Hindawi
International Journal of Differential Equations
Volume 2023, Article ID 3493912, 10 pages
https://doi.org/10.1155/2023/3493912

Research Article

Solving the Fractional Schrödinger Equation with Singular Initial Data in the Extended Colombeau Algebra of Generalized Functions

Ali El Mfadel,[1,2] **Said Melliani**,[1] **Abdellah Taqbibt**,[1] **and M'hamed Elomari**[1]

[1]*Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco*
[2]*Superior School of Technology, Sultan Moulay Slimane University, Khenifra, Morocco*

Correspondence should be addressed to Ali El Mfadel; elmfadelali@gmail.com

Received 21 July 2022; Revised 12 October 2022; Accepted 7 November 2022; Published 2 May 2023

Academic Editor: Andrea Scapellato

This manuscript aims to highlight the existence and uniqueness results for the following Schrödinger problem in the extended Colombeau algebra of generalized functions. $\begin{cases} 1/\imath\partial/\partial t\mathbf{u}(t,x) - \triangle\mathbf{u}(t,x) + v(x)\mathbf{u}(t,x) = 0, t\in\mathbb{R}^+, x\in\mathbb{R}^n, \\ v(x) = \delta(x), \\ \mathbf{u}(0,x) = \delta(x), \end{cases}$ where δ is the Dirac distribution. The proofs of our main results are based on the Gronwall inequality and regularization method. We conclude our article by establishing the association concept of solutions.

1. Introduction

In the theory of distributions, it is well known that the multiplication of two distributions is not always well defined; for more details, see [1]. However, until 1980, this operation was not used in a rigorous way. Following, Schwarz's renowned impossibilities relating to the build of a commutative, associative, and differential algebra in which we can inject the space of distributions $\mathscr{D}'$. In [2], Colombeau was able to answer to this question by replacing continuous functions with indefinitely differentiable functions. The thing that has allowed us to investigate nonlinear differential equations formed from natural occurrences with parameters in the sense of distributions in a rational manner; for that, see [1]. We are talking about an algebra that is defined as the quotient of the algebra of moderate functions by the algebra of negligible functions. So, it should come as no surprise that the following presentations' main topic will be to raise componentwise definitions to the level of equivalence classes. The derivative operator is one of the most essential operators, and its description in this context allows us to investigate differential equations in terms of distributions. Unfortunately, according to the definition, it does not pose a difficulty for generalized functions, which is not the case for the fractional derivative. In [3], Mirjana presents a method for dealing with fractional derivatives including singularities based on Colombeau's idea of algebras of generalized functions. The Colombeau algebra of generalized functions is extended to fractional derivatives by the same author. It is used to solve ODEs and PDEs with entire and fractional derivatives in terms of temporal and spacial variables; see [4–8]. The study of fractional order integral and derivative operators over real and complex domains, as well as their applications, is the subject of fractional calculus. The features of fractional derivatives make them ideal for modeling complicated systems. The ordinary derivative is a local operator, whereas fractional operators are nonlocal. Fractional derivatives exhibit nonlocal dynamics as a result of this, i.e. the process dynamics have some memory; for more information, see [9–12]. When the nonlinear term is a L^∞loc-function that does not satisfy the Lipschitz condition, the nonlinear Schrödinger equation with singular potential and initial data is studied in [13]. Recently, in [14], Benmerrous et al. established the existence and uniqueness of solutions for the Schrödinger equation with singular potential and initial data in the Colombeau algebra. The reader can consult articles as well [15, 16] and the references therein for more details on the fractional Schrödinger equation.

Motivated by the above works precisely by [14], we study the existence and uniqueness results for the Schrödinger equation with singular initial data in the extended Colombeau algebra of generalized functions. Also, we give the association of solutions.

The present article is organized as follows: In Section 2, we recall some fundamental properties of the generalized function theory. Section 3 is consecrated for the proofs of the existence and uniqueness of solutions for the Schrödinger problem (12) in the Colombeau algebra and the extended Colombeau algebra. We conclude this article by proving the association of solutions.

2. Preliminaries

To define the full algebra of Colombeau, for $r \in \mathbb{N}^*$ we define

$$\sum_r = \left\{\mu \in \frac{\mathcal{D}(\mathbb{R}^n)}{\int \mathbb{R}n^{\mu(y)}} dy = 1, \int_{\mathbb{R}^n} y^\beta \mu(y) \mathrm{d}y = 0 \text{ for } 1 \leq |\beta \leq r\right\}, \quad r = 1, 2\ldots, \tag{1}$$

We denote by

$$\begin{aligned} \xi_{\mathfrak{m}}^{\mathfrak{s}}(\mathbb{R}^n) &= \left\{(\mathfrak{u}_\epsilon)_{\epsilon\in(0,1)} \subset \frac{\xi(\mathbb{R}^n)}{\forall \mathcal{K}} \subset \ \subset \mathbb{R}^n, \forall p \in \mathbb{N}, \exists m \in \mathbb{N} \sup_{x\in\mathcal{K}} \left|\mathfrak{D}^p \mathfrak{u}_\epsilon(x)\right| = \mathcal{O}_{\epsilon\longrightarrow 0}(\epsilon^{-m})\right\}, \\ \aleph^{\mathfrak{s}}(\mathbb{R}^n) &= \left\{(\mathfrak{u}_\epsilon)_{\epsilon\in(0,1)} \subset \frac{\xi(\mathbb{R}^n)}{\forall \mathcal{K}} \subset \ \subset \mathbb{R}^n, \forall p \in \mathbb{N}, \forall m \in \mathbb{N} \sup_{x\in\mathcal{K}} \left|\mathfrak{D}^p \mathfrak{u}_\epsilon(x)\right| = \mathcal{O}_{\epsilon\longrightarrow 0}(\epsilon^{m})\right\}, \end{aligned} \tag{2}$$

