}^n) = \left\{ u: \Sigma_1 \times \mathbb{R}^n \rightarrow \mathbb{C} : \mu_{\varepsilon}(x) \text{ is } \mathcal{O}(\varepsilon^m) \text{ respect to the second variable } x \right\}.
\]

The full Colombeau algebra is defined by

\[
\mathcal{G}(\mathbb{R}^n) = \frac{\xi^\varepsilon_m(\mathbb{R}^n)}{\Xi^\varepsilon(\mathbb{R}^n)}
\]

(4)

To solve ODE and DED with integer and fractional derivatives with initial data distributions, we need to recall the definition of the extension of the fractional derivative in the Colombeau algebra.

We denote the set of all extended moderate functions by

\[
\xi^\varepsilon_m(\mathbb{R}) = \left\{ \left( u, \varepsilon \right) \subset (0, 1) \subset \mathcal{D}(\mathbb{R}) : \forall \beta \in \mathbb{R^+}, \exists m \in \mathbb{N} \sup_{x \in \mathcal{X}} |D^\beta u_{\varepsilon}(x)| = O(\varepsilon^m) \right\},
\]

\[
\xi(\mathbb{R}) = \left\{ u: \Sigma_1 \times \mathbb{R} \rightarrow \mathbb{C} : \mu_{\varepsilon}(x) \text{ is } \mathcal{O}(\varepsilon^m) \text{ respect to the second variable } x \right\}.
\]

The set of all extended negligible functions by

\[
\Xi^\varepsilon(\mathbb{R}) = \left\{ \left( u, \varepsilon \right) \subset (0, 1) \subset \mathcal{D}(\mathbb{R}) : \forall \beta \in \mathbb{R^+}, \forall \mu \in \mathbb{N} \sup_{x \in \mathcal{X}} |D^\beta u_{\varepsilon}(x)| = O(\varepsilon^m) \right\},
\]

and the set of all extended negligible functions by

\[
\Xi^\varepsilon(\mathbb{R}) = \left\{ \left( u, \varepsilon \right) \subset (0, 1) \subset \mathcal{D}(\mathbb{R}) : \forall \beta \in \mathbb{R^+}, \forall \mu \in \mathbb{N} \sup_{x \in \mathcal{X}} |D^\beta u_{\varepsilon}(x)| = O(\varepsilon^m) \right\}.
\]
The extended Colombeau algebra of generalized functions is the factor set.

\[ \mathcal{G}^\varepsilon (\mathbb{R}) = \frac{\mathcal{E}^\varepsilon (\mathbb{R})}{\mathcal{N}^\varepsilon (\mathbb{R})}. \]  

(7)

Now, we give the definitions of the fractional calculus theory.

Definition 1 (see [17, 18]). The fractional integral is defined as follows:

\[ \mathcal{I}^\beta g(\varepsilon) = \frac{1}{\Gamma (\beta)} \int_0^\varepsilon (\varepsilon - r)^{\beta - 1} g(r)dr, \beta > 0. \]  

(8)

The fractional derivative of order \( \beta > 0 \) in the Caputo sense is defined as follows:

\[ \mathcal{D}^\beta g(\varepsilon) = \frac{1}{\Gamma (\beta - p)} \int_0^\varepsilon (\varepsilon - r)^{\beta - p - 1} g^{(p)}(r)dr, 1 < \beta < p, p \in \mathbb{N}^*. \]  

(9)

We end this section by recalling the association relation on the Colombeau algebra \( \mathcal{G}^\varepsilon \). It identifies elements of \( u \in \mathcal{G}^\varepsilon \) if they coincide in the weak limit.

Definition 2 (see [19]). Let \( u_1, u_2 \in \mathcal{G}^\varepsilon (\mathbb{R}^n) \) such that \( u_{1, \varepsilon} \) and \( u_{2, \varepsilon} \) are their representatives, respectively. We say that \( u_1 \) and \( u_2 \) are associated in \( \mathcal{G}^\varepsilon (\mathbb{R}^n) \), and we write \( u_1 \approx u_2 \), if for every \( \mu \in \mathcal{D} (\mathbb{R}^n) \).

