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Research Article

Perturbed Keplerian Hamiltonian Systems

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This paper deals with a class of perturbation planar Keplerian Hamiltonian systems, by exploiting the nondegeneracy properties of the circular solutions of the planar Keplerian Hamiltonian systems, and by applying the implicit function theorem, we show that noncollision periodic solutions of such perturbed system bifurcate from the manifold of circular solutions for the Keplerian Hamiltonian system.

1. Introduction

In this paper, we study a class of Hamiltonian systems obtained as perturbation of Keplerian Hamiltonian $K(p,q) = 1/2\|p\|^2 - \|q\|^{-1}$. More precisely, we consider the Hamiltonians of the following form:

$$H(p,q,\varepsilon) = \frac{1}{2}\|p\|^2 - \|q\|^{-1} - \frac{\varepsilon}{2}\langle Aq, p\rangle + \varepsilon^2\left(\frac{\|A_q\|^2}{4} + V(\varepsilon,q)\right), \tag{1}$$

where $p, q \in \mathbb{R}^2, \varepsilon > 0$ is a perturbation parameter, A is a skew-symmetric matrix $A^* = -A$, and V is even in q. The corresponding Hamiltonian system is the following:

$$\ddot{q} + \frac{q}{\|q\|^3} + \varepsilon\left(Aq + V_q'(\varepsilon,q)\right) = 0. \tag{2}$$

For $\varepsilon = 0$, equation (2) becomes

$$\ddot{q} + \frac{q}{\|q\|^3} = 0. \tag{3}$$

Our aim is to seek noncollision periodic solutions of (2), and we wish to connect them to the circular solutions of the unperturbed system (3). These types of perturbation systems have been the focus of interest by a number of authors and the references therein [1–5]. We mention in particular the works of Poincaré, regarding the three-body problem (these orbits were called "first-view sort solutions") [6], and the studies by Ambrosetti et al. [7] and Celletti et al. [8] which showed the existence of a skew-$T/2$ periodic solution of the following problem:

$$\ddot{q} + \frac{q}{\|q\|^3} + V_q'(\varepsilon,q) = 0. \tag{4}$$

The essential difficulty in studying this problem is the free action of the S^1-group acting on equation (2) (if q is a solution of (2), then $q_s = q(.+s)$ is also a solution of (2), for all $s \in S^1$). To overcome this difficulty, we seek solutions of equation (2) near the circular orbits of the Keplerian system (3). These circular orbits are the more stable solutions and by exploiting their nondegeneracy property, we neutralize the free action of the S^1-group ([9, 10]). The degenerate solutions of the Keplerian problem are the least stable solutions (KAM theory [11]); we cannot dominate the invariance of the problem under the action of the group S^1 in the neighborhood of these solutions.

The proofs rely on the implicit function theorem and the nondegeneracy of the circular orbits for (3) in the space of the skew-$T/2$ periodic functions. Let $\Omega = \mathbb{R}^2/\{0\}$ and $V\colon \mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ satisfy the following:

$$\begin{aligned}&(V_1) V \in C^2(\mathbb{R}\times\Omega,\mathbb{R}),\\&(V_2) V(\varepsilon,-q) = V(\varepsilon,q).\end{aligned} \tag{5}$$

For all, $q \in \Omega, \varepsilon \in \mathbb{R}$.

We consider the following perturbed system of ordinary differential equations:

$$\begin{cases}\ddot{q} + \dfrac{q}{\|q\|^3} + \varepsilon(t+\varepsilon,q) = 0,\\ q(0) = q(T),\\ \dot{q}(0) = \dot{q}(T),\end{cases} \tag{6}$$

where A is a skew-symmetric matrix ($A^* = -A$), $\varepsilon \in \mathbb{R}$, and $T > 0$ is a fixed period.

The unperturbed system corresponding to (2) is the following:

$$\begin{cases}\ddot{q} + \dfrac{q}{\|q\|^3} = 0,\\ q(0) = q(T),\\ \dot{q}(0) = \dot{q}(T).\end{cases} \tag{7}$$

By a noncollision orbit of (6), we mean a solution of (6) such that $q(t) = 0$ for all t. We will say that $q \in H^1(\mathbb{R}/T\mathbb{Z},\mathbb{R}^2)$ is skew-$T/2$ periodic if $q(t+T/2) = -q(t)$ for every t. The following result holds.