where

$$\begin{cases} \mu_\epsilon(x) = \dfrac{1}{\epsilon^n}\mu\left(\dfrac{x}{\epsilon}\right), \forall \mu \in \mathcal{D}(\mathbb{R}^n), \\ \mathfrak{u}_\epsilon(x) = \mathfrak{u}(\mu_\epsilon, x) \forall \mu \in \Sigma_1, \end{cases} \tag{3}$$

$$\xi(\mathbb{R}^n) = \left\{\mathfrak{u}\colon \Sigma_1 \times \mathbb{R}^n \longrightarrow \frac{\mathbb{C}}{\mathfrak{u}(\mu, x) \text{is } \mathcal{C}^\infty \text{respect to}} \text{the second variable } x\right\}.$$

The full Colombeau algebra is defined by

$$\mathcal{G}^{\mathfrak{s}}(\mathbb{R}^n) = \frac{\xi_{\mathfrak{m}}^{\mathfrak{s}}(\mathbb{R}^n)}{\aleph^{\mathfrak{s}}(\mathbb{R}^n)}. \tag{4}$$

To solve ODE and DED with integer and fractional derivatives with initial data distributions, we need to recall the definition of the extension of the fractional derivative in the Colombeau algebra.

We denote the set of all extended moderate functions by

$$\xi_{\mathfrak{m}}^{e}(\mathbb{R}) = \left\{(\mathfrak{u}_\epsilon)_{\epsilon\in(0,1)} \subset \frac{\xi(\mathbb{R})}{\forall \mathcal{K}} \subset \ \subset \mathbb{R}, \forall \beta \in \mathbb{R}^+, \exists n \in \mathbb{N} \sup_{x\in\mathcal{K}} \left|\mathfrak{D}^\beta \mathfrak{u}_\epsilon(x) = \mathcal{O}_{\epsilon\longrightarrow 0}(\epsilon^{-n})\right.\right\}, \tag{5}$$

and the set of all extended negligible functions by

$$\aleph^{e}(\mathbb{R}) = \left\{(\mathfrak{u}_\epsilon)_{\epsilon\in(0,1)} \subset \frac{\xi(\mathbb{R})}{\forall \mathcal{K}} \subset \ \subset \mathbb{R}, \forall \beta \in \mathbb{R}^+, \forall p \in \mathbb{N} \sup_{x\in\mathcal{K}} \left|\mathfrak{D}^\beta \mathfrak{u}_\epsilon(x) = \mathcal{O}_{\epsilon\longrightarrow 0}(\epsilon^{p})\right.\right\}. \tag{6}$$

The extended Colombeau algebra of generalized functions is the factor set.

$$\mathcal{G}^e(\mathbb{R}) = \frac{\xi_m^e(\mathbb{R})}{\aleph^e(\mathbb{R})}. \tag{7}$$

Now, we give the definitions of the fractional calculus theory.

Definition 1 (see [17, 18]). The fractional integral is defined as follows:

$$\mathfrak{J}^\beta g(s) = \frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} g(\tau) \mathrm{d}\tau, \beta > 0. \tag{8}$$

The fractional derivative of order $\beta > 0$ in the Caputo sense is defined as follows:

$$\mathfrak{D}^\beta g(s) = \frac{1}{\Gamma(p-\beta)} \int_0^s \frac{g^{(p)}(\tau) d\tau}{(s-\tau)^{\beta+1-p}}, p-1<\beta<p, p \in \mathbb{N}^*. \tag{9}$$

The fractional derivative and the fractional integral are injected into the extension of the Colombeau algebra $\mathcal{G}^e$ which is given by the formulas in the following proposition:

Proposition 1 (see [14]). *Let* $\mathfrak{u} = [(\mathfrak{u}_\epsilon)_\epsilon] \in \mathcal{G}^e(\mathbb{R})$, *then*

$$\begin{aligned}\mathfrak{J}^\beta \mathfrak{u}(s) &= \left[\left(\mathfrak{J}^\beta \mathfrak{u}_\epsilon(s)\right)_\epsilon\right] \in \mathcal{G}^e(\mathbb{R}),\\ \mathfrak{D}^\beta \mathfrak{u}(s) &= \left[\left(\mathfrak{D}^\beta \mathfrak{u}_\epsilon(s)\right)_\epsilon\right] \in \mathcal{G}^e(\mathbb{R}).\end{aligned} \tag{10}$$

We end this section by recalling the association relation on the Colombeau algebra $\mathcal{G}^s$. It identifies elements of $\mathfrak{u} \in \mathcal{G}^s$ if they coincide in the weak limit.