\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} (u_{1, \varepsilon} - u_{2, \varepsilon}) \mu(x)dx = 0. \]  

(11)

3. Main Results

The nonlinear Schrödinger equation with singular potential and initial data is considered.

\[
\begin{aligned}
& \frac{1}{i} \frac{\partial}{\partial t} u(t, x) - \Delta u(t, x) + \nu(x) u(t, x) = 0, t \in \mathbb{R}^+, x \in \mathbb{R}^n; \\
& \nu(x) = \delta(x); \\
& u(0, x) = \delta(x).
\end{aligned}
\]  

(12)

where \( x \in \mathbb{R}^n, \mu \in \mathcal{A}_1, \mu(x) \geq 0, x \in \mathbb{R}^n \).

3.1. Existence and Uniqueness Results in the Colombeau Algebra

Theorem 1. Let equation (12) have the regularized equation:

\[
\begin{aligned}
& \frac{1}{i} \frac{\partial}{\partial t} u_{\varepsilon}(t, x) - \Delta u_{\varepsilon}(t, x) + \nu_{\varepsilon}(x) u_{\varepsilon}(t, x) = 0, t \in \mathbb{R}^+, x \in \mathbb{R}^n; \\
& \nu_{\varepsilon}(x) = \delta_{\varepsilon}(x); \\
& u_{\varepsilon}(0, x) = \delta_{\varepsilon}(x),
\end{aligned}
\]  

(15)

where \( \nu_{\varepsilon} \) and \( u_{0, \varepsilon} \) are regularized of \( \nu \) and \( u_0 \), respectively.
Then, the problem (12) has a unique solution in $\mathcal{B}^s(\mathbb{R}^n \times \mathbb{R}^n)$.

$$u_\epsilon(t, x) = \int_{\mathbb{R}^n} \kappa_n(t, x - y) u_{0,\epsilon}(y) dy + \int_0^t \int_{\mathbb{R}^n} \kappa_n(t - \tau, x - y) v_\epsilon(\tau, y) u_\epsilon(\tau, y) dy \, d\tau,$$  

(16)

where $\kappa_n(t, x)$ is the heat kernel.

**Proof.** The integral solution of equation (15) (see [13]) is given by.

$$\left\| u_\epsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq C \left\| \kappa_n(t, x - \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \| u_{0,\epsilon} \|_{L^\infty(\mathbb{R}^n)} + C \int_0^t \left\| \kappa_n(t - \tau, x - \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \| v_\epsilon(\tau, \cdot) \|_{L^\infty(\mathbb{R}^n)} d\tau,$$

(17)

By Gronwall inequality,

$$\left\| u_\epsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq C \left\| \ln \epsilon \right\|_{b_n} \exp(C \left\| \ln \epsilon \right\|_{b_n}).$$

(18)

Then, there exists $N > 0$ such that

$$\frac{\partial}{\partial x_j} u_\epsilon(t, x) = \int_{\mathbb{R}^n} \kappa_n(t, x - y) \frac{\partial}{\partial y_j} u_{0,\epsilon}(y) dy + \int_0^t \int_{\mathbb{R}^n} \kappa_n(t - \tau, x - y) \left( \frac{\partial}{\partial y_j} v_\epsilon(\tau, y) u_\epsilon(\tau, y) + v_\epsilon(\tau, y) \frac{\partial}{\partial y_j} u_\epsilon(\tau, y) \right) dy \, d\tau.$$

(20)