Theorem 1 (see [7, 8, 12]). *There exists $\varepsilon_0 > 0$ such that $\forall|\varepsilon| < \varepsilon_0$, the perturbed system (6), has at least one noncollision symmetric (skew-$T/2$ periodic) orbit near the circular orbit of (6).*

1.1. Bifurcation in the Nondegenerate Case. Actually, we wish to relate the skew-$T/2$ periodic solutions of (6) to the T-periodic circular solutions of (7). Let

$$S^1 = \frac{\mathbb{R}}{\mathbb{Z}, H_1^1 = \left(S^1,\mathbb{R}^2\right)}, \tag{8}$$

and

$$E_1 = \left\{u \in H_1^1 \text{ such that } u\left(t+\frac{1}{2}\right) = -u(t), \forall t \in S^1\right\}. \tag{9}$$

We consider the open subset of E_1 defined by

$$\Lambda_1 = \left\{u \in E_1 \text{ such that } u(t) \neq 0, \forall t \in S^1\right\}. \tag{10}$$

On $\Lambda \times \mathbb{R} \times \mathbb{R}_+^*$, we define a functional Φ by setting

$$\Phi(u,\varepsilon,T) = \int_0^1 \left\{\frac{\|u(t)\|^2}{2T} + \frac{1}{T^2\|u(t)\|} - \frac{\varepsilon}{2} < A\dot{u}(t), u(t) > -\, \varepsilon \frac{V(\varepsilon, Tu(t))}{T}\right\} \mathrm{dt}. \tag{11}$$

Lemma 1. *The functions Φ belongs to $C^2(\Lambda_1 \times \mathbb{R} \times \mathbb{R}_+^*, \mathbb{R})$ and for all $h, k \in E_1$, we have*

$$\begin{aligned}
&(a)\ \phi'_u(u,\varepsilon,T)h = -\int_0^1 \left\{\left\langle \frac{\ddot{u}(t)}{T} + \frac{u(t)}{T^2\|u(t)\|^3} + \varepsilon\left(A\dot{u}(t) + V'_u(\varepsilon, Tu(t))\right), h(t)\right\rangle\right\} \mathrm{dt},\\
&(b)\ \phi'_T(u,\varepsilon,T) = -\int_0^1 \left\{\left\langle \frac{\|\dot{u}(t)\|^2}{2T^2} + \frac{2}{T^3\|u(t)\|} + \varepsilon \frac{V'_u(\varepsilon,Tu)}{T}, u(t)\right\rangle \left(-\frac{V(\varepsilon,Tu)}{T^2}\right)\right\} \mathrm{dt},\\
&(c)\ \phi'_\varepsilon(u,\varepsilon,T) = -\int_0^1 \left\{\frac{1}{2}(A\dot{u},u) + \frac{V(\varepsilon,Tu)}{T} + \varepsilon V'_\varepsilon(\varepsilon,Tu)\right\} \mathrm{dt},\\
&(d)\ \phi''_{uu}(u,\varepsilon,T) = -\int_0^1 \left\{ < \frac{\ddot{h}(t)}{T} + \frac{u(t)}{T^2\|u(t)\|^3} + \left[h(t) - 3\frac{u(t)\langle u(t),h(t)\rangle}{\|u(t)\|^2}\right] = +\varepsilon\left(A\dot{t} + TV''_{uu}(\varepsilon,Tu)h,k\right)\right\rangle\right\} \mathrm{dt},\\
&(e)\ \phi'_{Tu}(T,\varepsilon,u)h = \int_0^1 \left\{\left\langle \frac{\ddot{u}(t)}{T^2} + \frac{2u(t)}{T^3\|u(t)\|^3} - \varepsilon V''_{uu}(\varepsilon,Tu)u, h(t)\right\rangle\right\} \mathrm{dt},\\
&(f)\ \phi'_{\varepsilon u}(T,\varepsilon,u)h = -\int_0^1 \left\{\left\langle V''_{\varepsilon u}(\varepsilon,Tu) + A\dot{u}(t), h(t)\right\rangle\right\} \mathrm{dt}.
\end{aligned} \tag{12}$$

Proof. The proof is left to the reader. □

We now show the following lemma.