Definition 2 (see [19]). Let $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{G}^s(\mathbb{R}^n)$ such that $\mathfrak{u}_{1,\epsilon}$ and $\mathfrak{u}_{2,\epsilon}$ are their representatives, respectively. We say that $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are associated in $\mathcal{G}^s(\mathbb{R}^n)$, and we write $\mathfrak{u}_1 \approx \mathfrak{u}_2$, if for every $\mu \in \mathcal{D}(\mathbb{R}^n)$.

$$\lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^n} \left(\mathfrak{u}_{1,\epsilon} - \mathfrak{u}_{2,\epsilon}\right) \mu(x) \mathrm{d}x = 0. \tag{11}$$

3. Main Results

The nonlinear Schrödinger equation with singular potential and initial data is considered.

$$\begin{cases} \dfrac{1}{i}\dfrac{\partial}{\partial t}\mathfrak{u}(t,x) - \triangle \mathfrak{u}(t,x) + v(x)\mathfrak{u}(t,x) = 0, t \in \mathbb{R}^+, x \in \mathbb{R}^n; \\ v(x) = \delta(x); \\ \mathfrak{u}(0,x) = \delta(x). \end{cases} \tag{12}$$

We shall use the regularization for the initial and Dirac measure.

$$v_\epsilon(x) = \delta_\epsilon(x) = (\mu_\epsilon(x)) = |\ln\epsilon|^{cn} \mu(x|\ln\epsilon|^c), c > 0, \tag{13}$$

$$\mathfrak{u}_{0,\epsilon}(x) = |\ln\epsilon|^{an} \mu(x|\ln\epsilon|^a), a > 0, \tag{14}$$

where $x \in \mathbb{R}^n, \mu \in A_1, \mu(x) \geq 0, x \in \mathbb{R}^n$.

3.1. Existence and Uniqueness Results in the Colombeau Algebra

Theorem 1. *Let equation (12) have the regularized equation:*

$$\begin{cases} \dfrac{1}{i}\dfrac{\partial}{\partial t}\mathfrak{u}_\epsilon(t,x) - \triangle \mathfrak{u}_\epsilon(t,x) + v_\epsilon(x)\mathfrak{u}_\epsilon(t,x) = 0, t \in \mathbb{R}^+, x \in \mathbb{R}^n; \\ v_\epsilon(x) = \delta_\epsilon(x); \\ \mathfrak{u}_\epsilon(0,x) = \delta_\epsilon(x), \end{cases} \tag{15}$$

where v_ϵ *and* $\mathfrak{u}_{0,\epsilon}$ *are regularized of* v *and* $\mathfrak{u}_0$, *respectively.*

Then, the problem (12) has a unique solution in $\mathcal{G}^{\mathfrak{s}}(\mathbb{R}^+ \times \mathbb{R}^n)$.

Proof. The integral solution of equation (15) (see [13]) is given by.

$$\mathbf{u}_\epsilon(t,x) = \int_{\mathbb{R}^n} \kappa_n(t,x-y)\mathbf{u}_{0,\epsilon}(y)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \kappa_n(t-\tau,x-y)v_\epsilon(y)\mathbf{u}_\epsilon(\tau,y)\mathrm{d}y\,\mathrm{d}\tau, \tag{16}$$

where $\kappa_n(t,x)$ is the heat kernel.

Then,

$$\begin{aligned}\left\|\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} &\le \left\|\kappa_n(t,x-.)\right\|_{L^1}\left\|\mathbf{u}_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ &\quad + \int_0^t \left\|\kappa_n(t-\tau,x-.)\right\|_{L^1}\left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\mathbf{u}_\epsilon(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}\mathrm{d}\tau \\ &\le C\left\|\mathbf{u}_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} + C\left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\int_0^t \left\|\mathbf{u}_\epsilon(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}\mathrm{d}\tau.\end{aligned} \tag{17}$$

By Gronwall inequality,

$$\left\|\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \le C\,|\ln\epsilon|^{an}\exp\left(CT\,|\ln\epsilon|^{bn}\right). \tag{18}$$

Then, there exists $N>0$ such that

$$\left\|\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \le C\epsilon^{-N}. \tag{19}$$

For the first derivative to $x_i, i \in \{1,n\}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial x_j}\mathbf{u}_\epsilon(t,x) &= \int_{\mathbb{R}^n} \kappa_n(t,x-y)\frac{\partial}{\partial y_j}\mathbf{u}_{0,\epsilon}(y)\mathrm{d}y \\ &\quad + \int_0^{\mathrm{t}}\int_{\mathbb{R}^n}\kappa_n(t-\tau,x-y)\left(\frac{\partial}{\partial y_j}v_\epsilon(y)\mathbf{u}_\epsilon(\tau,y) + v_\epsilon(y)\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon(\tau,y)\right)\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{20}$$

Then,

$$\begin{aligned}\left\|\frac{\partial}{\partial x_j}\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} &\le \left\|\kappa_n(t,x-.)\right\|_{L^1}\left\|\frac{\partial}{\partial y_j}\mathbf{u}_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ &\quad + \int_0^t \left\|\kappa_n(t-\tau,x-.)\right\|_{L^1}\left(\left\|\frac{\partial}{\partial y_j}v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty} \right. \\ &\quad \left. + \left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}\right)d\tau \\ &\le C\,|\ln\epsilon|^{a(n+1)} + C\int_0^t\left(|\ln\epsilon|^{b(n+1)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty} \right. \\ &\quad \left. + |\ln\epsilon|^{bn}\,\left|\left\|\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon(\tau,\right\|_{L^\infty(\mathbb{R}^n)}\right)d\tau \\ &\le C\left(|\ln\epsilon|^{a(n+1)} + T\,|\ln\epsilon|^{b(n+1)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\right) \\ &\quad + C\,|\ln\epsilon|^{bn}\,\left|\int_0^t\right\|\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon(\tau,.)\|_{L^\infty(\mathbb{R}^n)}\mathrm{d}\tau.\end{aligned} \tag{21}$$