Then,

$$\left\| \frac{\partial}{\partial x_j} u_\epsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq \left\| \kappa_n(t, x - \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \left\| \frac{\partial}{\partial y_j} u_{0,\epsilon} \right\|_{L^\infty(\mathbb{R}^n)} + C \int_0^t \left\| \kappa_n(t - \tau, x - \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \left\| \frac{\partial}{\partial y_j} u_\epsilon(\tau, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} d\tau.$$

(21)

By the previous step, there exist $N > 0$ such that

$$\left\| \frac{\partial}{\partial x_j} u_\epsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq C \left\| \ln \epsilon \right\|_{a(n+1)} + T \left\| \ln \epsilon \right\|_{b(n+1)} \| u_\epsilon \|_{L^\infty} \exp(C \left\| \ln \epsilon \right\|_{b_n}).$$

(22)
\[ \left\| \frac{\partial}{\partial x_j} u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq C_\varepsilon^{-N}. \quad (23) \]

For the second derivative, for \( y_j, j \in \{1, n\}, \) we get

\[ \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_\varepsilon(t, x) = \int_{\mathbb{R}^n} \kappa_\varepsilon(t, x - y) \left( \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_j} u_{\varepsilon, 0}(y) \right) dy + \int_0^t \int_{\mathbb{R}^n} \left( \kappa_\varepsilon(t - \tau, x - y) \left( \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_j} u_{\varepsilon}(\tau, y) \right) + \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_j} u_{\varepsilon}(\tau, y) \right) dy \right] d\tau. \quad (24) \]

So,

\[ \left\| \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq \left\| \kappa_\varepsilon(t, x - y) \right\|_{L^1} \left\| \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_j} u_{\varepsilon, 0}(\cdot) \right\|_{L^\infty(\mathbb{R}^n)} 
+ \int_0^t \left\| \kappa_\varepsilon(t - \tau, x - y) \right\|_{L^1} \left( \left\| \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_j} u_{\varepsilon}(\tau, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} + \left\| \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_j} u_{\varepsilon}(\tau, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \right) \right] d\tau \]

\[ \leq C \left( | \ln \varepsilon | a_{(n+2)} + | \ln \varepsilon | b_{(n+1)} \right) \left\| u_{\varepsilon} \right\|_{L^\infty} 
+ \int_0^t \left( | \ln \varepsilon | b_{(n+1)} \right) \left\| u_{\varepsilon} \right\|_{L^\infty} d\tau \]

Using Gronwall inequality, we obtain the following equation:

\[ \left\| \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_\varepsilon(t, \cdot) \right\|_{L^\infty(\mathbb{R}^n)} \leq C \exp \left( C \left( | \ln \varepsilon | b_{(n+1)} \right) \left( | \ln \varepsilon | a_{(n+2)} + | \ln \varepsilon | b_{(n+1)} \right) \left\| u_{\varepsilon} \right\|_{L^\infty} \right) \quad (26) \]
By the previous step, there exist \( N > 0 \) such that
\[
\left\| \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_{\epsilon}(t,.) \right\|_{L^\infty(R^d)} \leq C e^{-N}. \tag{27}
\]

Let us prove the uniqueness. Suppose that there exist two solutions \( u_1, u_2 \) to problem (12), then we obtain the following equation:

\[
\begin{cases}
\frac{1}{t} \frac{\partial}{\partial t} (u_{1,\epsilon}(t,x) - u_{2,\epsilon}(t,x)) = \Delta (u_{1,\epsilon}(t,x) - u_{2,\epsilon}(t,x)) - \nu_\epsilon(x)(u_{1,\epsilon}(t,x) - u_{2,\epsilon}(t,x)), \\
u_{1,\epsilon}(0,x) = u_{2,\epsilon}(0,x) + N_{0,\epsilon}(x),
\end{cases}
\tag{28}
\]

where \( N_{\epsilon}(t,x) \in \mathcal{N}(R^+ \times R^n), N_{0,\epsilon}(x) \in \mathcal{N}(R^n) \). Then,