Lemma 2. *The following statements are equivalent:*

(i) $\Phi_u'(u,\varepsilon,T)=0$

(ii) $q(t)=Tu(tT^{-1})$ *is a noncollision skew* $-$ $T/2$ *periodic solution of (6)*

Proof. We prove the lemma in two steps: □

Step 1. $q\in C^2(S^T,\mathbb{R}^2)$. The equation, $\phi_u'(u,\varepsilon,T)=0$, means for every $h\in E_1$,

$$\int_0^1\langle\frac{\ddot{u}(t)}{T}+\frac{u(t)}{T^2\|u(t)\|^3}+\varepsilon\left(A\dot{u}(t)+V_u'(\varepsilon,Tu(t))\right),h(t)\rangle \mathrm{dt}. \tag{13}$$

Therefore,

$$\int_0^1\langle\frac{\ddot{u}(t)}{T},h(t)\rangle \mathrm{dt}=-\int_0^1\langle\frac{u(t)}{T^2\|u(t)\|^3}+\varepsilon\left(A\dot{u}(t)+V_u'(\varepsilon,Tu(t))\right),h(t)\rangle \mathrm{dt}. \tag{14}$$

Let

$$h=h_1+h_2, \tag{15}$$

where

$$h_1=\frac{h(t)+h(t+1/2)}{2},\quad h_1\text{ is }\frac{1}{2}\text{ periodic}, \tag{16}$$

and

$$h_2=\frac{h(t)-h(t+1/2)}{2},\quad h_2\text{ is skew }\frac{1}{2}\text{ periodic}. \tag{17}$$

Hence,

$$\int_0^1\langle\frac{\ddot{u}(t)}{T},h(t)\rangle \mathrm{dt}=\int_0^1\langle\frac{\ddot{u}(t)}{T},h_1(t)\rangle \mathrm{dt}+\int_0^1\langle\frac{\ddot{u}(t)}{T},h_2(t)\rangle \mathrm{dt}. \tag{18}$$

It is clear that

$$\int_0^1\langle\frac{\ddot{u}(t)}{T},h_1(t)\rangle \mathrm{dt}=0. \tag{19}$$

Therefore,

$$\begin{aligned}\int_0^1\langle\frac{\ddot{u}(t)}{T},h(t)\rangle \mathrm{dt}&=\int_0^1\langle\frac{\ddot{u}(t)}{T},h_2(t)\rangle \mathrm{dt},\\&=-\int_0^1\langle\frac{u(t)}{T^2\|u(t)\|^3}+\varepsilon(A\dot{u}(t)\\&\quad+V_u'(\varepsilon,Tu(t))),h(t)\rangle \mathrm{dt},\\&<+\infty.\end{aligned} \tag{20}$$

This implies $q\in H^2(s^T,\mathbb{R}^2)$ and $q\in C^2(S^T,\mathbb{R}^2)$.

Step 2. q is a noncollision skew $-1/2$ periodic solution. Since $u\in\Lambda_1$, it is skew-$T/2$ periodic.

Denote by $E_1^{\perp}$ the orthogonal subspace in H_1^1 to E_1 and

$$\varphi(t)=\frac{\ddot{u}(t)}{T}+\frac{u(t)}{T^2\|u(t)\|^3}+\varepsilon\left(A\dot{u}(t)+V_u'(\varepsilon,Tu(t))\right). \tag{21}$$

From (13), it follows that

$$\varphi\in E_1^{\perp}. \tag{22}$$

Using (V_2), one finds that

$$\varphi\left(t+\frac{1}{2}\right)=-\varphi(t). \tag{23}$$

Hence,

$$\varphi\in E_1\cap E_1^{\perp}, \tag{24}$$

and thus, $\varphi=0$,

$$q(t)=Tu\left(tT^{-1}\right). \tag{25}$$

Then,

$$\begin{aligned}\dot{q}&=\dot{u}\left(tT^{-1}\right),\\ \ddot{q}(t)&=\frac{\ddot{u}\left(tT^{-1}\right)}{T}.\end{aligned} \tag{26}$$

By substituting in $\phi=0$, we obtain

$$\ddot{q}+\frac{q}{\|q\|^3}+\varepsilon\left(A\dot{q}+V_q'(\varepsilon,q)\right)=0. \tag{27}$$

Moreover,

$$\begin{aligned}q\left(\frac{T}{2}\right)&=-q(0),\\ \dot{q}\left(\frac{T}{2}\right)&=-\dot{q}(0).\end{aligned} \tag{28}$$

This completes the proof.