By Gronwall inequality,

$$\left\|\frac{\partial}{\partial x_j}\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \le C\left(|\ln\epsilon|^{a(n+1)} + T\,|\ln\epsilon|^{b(n+1)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\right)\exp\left(CT\,|\ln\epsilon|^{bn}\right). \tag{22}$$

By the previous step, there exist $N>0$ such that

$$\left\| \frac{\partial}{\partial x_j} \mathbf{u}_\epsilon (t,.) \right\|_{L^\infty(\mathbb{R}^n)} \leq C\epsilon^{-N}. \tag{23}$$

For the second derivative, for $y_j, j \in \{1, n\}$, we get

$$\begin{aligned} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\mathbf{u}_\epsilon (t,x) = &\int_{\mathbb{R}^n} \kappa_n (t, x-y)\left(\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_j}\mathbf{u}_{0,\epsilon}(y)\mathrm{d}y\right. \\ &+ \int_0^t \left[\int_{\mathbb{R}^n}\left(\kappa_n (t-\tau, x-y)\left(\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_j} v_\epsilon (y)\mathbf{u}_\epsilon (\tau, y)\right.\right.\right. \\ &+ \frac{\partial}{\partial y_j} v_\epsilon (y)\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon (\tau, y) + \frac{\partial}{\partial y_j} v_\epsilon (y)\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon (\tau, y) \\ &\left.\left.+ v_\epsilon (y)\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon (\tau, y)\right)\mathrm{d}y\right]\mathrm{d}\tau. \end{aligned} \tag{24}$$

So,

$$\begin{aligned} \left\|\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\mathbf{u}_\epsilon (t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq &\left\|\kappa_n (t, x-y)\right\|_{L^1}\left\|\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\mathbf{u}_{0,\epsilon}(.)\right\|_{L^\infty(\mathbb{R}^n)} \\ &+ \int_0^t \left\|\kappa_n (t-\tau, x-.)\right\|_{L^1}\left(\left\|\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_j} v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\right. \\ &+ \left\|\frac{\partial}{\partial y_j} v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon\right\|_{L^\infty} + \left\|\frac{\partial}{\partial y_i} v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon\right\|_{L^\infty} \\ &\left.+ \left\|v_\epsilon (.)\right\|_{L^\infty}\left\|\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon (\tau, .)\right\|_{L^\infty(\mathbb{R}^n)}\right)\mathrm{d}\tau \\ \leq &C\left(|\ln\epsilon|^{a(n+2)} + |\ln\epsilon|^{b(n+1)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\right. \\ &\left.+ |\ln\epsilon|^{b(n+1)}\left\|\frac{\partial}{\partial y_i}\mathbf{u}_\epsilon\right\|_{L^\infty} + |\ln\epsilon|^{b(n+1)}\left\|\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon\right\|_{L^\infty}\right) \\ &+ C|\ln\epsilon|^{bn}\int_0^t\left\|\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon (\tau, .)\right\|_{L^\infty(\mathbb{R}^n)}\mathrm{d}\tau. \end{aligned} \tag{25}$$

Using Gronwall inequality, we obtain the following equation:

$$\begin{aligned} \left\|\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\mathbf{u}_\epsilon (t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq &C\exp\left(\mathrm{CT}|\ln\epsilon|^{bn}\right)\left(|\ln\epsilon|^{a(n+2)} + |\ln\epsilon|^{b(n+1)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\right. \\ &\left.+ |\ln\epsilon|^{b(n+1)}\left\|\frac{\partial}{\partial y_i}\mathbf{u}_\epsilon\right\|_{L^\infty} + |\ln\epsilon|^{b(n+1)}\left\|\frac{\partial}{\partial y_j}\mathbf{u}_\epsilon\right\|_{L^\infty}\right). \end{aligned} \tag{26}$$

By the previous step, there exist $N > 0$ such that

$$\left\| \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} \mathfrak{u}_\epsilon (t,.) \right\|_{L^\infty(\mathbb{R}^n)} \leq C\epsilon^{-N}. \quad (27)$$

Let us prove the uniqueness. Suppose that there exist two solutions $\mathfrak{u}_1, \mathfrak{u}_2$ to problem (12), then we obtain the following equation:

$$\begin{cases} \frac{1}{i}\frac{\partial}{\partial t}\left(\mathfrak{u}_{1,\epsilon}(t,x) - \mathfrak{u}_{2,\epsilon}(t,x)\right) = \triangle\left(\mathfrak{u}_{1,\epsilon}(t,x) - \mathfrak{u}_{2,\epsilon}(t,x)\right) - v_\epsilon(x)\left(\mathfrak{u}_{1,\epsilon}(t,x) - \mathfrak{u}_{2,\epsilon}(t,x)\right), \\ \mathfrak{u}_{1,\epsilon}(0,x) = \mathfrak{u}_{2,\epsilon}(0,x) + N_{0,\epsilon}(x), \end{cases} \quad (28)$$

where $N_\epsilon(t,x) \in \aleph^{\mathfrak{s}}(\mathbb{R}^+ \times \mathbb{R}^n)$, $N_{0,\epsilon}(x) \in \aleph^{\mathfrak{s}}(\mathbb{R}^n)$. Then,