\[
u_{1,\epsilon}(t,x) - u_{2,\epsilon}(t,x) = \int_{R^n} \kappa_{\epsilon}(t,x-y)N_{0,\epsilon}(y)dy + \int_0^t \int_{R^n} \kappa_{\epsilon}(t-\tau,x-y)\nu_{\epsilon}(y)(u_{1,\epsilon}(\tau,y) - u_{2,\epsilon}(\tau,y))dyd\tau
\tag{29}
\]

and thus

\[
\left\| u_{1,\epsilon}(t,.) - u_{2,\epsilon}(t,.) \right\|_{L^\infty(R^d)} \leq \left\| \kappa_{\epsilon}(t,x-) \right\|_{L^1} \left\| N_{0,\epsilon} \right\|_{L^\infty(R^n)}
\]

\[
+ \left\| \kappa_{\epsilon}(t,x-) \right\|_{L^1} \int_0^t \left\| \nu_{\epsilon} \right\|_{L^\infty(R^n)} \left\| u_{1,\epsilon}(\tau,.) - u_{2,\epsilon}(\tau,.) \right\|_{L^\infty(R^d)} d\tau
\]

\[
+ \left\| \kappa_{\epsilon}(t,x-) \right\|_{L^1} \int_0^t \left\| \nu_{\epsilon} \right\|_{L^\infty(R^n)} \left\| u_{1,\epsilon}(\tau,.) - u_{2,\epsilon}(\tau,.) \right\|_{L^\infty(R^d)} d\tau
\tag{30}
\]

By Gronwall inequality,

\[
\left\| u_{1,\epsilon}(t,.) - u_{2,\epsilon}(t,.) \right\|_{L^\infty(R^d)} \leq C \left( \left\| N_{0,\epsilon} \right\|_{L^\infty(R^n)} + \left\| N_{\epsilon} \right\|_{L^\infty} \right) \exp \left( C t \left\| \nu_{\epsilon} \right\|_{L^\infty(R^n)} \right).
\tag{31}
\]

Then,

\[
\left\| u_{1,\epsilon}(t,.) - u_{2,\epsilon}(t,.) \right\|_{L^\infty(R^d)} \leq C e^{-t}, \forall q \in \mathbb{N}.
\tag{32}
\]

Then, the problem (12) has a unique solution in \( \mathcal{G}(R^+ \times R^n) \).

---

3.2. Existence and Uniqueness Results in the Extension of the Colombeau Algebra. In a framework of extended algebra of generalized functions, we prove the existence uniqueness result for nonlinear Schrödinger equations with singular potential and initial data and an equation driven by the fractional derivative of the delta distribution. It means that
we prove the moderateness and the negligibility for entire and fractional derivatives to the spatial variable $x$.

Let the following problem

$$
\begin{aligned}
\frac{1}{i} \frac{\partial}{\partial t} u_\epsilon(t, x) - \Delta u_\epsilon(t, x) + v_\epsilon(x) u_\epsilon(t, x) &= 0, t \in \mathbb{R}^+, x \in \mathbb{R}^n, \\
v_\epsilon(x) &= \delta(x), \\
u_\epsilon(0, x) &= \delta(x),
\end{aligned}
$$

be the regularized equation of (12) such that $v_\epsilon$ and $u_{0, \epsilon}$ are regularizations of $v$ and $u_0$, respectively. Then, the problem (12) has a unique solution in $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$.