1.2. Finding Critical Points for $\phi(\cdot,\varepsilon,T)$. Let $T_0>0$, and

$$\mathbb{Y}=\left\{u\in\Lambda_1,u(t)=\frac{r}{T_0}\left(\xi e^{2i\pi t}+\overline{\xi}e^{-2i\pi t}\right),\xi\in\mathbb{C}^2,\langle\xi,\xi\rangle=\frac{1}{2}\text{ and }\langle\xi,\overline{\xi}\rangle=0\right\}, \tag{29}$$

where

$$\frac{1}{r^3} = \omega_0^2 \left(\omega_0 = \frac{2\pi}{T_0} \right), \tag{30}$$

is a manifold of critical points for $\phi(\cdot, 0, T_0)$ (that is, $\phi_u'(u, 0, T_0) = 0$, for every $u \in \mathbb{Y}_0$). Using Lemma 2, we find that

$$Z_0 = \left\{ q \in \Lambda_{T_0} : q(t) = r\left(\xi e^{i\varepsilon_0 t}\right), \xi \in \mathbb{C}^2 \langle \xi, \xi \rangle = \frac{1}{2} \text{ and } \langle \xi, \overline{\xi} \rangle = 0 \right\}, \tag{31}$$

is a manifold of circular solutions for the unperturbed system,

$$\begin{cases} \ddot{q} + \dfrac{q}{\|q\|^3} = 0, \\ q(0) = q(T_0), \\ \dot{q}(0) = \dot{q}(T). \end{cases} \tag{32}$$

We wish to investigate the situation around $(q_0, 0, T_0), q_0 \in \mathbb{Z}_0$ by applying the inverse function theorem. For this, we need to know more about the derivative of ϕ_u' and ϕ_{uu}''.

Lemma 3. *$\phi_{uu}''(u, 0, T_0)$ is the Fredholm operator of index zero, $\forall u \in \mathbb{Y}_0$.*

Proof. Letting $u \in \mathbb{Y}_0$, for all h and k in E_1, we have

$$\begin{aligned}(\phi_{uu}''(u, 0, T_0)h, k) &= \frac{1}{T_0} \int_0^1 \langle h(t), k(t) \rangle \mathrm{d}t \\ &+ \frac{3}{r^5 T_0^2} \int_0^1 \langle u(t), k(t) \rangle \langle u(t), k(t) \rangle \mathrm{d}t.\end{aligned} \tag{33}$$

Then,

$$(\phi_{uu}''(u, 0, T_0)h, k, h, k) = (h, k)_{H_1^1} - \left(1 + \frac{1}{r^3 T_0^2}\right) \int_0^T \langle h(t), k(t) \rangle \mathrm{d}t + \frac{3}{r^5 T_0^5} \int_0^1 \langle u(t), h(t) \rangle \langle u(t), k(t) \rangle \mathrm{d}t. \tag{34}$$

Hence,

$$\phi_{uu}''(u, 0, T_0) = I_{E_r} - \mathfrak{J}, \tag{35}$$

where $\mathfrak{J}$ is a linear operator from E_1 into E_1 and defined by

$$(\mathfrak{J}(h), k) = \left(1 + \frac{1}{r^3 T_0^2}\right) \int_0^T \langle h(t), k(t) \rangle \mathrm{d}t - \left(\frac{3}{r^5 T_0^5}\right) \int_0^1 \langle u(t), h(t) \rangle \langle u(t), k(t) \rangle \mathrm{d}t. \tag{36}$$

It is easy to verify that $\mathfrak{J}$ is a compact operator. S° $\phi_{uu}''(u, 0, T_0)$ is the Fredholm operator of index zero.

Thus, the proof is complete.