$$\begin{aligned} \mathfrak{u}_{1,\epsilon}(t,x) - \mathfrak{u}_{2,\epsilon}(t,x) &= \int_{\mathbb{R}^n} \kappa_n(t,x-y) N_{0,\epsilon}(y)\mathrm{d}y \\ &+ \int_0^t \int_{\mathbb{R}^n} \kappa_n(t-\tau,x-y) v_\epsilon(y)\left(\mathfrak{u}_{1\epsilon}(\tau,y) - \mathfrak{u}_{2\epsilon}(\tau,y)\right)\mathrm{d}y\,\mathrm{d}\tau \\ &+ \int_0^t \int_{\mathbb{R}^n} \kappa_n(t-\tau,x-y) N_\epsilon(\tau,y)\mathrm{d}y\,\mathrm{d}\tau, \end{aligned} \quad (29)$$

and thus

$$\begin{aligned} \left\|\mathfrak{u}_{1,\epsilon}(t,.) - \mathfrak{u}_{2,\epsilon}(t,.)\right\|_{L^\infty(\mathbb{R}^n)} &\leq \left\|\kappa_n(t,x-.)\right\|_{L^1} \left\|N_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ &+ \left\|\kappa_n(t,x-.)\right\|_{L^1} \int_0^t \left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)} \left\|\mathfrak{u}_{1\epsilon}(\tau,.) - \mathfrak{u}_{2\epsilon}(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}\mathrm{d}\tau \\ &+ \left\|\kappa_n(t,x-.)\right\|_{L^1} \left\|N_\epsilon\right\|_{L^\infty} \\ &\leq C\left(\left\|N_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} + \left\|N_\epsilon\right\|_{L^\infty}\right) \\ &+ \left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)} \int_0^t \left\|\mathfrak{u}_{1\epsilon}(\tau,.) - \mathfrak{u}_{2\epsilon}(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)} \mathrm{d}\tau. \end{aligned} \quad (30)$$

By Gronwall inequality,

$$\left\|\mathfrak{u}_{1,\epsilon}(t,.) - \mathfrak{u}_{2,\epsilon}(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq C\left(\left\|N_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} + \left\|N_\epsilon\right\|_{L^\infty}\right) \exp\left(\mathrm{CT}\left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\right). \quad (31)$$

Then,

$$\left\|\mathfrak{u}_{1,\epsilon}(t,.) - \mathfrak{u}_{2,\epsilon}(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq C\epsilon^q, \forall q \in \mathbb{N}. \quad (32)$$

Then, the problem (12) has a unique solution in $\mathcal{G}^{\mathfrak{s}}(\mathbb{R}^+ \times \mathbb{R}^n)$. □

3.2. Existence and Uniqueness Results in the Extension of the Colombeau Algebra. In a framework of extended algebra of generalized functions, we prove the existence uniqueness result for nonlinear Schrödinger equations with singular potential and initial data and an equation driven by the fractional derivative of the delta distribution. It means that we prove the moderateness and the negligibility for entire and fractional derivatives to the spatial variable x.

Theorem 2. *Let the following problem*

$$\begin{cases} \frac{1}{i}\frac{\partial}{\partial t}\mathbf{u}_\epsilon(t,x) - \triangle \mathbf{u}_\epsilon(t,x) + v_\epsilon(x)\mathbf{u}_\epsilon(t,x) = 0, t \in \mathbb{R}^+, x \in \mathbb{R}^n, \\ v_\epsilon(x) = \delta_\epsilon(x), \\ \mathbf{u}_\epsilon(0,x) = \delta_\epsilon(x), \end{cases} \quad (33)$$

be the regularized equation of (12) such that v_ϵ *and* $\mathbf{u}_{0,\epsilon}$ *are regularizations of* v *and* $\mathbf{u}_0$*, respectively. Then, the problem (12) has a unique solution in* $\mathscr{G}^e(\mathbb{R}^+ \times \mathbb{R}^n)$*.*

Proof. We prove only the fractional part since the entire part is already proved in Theorem 1. The fractional derivative, $\mathfrak{D}^\beta, 0<\beta<1$, is considered. Without loss of generality, the same holds for $m-1<\beta<m, m \in \mathbb{N}$. The fractional derivative is taken to the spatial variable to equation (12). We have

$$\begin{aligned}\mathfrak{D}^\beta(\mathbf{u}_\epsilon(t,x)) = &\int_{\mathbb{R}^n}\kappa_n(t,x-y)\mathfrak{D}^\beta\mathbf{u}_{0,\epsilon}(y)\mathrm{d}y \\ &+ \int_0^t\int_{\mathbb{R}^n}\kappa_n(t-\tau,x-y)\mathfrak{D}^\beta v_\epsilon(y)\mathbf{u}_\epsilon(\tau,y)\mathrm{d}y\,\mathrm{d}\tau \\ &+ \int_0^t\int_{\mathbb{R}^n}\kappa_n(t-\tau,x-y)v_\epsilon(y)\mathfrak{D}^\beta\mathbf{u}_\epsilon(\tau,y)\mathrm{d}y\,\mathrm{d}\tau.\end{aligned} \quad (34)$$

Then,

$$\begin{aligned}\left\|\mathfrak{D}^\beta\left(\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq &\left\|\kappa_n(t,x-.)\right\|_{L^1}\left\|\mathfrak{D}^\beta\mathbf{u}_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} \\ &+\left\|\kappa_n(t-\tau,x-.)\right\|_{L^1}\int_0^t\left\|\mathfrak{D}^\beta v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\mathbf{u}_\xi(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}d\tau \\ &+\left\|\kappa_n(t-\tau,x-.)\right\|_{L^1}\int_0^t\left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\mathfrak{D}^\beta\mathbf{u}_\epsilon(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}d\tau \\ \leq &C\left\|\mathfrak{D}^\beta\mathbf{u}_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} + T\left\|\mathfrak{D}^\beta v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\Big) \\ &+\left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\int_0^t\left\|\mathfrak{D}^\beta\mathbf{u}_\epsilon(\tau,.)\right\|_{L^\infty(\mathbb{R}^n)}d\tau.\end{aligned} \quad (35)$$