**Proof.** We prove only the fractional part since the entire part is already proved in Theorem 1. The fractional derivative, $\mathcal{D}^\beta, 0 < \beta < 1$, is considered. Without loss of generality, the same holds for $m - 1 < \beta < m, m \in \mathbb{N}$. The fractional derivative is taken to the spatial variable to equation (12). We have

$$
\begin{aligned}
\|\mathcal{D}^\beta(u_\epsilon(t, \cdot))\|_{L^\infty(\mathbb{R}^n)} &\leq \|\kappa_\epsilon(t, x - \cdot)\|_{L^1} \|\mathcal{D}^\beta u_{0, \epsilon}\|_{L^\infty(\mathbb{R}^n)} \\
&+ \|\kappa_\epsilon(t - \tau, x - \cdot)\|_{L^1} \int_0^t \|\mathcal{D}^\beta v_\epsilon(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} d\tau \\
&+ \|\kappa_\epsilon(t - \tau, x - \cdot)\|_{L^1} \int_0^t \|v_\epsilon(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} \|\mathcal{D}^\beta u_\epsilon(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} d\tau \\
&\leq C \|\mathcal{D}^\beta u_{0, \epsilon}\|_{L^\infty(\mathbb{R}^n)} + T \|\mathcal{D}^\beta v_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_\epsilon\|_{L^\infty(\mathbb{R}^n)} \\
&+ \|v_\epsilon\|_{L^\infty(\mathbb{R}^n)} \int_0^t \|\mathcal{D}^\beta u_\epsilon(\tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} d\tau.
\end{aligned}
$$

By Gronwall inequality, we get

$$
\|\mathcal{D}^\beta(u_\epsilon(t, \cdot))\|_{L^\infty(\mathbb{R}^n)} \leq C \exp\left(CT \|v_\epsilon\|_{L^\infty(\mathbb{R}^n)} \right) \left( \|\mathcal{D}^\beta u_{0, \epsilon}\|_{L^\infty(\mathbb{R}^n)} + T \|\mathcal{D}^\beta v_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_\epsilon\|_{L^\infty(\mathbb{R}^n)} \right).
$$

From Theorem 1 and hypothesis (13), we obtain

$$
\|\mathcal{D}^\beta(u_\epsilon(t, \cdot))\|_{L^\infty(\mathbb{R}^n)} \leq C \exp\left(CT \|v_\epsilon\|_{L^\infty(\mathbb{R}^n)} \right) \left( C_{\beta, T} |\ln \epsilon|^{a(n+1)} + T C_{\beta, T} |\ln \epsilon|^{b(n+1)} \|u_\epsilon\|_{L^\infty} \right).
$$

Then, there exist $N > 0$ such that

$$
\|\mathcal{D}^\beta(u_\epsilon(t, \cdot))\|_{L^\infty(\mathbb{R}^n)} \leq C \epsilon^{-N}.
$$
It follows moderateness for the fractional derivatives in the space \( \mathcal{D}^q (\mathbb{R}^n) \). For uniqueness, let \( 0 < \beta < 1 \), and apply \( \mathcal{D}^\beta \) to (29) we get:

\[
\mathcal{D}^\beta (w_1 (t, x) - w_2 (t, x)) = \int_{\mathbb{R}^n} \kappa_n (t, x - y) \mathcal{D}^\beta N_{0, \epsilon} (y) dy \\
+ \int_0^t \int_{\mathbb{R}^n} \kappa_n (t - \tau, x - y) \mathcal{D}^\beta v (y) (u_{1, \epsilon} (\tau, y) - u_{2, \epsilon} (\tau, y)) dy d\tau \\
+ \int_0^t \int_{\mathbb{R}^n} \kappa_n (t - \tau, x - y) v (y) \mathcal{D}^\beta (u_{1, \epsilon} (\tau, y) - u_{2, \epsilon} (\tau, y)) dy d\tau \\
+ \int_0^t \int_{\mathbb{R}^n} \kappa_n (t - \tau, x - y) \mathcal{D}^\beta N_{\epsilon} (\tau, y) dy d\tau.
\]