The preceding lemma implies

$$E_1 = \operatorname{Ker}\phi_{uu}''(u, 0, T_0) \oplus \operatorname{Img}\phi_{uu}''(u, 0, T_0), \forall u \in \mathbb{Y}_0. \tag{37}$$

We deduce that, $\phi_{uu}''(u, 0, T_0)$ cannot be an onto function. The ultimate reason for this lies in the fact that the function ϕ is invariant by the S^1- action which sends $u(t)$ into $u(t+\theta) \in S^1$ and this induces degeneracy in the derivatives. We can estimate $\operatorname{Ker}\phi_{uu}''(u, 0, T_0)$ by relating ϕ_{uu}'' to the linearized equation (32) around u. This is done as in the following lemma. □

Lemma 4. *Let $u_0 \in \mathbb{Y}_0$. The following two conditions are equivalent:*

(a) $\phi_{uu}''(u_0, 0, T_0)k = 0$.

(b) $h(t) = T_0 k(tT^{-1})$ is a skew $-T_0/2$ periodic solution of

$$\ddot{h} + \frac{1}{\|q_0\|^3} \left[h - \frac{3q_0 \langle q_0, h \rangle}{\|q_0\|^2} \right] = 0, \tag{38}$$

where

$$q_0(t) = T_0 u_0\left(tT_0^{-1}\right). \tag{39}$$

Proof. Let $k \in E_1$, such that $\phi_{uu}''(u_0, 0, T_0)k = 0$. By reasoning as in the proof of Lemma 2, we obtain

$$k \in C^2\left(S^1, \mathbb{R}^2\right), \tag{40}$$

and

$$\frac{\ddot{k}}{T_0} + \frac{1}{T_0^2 \|u_0\|^3} \left[k - \frac{3u_0 \langle q_0, k \rangle}{\|u_0\|^2} \right] = 0. \tag{41}$$

Set

$$\begin{aligned} h(t) &= T_0 u\left(tT_0^{-1}\right), \\ q_0(t) &= T_0 u_0\left(tT_0^{-1}\right). \end{aligned} \tag{42}$$

By substituting (42) in (41), we obtain,

$$\ddot{h} + \frac{1}{\|q_0\|^3}\left[q - \frac{3q_0\langle q_0, h\rangle}{\|q_0\|^2}\right] = 0. \tag{43}$$

This completes the proof.

More precisely, we have the following lemma. □

Lemma 5

$$\mathrm{Ker}\phi''_{uu}(u_0, 0, T_0) = T_{u_0}\mathbb{Y}_0 = \langle \dot{u}\rangle, \forall u_0 \in \mathbb{Y}_0. \tag{44}$$

Proof. According to the above lemma, the dimension of $\mathrm{Ker}\phi''_{uu}(u_0, 0, T_0)$ is equal to that of the set of the solutions for equation (38). Let h be the solution of (38), and by using the fact that $\|q_0\| = r$, equation (38) reduces to

$$-\ddot{h} = \omega_0^2\left(h - \frac{3q_0\langle q_0, h\rangle}{\|r\|^2}\right). \tag{45}$$

Set

$$h(t) = \sum_{k\in\mathbb{Z}} c_k e^{ik\omega t}, \quad c_k \in \mathbb{C}^2, \tag{46}$$

since $k \in E_1$, then $c_{2k} = 0, \forall k \in \mathbb{Z}$. By substituting, we find

$$\sum_{k\in\mathbb{Z}}\left(k^2 - 1\right)c_k e^{ik\omega t} = -3\sum_{k\in\mathbb{Z}}\left\{\langle c_k, \xi\rangle\xi + \langle c_{k+2}, \xi\rangle\,\overline{\xi} + \langle c_{k-2}, \overline{\xi}\rangle\xi + \langle c_k, \overline{\xi}\rangle\,\overline{\xi}\right\}e^{ik\omega t}, \tag{47}$$

from which we deduce the following equation for c_k, where $k \in \mathbb{Z}$,

$$\begin{aligned} &(a)\ \text{for all}\, k \neq \pm 1, \\ &(b)\ \langle c_1, \xi > + < c_{-1}, \overline{\xi}\rangle = 0, \\ &(c)\ \langle c_1, \overline{\xi}\rangle = 0. \end{aligned} \tag{48}$$

It is easy to see that conditions $(a) - (c)$ define the tangent manifold to $\mathbb{Y}_0$. According to the definition of a nondegenerate critical manifold, this means that $\mathbb{Z}_0$ is a nondegenerate critical:

$$\begin{aligned} &\mathrm{Ker}\phi''_{uu}(u_0, 0, T_0) = T_{q_0}\mathbb{Y}_0, \\ &\dot{u}_0 \in \mathrm{Ker}\phi''_{uu}(u_0, 0, T_0)\ \text{and}\ \dim T_{u_0}\mathbb{Y}_0 = 1. \end{aligned} \tag{49}$$

Therefore,

$$\mathrm{Ker}\phi''_{uu}(u_0, 0, T_0) = \langle \dot{u}_0\rangle. \tag{50}$$

This completes the proof.