By Gronwall inequality, we get

$$\begin{aligned}\left\|\mathfrak{D}^\beta\left(\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq &C\exp\left(\mathrm{CT}\left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\right) \\ &\cdot\left(\left\|\mathfrak{D}^\beta\mathbf{u}_{0,\epsilon}\right\|_{L^\infty(\mathbb{R}^n)} + T\left\|\mathfrak{D}^\beta v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\right).\end{aligned} \quad (36)$$

From Theorem 1 and hypothesis (13), we obtain

$$\begin{aligned}\left\|\mathfrak{D}^\beta\left(\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq &C\exp\left(\mathrm{CT}\left\|v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\right) \\ &\cdot\left(C_{\beta,T}\mid\ln\epsilon\mid^{a(n+1)} + \mathrm{TC}_{\beta,T}\mid\ln\epsilon\mid^{b(n+1)}\left\|\mathbf{u}_\epsilon\right\|_{L^\infty}\right).\end{aligned} \quad (37)$$

Then, there exist $N>0$ such that

$$\left\|\mathfrak{D}^\beta\left(\mathbf{u}_\epsilon(t,.)\right\|_{L^\infty(\mathbb{R}^n)} \leq C\epsilon^{-N}. \quad (38)$$

It follows moderateness for the fractional derivatives in the space $\mathcal{G}^e(\mathbb{R}^+ \times \mathbb{R}^n)$. For uniqueness, let $0 < \beta < 1$, and apply $\mathfrak{D}^\beta$ to (29) we get:

$$\begin{aligned}\mathfrak{D}^\beta\left(\mathfrak{u}_{1,\epsilon}(t,x) - \mathfrak{u}_{2,\epsilon}(t,x)\right) = &\int_{\mathbb{R}^n} \kappa_n(t, x-y)\mathfrak{D}^\beta N_{0,\epsilon}(y)\mathrm{d}y \\ &+ \int_0^t \int_{\mathbb{R}^n} \kappa_n(t-\tau, x-y)\mathfrak{D}^\beta v_\epsilon(y)\left(\mathfrak{u}_{1\epsilon}(\tau,y) - \mathfrak{u}_{2\epsilon}(\tau,y)\right)\mathrm{d}y\mathrm{d}\tau \\ &+ \int_0^t \int_{\mathbb{R}^n} \kappa_n(t-\tau, x-y)v_\epsilon(y)\mathfrak{D}^\beta\left(\mathfrak{u}_{1\epsilon}(\tau,y) - \mathfrak{u}_{2\epsilon}(\tau,y)\right)\mathrm{d}y\mathrm{d}\tau \\ &+ \int_0^t \int_{\mathbb{R}^n} \kappa_n(t-\tau, x-y)\mathfrak{D}^\beta N_\epsilon(\tau,y)\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{39}$$

So,

$$\begin{aligned}\left\|\mathfrak{D}^\beta\left(\mathfrak{u}_{1,\epsilon}(t,.) - \mathfrak{u}_{2,\epsilon}(t,.)\right)\right\|_{L^\infty} \le &\left\|\kappa_n(t,x-.)\right\|_{L^1}\left\|\mathfrak{D}^\beta N_{0,\epsilon}\right\|_{L^\infty} \\ &+ \| \kappa_n(t-\tau, x-.)\|_{L^1} \int_0^t \| \mathfrak{D}^\beta v_\epsilon\|_{L^\infty} \| \mathfrak{u}_{1\epsilon}(\tau,.) - \mathfrak{u}_{2\epsilon}(\tau,.)\|_{L^\infty}\mathrm{d}\tau \\ &+ \| \kappa_n(t-\tau, x-.)\|_{L^1} \int_0^t \| v_\epsilon\|_{L^\infty} \| \mathfrak{D}^\beta\left(\mathfrak{u}_{1\epsilon}(\tau,.) - \mathfrak{u}_{2\epsilon}(\tau, \|_{L^\infty}\mathrm{d}\tau \\ &+ \| \kappa_n(t-\tau, x-.)\|_{L^1} \int_0^t \| \mathfrak{D}^\beta N_\epsilon(\tau,.)\|_{L^\infty}\mathrm{d}\tau \\ \le &C\left[N_{0,\epsilon}\|_{L^\infty} + T \| \mathfrak{D}^\beta v_\epsilon\|_{L^\infty} \| \mathfrak{u}_{1\epsilon} - \mathfrak{u}_{2\epsilon}\|_{L^\infty} d\tau + \| \mathfrak{D}^\beta N_\epsilon\|_{L^\infty}\right] \\ &+ C \| v_\epsilon\|_{L^\infty(\mathbb{R}^n)} \int_0^t \| \mathfrak{D}^\beta\left(\mathfrak{u}_{1\epsilon}(\tau,.) - \mathfrak{u}_{2\epsilon}(\tau, \|_{L^\infty(\mathbb{R}^n)}\mathrm{d}\tau.\end{aligned} \tag{40}$$