So,

\[
\left\| \mathcal{D}^\beta (w_1 (t, \cdot) - w_2 (t, \cdot)) \right\|_{L^\infty} \leq \left\| \kappa_n (t, x - \cdot) \right\|_{L^1} \left\| \mathcal{D}^\beta N_{0, \epsilon} \right\|_{L^\infty} \\
+ \left\| \kappa_n (t - \tau, x - \cdot) \right\|_{L^1} \int_0^t \left\| \mathcal{D}^\beta v \right\|_{L^\infty} \left\| u_{1, \epsilon} (\tau, \cdot) - u_{2, \epsilon} (\tau, \cdot) \right\|_{L^\infty} d\tau \\
+ \left\| \kappa_n (t - \tau, x - \cdot) \right\|_{L^1} \int_0^t \left\| \mathcal{D}^\beta N_{\epsilon} \right\|_{L^\infty} \left\| u_{1, \epsilon} (\tau, \cdot) - u_{2, \epsilon} (\tau, \cdot) \right\|_{L^\infty} d\tau
\]

By Gronwall inequality,

\[
\left\| \mathcal{D}^\beta (u_{1, \epsilon} (t, \cdot) - u_{2, \epsilon} (t, \cdot)) \right\|_{L^\infty (\mathbb{R}^n)} \leq C \exp \left( \int_{\mathbb{R}^n} \left\| v \right\|_{L^\infty (\mathbb{R}^n)} \right) \left( N_{0, \epsilon} \right)_{L^\infty (\mathbb{R}^n)} \\
+ T \left\| \mathcal{D}^\beta v \right\|_{L^\infty (\mathbb{R}^n)} \left\| u_{1, \epsilon} - u_{2, \epsilon} \right\|_{L^\infty (\mathbb{R}^n)} + \left\| \mathcal{D}^\beta N_{\epsilon} \right\|_{L^\infty (\mathbb{R}^n)}.
\]

Using Theorem 1, we obtain

\[
\left\| \mathcal{D}^\beta (u_{1, \epsilon} (t, \cdot) - u_{2, \epsilon} (t, \cdot)) \right\|_{L^\infty (\mathbb{R}^n)} \leq C e^{\beta T} N_{0, \epsilon}, \quad \forall q \in \mathbb{N},
\]

which completes the proof. \( \square \)

3.3. Association of Solutions. Let \( w_1 \) be the solution of the problem.

\[
\begin{cases}
\frac{1}{i} \frac{\partial}{\partial t} w_1 (t, x) - \Delta w_1 (t, x) = 0, \\
w_1 (0, x) = \delta (x),
\end{cases}
\]

and \( w_2 \) be a solution of the problem.

\[
\begin{cases}
\frac{1}{i} \frac{\partial}{\partial t} w_2 (t, x) - \Delta w_2 (t, x) + v (x) w_2 (t, x) = 0, \\
v (x) = \delta (x), w_2 (0, x) = 0.
\end{cases}
\]

Proposition 2. The generalized solution \( u \) of the problem (12) is associated with \( w_1 + w_2 \).

Proof. Let \( w_{1, \epsilon} \) be the classical solution to the problem.

\[
\begin{cases}
\frac{1}{i} \frac{\partial}{\partial t} w_{1, \epsilon} (t, x) - \Delta w_{1, \epsilon} (t, x) = 0, \\
w_{1, \epsilon} (0, x) = \delta_\epsilon (x),
\end{cases}
\]

\[
\begin{cases}
\frac{1}{i} \frac{\partial}{\partial t} w_{2, \epsilon} (t, x) = 0, \\
w_{2, \epsilon} (0, x) = \delta_\epsilon (x),
\end{cases}
\]

\[
\begin{cases}
\frac{1}{i} \frac{\partial}{\partial t} u (t, x) = 0, \\
u (0, x) = \delta (x)
\end{cases}
\]
and \( w_{2,\varepsilon} \) be the classical solution to the problem.

\[
\begin{aligned}
\frac{1}{i} \frac{\partial}{\partial t} w_{2,\varepsilon}(t, x) - \Delta w_{2,\varepsilon}(t, x) + v_{\varepsilon}(x) \left( w_{2,\varepsilon}(t, x) + m(t, x) \right) &= 0, \\
v_{\varepsilon}(x) &= \delta(x), \quad w_{2,\varepsilon}(0, x) = 0.
\end{aligned}
\]