We denote

$$\begin{aligned} N_{u_0} &= \mathrm{Ker}\phi''_{uu}(u_0, 0, T_0), u_0 \in \mathbb{Y}_0, \\ R_{u_0} &= \mathrm{Im}\phi''_{uu}(u_0, 0, T_0), u_0 \in \mathbb{Y}_0. \end{aligned} \tag{51}$$

This is a fact that $N_{u_0} = \langle \dot{u}_0\rangle$ gives us a considerable simplification. □

Lemma 6. *For (u, ε, T) close to $(u_0, 0, T_0)$, the following statements are equivalent:*

$$\begin{aligned} &(i)\ \phi'_u(u, \varepsilon, T) = 0. \\ &(ii)\ \phi'_u(u, \varepsilon, T) \in N_{u0}. \end{aligned} \tag{52}$$

Proof. $\mathrm{Ker}\phi''_{uu}(u_0, 0, T_0)$ is spanned by $\dot{u}_0$. The equation $\mathrm{Ker}\phi'_u(u, 0, T) \in \mathrm{Ker}\phi_{uu}{}''(u_0, 0, T_0)$,

$$\frac{\ddot{u}(t)}{T} + \frac{u(t)}{T^2\|u(t)\|^3} + \varepsilon\left(A\dot{u}(t) + V'_u(\varepsilon, Tu(t))\right) = \alpha\dot{u}_0. \tag{53}$$

Multiplying both sides by $\dot{u}$ and integrating, we get

$$\int_0^1\left\{\langle\frac{\ddot{u}(t)}{T}, \dot{u}(t)\rangle + \frac{\langle u(t), \dot{u}(t)}{T^2\|u(t)\|^3} + \varepsilon\langle\langle A\dot{u}(t), \dot{u}(t)\rangle + V'_u(\varepsilon, Tu(t), \dot{u}(t)\rangle\rangle\right\}\mathrm{dt} = \alpha\int_0^1\langle \dot{u}_0, \dot{u}(t)\rangle\mathrm{dt}. \tag{54}$$

$\langle A\dot{u}, \dot{u}\rangle = 0$ and if we integrate the first and second terms on the left by parts, we get zero. In the last term, we recognize the time derivative of $V(\varepsilon, Tu)$, which integrates away to zero. Finally, we get,

$$\alpha\int_0^1\langle \dot{u}_0, \dot{u}(t)\rangle\mathrm{dt} = 0. \tag{55}$$

If u is close to u_0, in E_1, the integral is strictly positive, and, hence, α must be zero.

We now state our main result. □

Theorem 2. *Let $q_0 \in \mathbb{Z}_0$. If*

$$\int_0^1 q_0(t)e^{i\omega_0 t}\mathrm{dt} \neq 0\ \text{in}\ \mathbb{C}^2, \tag{56}$$

then there are positive numbers r_0, ε, a neighborhood V of the path q_0 in $\mathbb{R}^2$, and a $\mathbb{C}^2$ map,

$$v : S_1 \times [-\varepsilon_0, \varepsilon_0] \times [T_0 - r_0, T_0 + r_0] \longrightarrow \mathbb{R}^2, \tag{57}$$

such that

$$q_0(t) = T_0 v\left(tT^{-1}, 0, T_0\right), \tag{58}$$

and for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ *and* T *such* $|T - T_0\| < r_0$ *the curve,*

$$q(t) = Tv\left(tT^{-1}, \varepsilon, T\right), \tag{59}$$

is a skew-$T/2$ *periodic solution of equation (6). Conversely, whenever* q *is a skew* $-T/2$ *periodic solution of (6), with* $\varepsilon \in [-\varepsilon, \varepsilon]$, $|T - T_0| < r_0$, *and* $q(t)$ *remaining in* V *for all* t, *then some* $\theta \in \mathbb{R}$ *can be found such that*