By Gronwall inequality,

$$\begin{aligned}\left\|\mathfrak{D}^\beta\left(\mathfrak{u}_{1,\epsilon}(t,.) - \mathfrak{u}_{2,\epsilon}\left(t, \|_{L^\infty(\mathbb{R}^n)} \le C\exp\left(\mathrm{CT}\|v_\epsilon\|_{L^\infty(\mathbb{R}^n)}\right)\right(N_{0,\epsilon}\|_{L^\infty(\mathbb{R}^n)} \right.\\ \left. + \mathrm{T}\left\|\mathfrak{D}^\beta v_\epsilon\right\|_{L^\infty(\mathbb{R}^n)}\|\mathfrak{u}_{1\epsilon} - \mathfrak{u}_{2\epsilon}\|_{L^\infty} + \left\|\mathfrak{D}^\beta N_\epsilon\right\|_{L^\infty}\right).\end{aligned} \tag{41}$$

Using Theorem 1, we obtain

$$\left\|\mathfrak{D}^\beta\left(\mathfrak{u}_{1,\epsilon}(t,.) - \mathfrak{u}_{2,\epsilon}\left(t, \|_{L^\infty(\mathbb{R}^n)} \le C\epsilon^q, \forall q \in \mathbb{N},\right.\right. \tag{42}$$

which completes the proof. □

3.3. Association of Solutions. Let w_1 be the solution of the problem.

$$\begin{cases}\dfrac{1}{i}\dfrac{\partial}{\partial t}w_1(t,x) - \triangle w_1(t,x) = 0, \\ w_1(0,x) = \delta(x),\end{cases} \tag{43}$$

and w_2 be a solution of the problem.

$$\begin{cases}\dfrac{1}{i}\dfrac{\partial}{\partial t}w_2(t,x) - \triangle w_2(t,x) + v(x)w_2(t,x) = 0, \\ v(x) = \delta(x), w_2(0,x) = 0.\end{cases} \tag{44}$$

Proposition 2. *The generalized solution u of the problem (12) is associated with* $\mathrm{w}_1 + \mathrm{w}_2$.

Proof. Let $w_{1,\varepsilon}$ be the classical solution to the problem.

$$\begin{cases}\dfrac{1}{i}\dfrac{\partial}{\partial t}w_{1,\varepsilon}(t,x) - \triangle w_{1,\varepsilon}(t,x) = 0, \\ w_{1,\varepsilon}(0,x) = \delta_\varepsilon(x),\end{cases} \tag{45}$$

and $w_{2,\varepsilon}$ be the classical solution to the problem.

$$\begin{cases} \frac{1}{i}\frac{\partial}{\partial t} w_{2,\varepsilon}(t,x) - \triangle w_2(t,x) + v_\varepsilon(x)\big(w_{2,\varepsilon}(t,x) + m(t,x)\big) = 0, \\ v_\varepsilon(x) = \delta(x), w_{2,\varepsilon}(0,x) = 0. \end{cases} \tag{46}$$

Then,

$$\begin{cases} \frac{1}{i}\frac{\partial}{\partial t}\big(\mathbf{u}_\varepsilon - w_{1,\varepsilon} - w_{2,\varepsilon}\big) - \triangle\big[\mathbf{u}_\varepsilon - w_{1,\varepsilon} - w_{2,\varepsilon}\big] + v_\varepsilon\big[\mathbf{u}_\varepsilon - w_{2,\varepsilon} - m\big] = 0, \\ \mathbf{u}_\varepsilon(0,x) - w_{1,\varepsilon}(0,x) - w_{2,\varepsilon}(0,x) = 0. \end{cases} \tag{47}$$

The integral solution is

$$\begin{aligned} \mathbf{u}_\varepsilon(t,x) - w_{1,\varepsilon}(t,x) - w_{2,\varepsilon}(t,x)\big) &= \int_0^t \int_{\mathbb{R}} \kappa_n(t-\tau, x-y) v_\varepsilon(y) \\ &\quad \cdot \big[\mathbf{u}_\varepsilon(\tau,y) - w_{2,\varepsilon}(\tau,y) - m(\tau,y)\big] \cdot \mathrm{d}y\,\mathrm{d}\tau \\ &= \int_0^t \int_{\mathbb{R}} \kappa_n(t-\tau, x-y) v_\varepsilon(y) \\ &\quad \cdot \big[\mathbf{u}_\varepsilon(\tau,y) - w_{1,\varepsilon}(\tau,y) - w_{2,\varepsilon}(\tau,y)\big]\mathrm{d}y\,\mathrm{d}\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} \kappa_n(t-\tau, x-y) v_\varepsilon(y) \\ &\quad \cdot \big[w_{1,\varepsilon}(\tau,y) - m(\tau,y)\big]\mathrm{d}y\,\mathrm{d}\tau. \end{aligned} \tag{48}$$

So,

$$\begin{aligned} \big\|\mathbf{u}_\varepsilon(t,.) - w_{1,\varepsilon}(t,.) - w_{2,\varepsilon}(t,.)\big\|_{L^\infty} &\le \int_0^t \|\kappa_n(t-\tau, x-.)\|_{L^1} \|v_\varepsilon\|_{L^\infty} \\ &\quad \cdot \big\|\big(w_{1,\varepsilon}(\tau,.) - m(\tau,.)\big)\big\|_{L^\infty} \mathrm{d}\tau \\ &\quad + \int_0^t \|\kappa_n(t-\tau, x-.)\|_{L^1} \|v_\varepsilon\|_{L^\infty} \\ &\quad \cdot \big\|\mathbf{u}_\varepsilon(\tau,.) - w_{1,\varepsilon}(\tau,.) - w_{2,\varepsilon}(\tau,.)\big\|_{L^\infty} \mathrm{d}\tau \\ &\le C\|v_\varepsilon\|_{L^\infty}\bigg[\int_0^t \big\|\big(w_{1,\varepsilon}(\tau,.) - m(\tau,.)\big)\big\|_{L^\infty}\mathrm{d}\tau \\ &\quad + \int_0^t \|\mathbf{u}_\varepsilon(\tau,.) - w_{1,\varepsilon}(\tau,.) - w_{2,\varepsilon}(\tau,.)\|_{L^\infty}\bigg]\mathrm{d}\tau. \end{aligned} \tag{49}$$