Then,

\[
\begin{aligned}
\frac{1}{i} \frac{\partial}{\partial t} \left( u_{\varepsilon} - w_{1,\varepsilon} - w_{2,\varepsilon} \right) - \Delta \left[ u_{\varepsilon} - w_{1,\varepsilon} - w_{2,\varepsilon} \right] + v_{\varepsilon} \left[ u_{\varepsilon} - w_{2,\varepsilon} - m \right] &= 0, \\
u_{\varepsilon}(0, x) - w_{1,\varepsilon}(0, x) - w_{2,\varepsilon}(0, x) &= 0.
\end{aligned}
\]

The integral solution is

\[
\begin{aligned}
\left( u_{\varepsilon}(t, x) - w_{1,\varepsilon}(t, x) - w_{2,\varepsilon}(t, x) \right) &= \int_0^t \int_{\mathbb{R}} \kappa_n(t - \tau, x - y) v_{\varepsilon}(y) \\
&\quad \cdot \left( u_{\varepsilon}(\tau, y) - w_{2,\varepsilon}(\tau, y) - m(\tau, y) \right) dy d\tau \\
&= \int_0^t \int_{\mathbb{R}} \kappa_n(t - \tau, x - y) v_{\varepsilon}(y) \\
&\quad \cdot \left( u_{\varepsilon}(\tau, y) - w_{1,\varepsilon}(\tau, y) - w_{2,\varepsilon}(\tau, y) \right) dy d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}} \kappa_n(t - \tau, x - y) v_{\varepsilon}(y) \\
&\quad \cdot \left( w_{1,\varepsilon}(\tau, y) - m(\tau, y) \right) dy d\tau.
\end{aligned}
\]

So,

\[
\begin{aligned}
\| u_{\varepsilon}(t, .) - w_{1,\varepsilon}(t, .) - w_{2,\varepsilon}(t, .) \|_{L^\infty} &\leq \int_0^t \| \kappa_n(t - \tau, x - .) \|_{L^1} \| v_{\varepsilon} \|_{L^\infty} \\
&\quad \cdot \left\| \left( u_{\varepsilon}(\tau, .) - w_{2,\varepsilon}(\tau, .) - m(\tau, .) \right) \right\|_{L^\infty} d\tau \\
&\quad + \int_0^t \| \kappa_n(t - \tau, x - .) \|_{L^1} \| v_{\varepsilon} \|_{L^\infty} \\
&\quad \cdot \left\| \left( u_{\varepsilon}(\tau, .) - w_{1,\varepsilon}(\tau, .) - w_{2,\varepsilon}(\tau, .) \right) \right\|_{L^\infty} d\tau \\
&\quad \leq C \| v_{\varepsilon} \|_{L^\infty} \left[ \int_0^t \left\| \left( u_{\varepsilon}(\tau, .) - w_{1,\varepsilon}(\tau, .) - w_{2,\varepsilon}(\tau, .) \right) \right\|_{L^\infty} d\tau \\
&\quad + \int_0^t \left\| \left( u_{\varepsilon}(\tau, .) - w_{1,\varepsilon}(\tau, .) - w_{2,\varepsilon}(\tau, .) \right) \right\|_{L^\infty} d\tau \right].
\end{aligned}
\]

Thanks to the Gronwall lemma, we get
by passing to the limit, and we obtain
\[ u = w_1 + w_2, \]
which completes the proof. \( \square \)

4. Conclusion

In this article, we studied the existence and uniqueness results for the Schrödinger equation with distribution type initial conditions in the extended Colombeau algebra of generalized functions. The existence theorem is proved by using some regularizations of the proposed problem and Gronwall inequality.

Data Availability

The data generated or analyzed to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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