$$q(t) = T_0 v\left(tT^{-1}\right) + (\theta, \varepsilon, T). \tag{60}$$

Proof. Set $u_0(\cdot) = tT_0^{-1} q_0(\cdot T_0)$. It is known that $\phi_{uu}(u_0, 0, T_0)$ is a Fredholm map of index zero. Split E_1 into $N_{u_0} \oplus R_{u_0}$; then, ϕ'_{uu} is an isomorphism of R_{u_0} onto itself. By the implicit function theorem, the equation

$$\phi'_u(u, \varepsilon, T) \in N_{u_0}, \tag{61}$$

determines $u \in R_{u_0}$ in terms of the remaining variables. These are (ε, T) and the component α of u in N_{u_0}. By this, we mean

$$\begin{aligned} u_1 &= u_0 + \alpha \dot{u}_0, \\ \forall u_1 &\in \left(u_0 + \alpha \dot{u}_0\right). \end{aligned} \tag{62}$$

By Lemma 6, this means that the equation $\phi'_u(u, \varepsilon, T) = 0$ can be solved in E_1 as follows in neighborhood of $(u_0, 0, T_0)$,

$$u = v_1(\alpha, \varepsilon, T). \tag{63}$$

We now replace α by more convenient variables.

For any $s \in S^1$, set $u^s(t) = u(t+s)$. If $\phi'_u(u, \varepsilon, T) = 0$, we also have $\phi'_u(u^s, \varepsilon, T) = 0$. We, thus, have an S^1-action which leaves our equations invariants, and we wish to find a coordinate system adapted to this group-invariance. For u near $u_0 \in E_1$, the complex number,

$$\left(\int_0^1 u_0(t) e^{2i\pi t}\right)_1 \left(\int_0^1 \overline{u}_0(t) e^{-2i\pi t}\right)_1, \tag{64}$$

has a well-defined argument $\theta(u)$, called the phase of u with respect to u_0. In the subsequent paragraphs, we will verify that

$$\theta(u^s) = \theta(u) + 2\pi s, \forall s \in S^1, \quad \text{(See [7])}. \tag{65}$$

I now claim that we can use (θ, ε, T) as a local coordinate system for $(u_0 + N_{u_O}) \times \mathbb{R} \times \mathbb{R}_T$ near $(u_o, 0, T_o)$. Computing the Jacobian at this point gives

$$\frac{D(\theta, \varepsilon, T)}{D(\alpha, \varepsilon, T)} = \frac{D(\theta)}{D(\alpha)} = \left(\theta'(u - 0), \dot{u}_0\right), \quad \text{(See [6])}. \tag{66}$$

Since

$$\left(\theta'(u_0), \dot{u}_0\right) = \frac{\mathrm{d}}{\mathrm{ds}}(\theta(u) + 2\pi s) = 2\pi, \tag{67}$$

so the Jacobian does not vanish. The equation now becomes

$$u = v_2(\alpha, \varepsilon, T). \tag{68}$$

Using Lemma 4 to translate in terms of q and q_0, we get the desired result. v is at least a C^2 map from $\mathbb{R} \times \mathbb{R}_+^*$ into the space $C^2(S^1, \mathbb{R}^2)$; it will then have a Taylor expansion. □

2. Conclusion

We can conclude that the class of planar perturbations are Keplerian Hamiltonian systems, as we have shown that the noncollision periodic solutions of this perturbed system radiate from the complex of circular solutions of the Keplerian Hamiltonian system. We have studied a class of Hamiltonian systems obtained as perturbation in the Keplerian Hamiltonian

$$K(p, q) = \frac{1}{2}\|p\|^2 - \|q\|^{-1}. \tag{69}$$

Our goal was to search for noncollision periodic solutions of (2), and we wish to relate them to circular solutions for the nonperturbed system (3). These kinds of systems were restless focus of a number of authors and the references therein [1–5]. We mention in particular, the work of Poincaré, on the three-body problem (these orbitals are called "first-view sort solutions").

In an effort to organize another piece of work into a paper, we determine the coefficients of the Taylor expansion u_p to the second order of the noncollision periodic solutions for the perturbed planar Keplerian Hamiltonian system, which is connected to Kepler Hamiltonian systems by a perturbation parameter. This Taylor expansion is made with respect to a perturbation term ε and the period T of the solution.

Data Availability

The authors claim that this work is a theoretical result, and there are no available data source.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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