Thanks to the Gronwall lemma, we get

$$\left\|\mathfrak{u}_{\varepsilon}(t,.)-w_{1,\varepsilon}(t,.)-w_{2,\varepsilon}(t,.)\right\|_{L^{\infty}}\leq C\left\|v_{\varepsilon}\right\|_{L^{\infty}}e^{\mathrm{CT}\left\|v_{\epsilon}\right\|_{L^{\infty}}}\cdot\int_{0}^{t}\left\|w_{1,\varepsilon}(\tau,.)-m(\tau,.)\right\|_{L^{\infty}}\mathrm{d}\tau, \tag{50}$$

by passing to the limit, and we obtain

$$\mathfrak{u}\approx w_{1}+w_{2}, \tag{51}$$

which completes the proof. □

4. Conclusion

In this article, we studied the existence and uniqueness results for the Schrödinger equation with distribution type initial conditions in the extended Colombeau algebra of generalized functions. The existence theorem is proved by using some regularizations of the proposed problem and Gronwall inequality.

Data Availability

The data generated or analyzed to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Research Notes in Mathematics, 1992.
[2] L. Schwartz, "Théorie des distributions à valeurs vectorielles. I," *Annales de l'Institut Fourier*, vol. 7, no. 7, pp. 1–141, 1957.
[3] M. Stojanovic, "Fondation of the fractional calculus in generalized function algebras," *Analysis and Applications*, vol. 10, no. 4, pp. 439–467, 2012.
[4] M. I. Abbas and M. A. Ragusa, "Nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel," *AIMS Mathematics*, vol. 7, no. 11, pp. 20328–20340, 2022.
[5] M. I. Abbas, "Investigation of Langevin equation in terms of generalized proportional fractional derivatives with respect to another function," *Filomat*, vol. 35, no. 12, pp. 4073–4085, 2021.
[6] Z. A. Khan, R. Gul, and K. Shah, "On impulsive boundary value problem with Riemann-Liouville fractional order derivative," *Journal of Function Spaces*, vol. 2021, pp. 1–11, 2021.
[7] J. Lee, J. M. Kim, Y. H. Kim, and A. Scapellato, "On multiple solutions to a nonlocal fractional p(\cdot)-Laplacian problem with concave–convex nonlinearitiesp-Laplacian problem with concave-convex nonlinearities," *Advances in Continuous and Discrete Models*, vol. 2022, no. 1, p. 14, 2022.
[8] M. Stojanovic, "Extension of Colombeau algebra to derivatives of arbitrary order ,D^{∞}. $\alpha\in R_{+}=\cup\{0\}$, Application to ODEs and PDEs with entire and fractional derivatives," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5458–5475, 2009.
[9] A. El Mfadel, S. Melliani, and M. Elomari, "Existence of solutions for nonlinear ψ-Caputo-type fractional hybrid differential equations with periodic boundary conditions," *Asia Pacific Journal of Management*, vol. 7, 2022.
[10] A. El Mfadel, S. Melliani, and M. Elomari, "Existence and uniqueness results for Caputo fractional boundary value problems involving the p-Laplacian operator," *UPB Scientific Bulletin, Series A*, vol. 84, no. 1, pp. 37–46, 2022.
[11] A. El Mfadel, S. Mellïanï, and M. Elomarï, "New existence results for nonlinear functional hybrid differential equations involving the ψ-Caputo fractional derivative," *Results in Nonlinear Analysis*, vol. 5, no. 1, pp. 78–86, 2022.
[12] Y. Zhou and F. Jiao, "Nonlocal Cauchy problem for fractional evolution equations," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 4465–4475, 2010.
[13] M. Stojanovic, "Nonlinear Schrödinger equation with singular potential and initial data," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 7, pp. 1460–1474, 2006.
[14] A. Benmerrous, L. S. Chadli, A. Moujahid, M. Elomari, and S. Melliani, "Generalized solution of Schrödinger equation with singular potential and initial data," *International Journal of Nonlinear Analysis and Applications*, vol. 13, no. 1, pp. 3093–3101, 2022.
[15] J. I. Diaz, D. Gomez-Castro, and J. L. Vazquez, "The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach," *Nonlinear Analysis*, vol. 177, pp. 325–360, 2018.
[16] E. M'hamed, S. Melliani, F. E. Bourhim, and A. El Mfadel, "Existence of mild solutions for fractional Schrodinger equations in extended Colombeau algebras," *International Journal of Nonlinear Analysis and Applications*, 2022.
[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical studies, Amsterdam, Netherlands, 2006.
[18] I. Podlubny, "Fractional differential equations," *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1999.
[19] J. F. Colombeau, *Elementary Introduction in New Generalized Functions*, North Holland, Amsterdam, Netherlands, 1985.