Hindawi
International Journal of Differential Equations
Volume 2023, Article ID 5561224, 9 pages
https://doi.org/10.1155/2023/5561224

Research Article

On Stabilizability of Nonbilinear Perturbed Descriptor Systems

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Received 3 March 2023; Revised 18 July 2023; Accepted 26 July 2023; Published 14 August 2023

Academic Editor: Md. Kamrujjaman

One way in which nonlinear descriptor systems of (index-k) naturally arise is through semiexplicit differential-algebraic equations. The study considers the nonbilinear dynamical systems which are described by the class of higher-index differential-algebraic equations (DAEs). Their nature is analysed both quantitatively and qualitatively, and stability characteristics are presented for their solution. Higher-index differential-algebraic systems seem to show inherent shaky around their solution manifolds. The often use of logarithmic norms is for the estimation of stability and perturbation bounds in linear ordinary differential equations (ODEs). The question of how to apply the notation of logarithmic norms to nonlinear DAEs has long been an open question. Other problem extensions including nonlinear dynamics and nonbilinear DAEs need subtle modification of the logarithmic norms. The logarithmic norm is combined by conceptual focus with the finite-time stability criterion in order to treat nonbilinear DAEs with the aim of covering some unbounded operators. This means we obtain the perturbation bounds from differential inequalities for a norm by the use of the relationship between Dini derivatives and semi-inner products. A numerical result obtained when tested on the nonbilinear mechanical system with a larger scale showed that the method was highly efficient and accurate and particularly suitable for nonbilinear DAEs.

1. Introduction

Differential-algebraic (called DAE, descriptor or singular) systems provide a classical state-space framework generalization allowing a simpler description of several physical phenomena, including mass and flow conservation, environmental and topological limits, and thermodynamical relations. The naturally described engineering applications by DAE systems include mechanical systems [1–3], a robot manipulator with a constrained end effector [4], and an electrical network with a nonlinear element [5].

Bilinear systems are a significant subtype of nonlinear systems with many applications in engineering, biology, and economics. There are several studies examining bilinear control systems [6–8].

The classical approaches to the DAE system stability study depend on the system index or organise the reduction techniques, through multiple time differentiations and algebraic manipulations, showing the underlying differential system depiction in which we can apply the classical results. A study in [9] was the first contribution to the system where there is an introduction of state-space equivalent forms for linear time-invariant DAE systems. A different study [10] includes a state-space realization for index three nonlinear DAE systems derived. Also, the feedback stabilization problem is solved through the techniques of linearization. A study in [11] has used a similar approach. Yet, the multiple algebraic equation differentiation and the demand for further algebraic manipulations which these methods require are poorly suitable for the scale of several engineering problems. Also, nonlinearities in the model equations and model uncertainties could prevent applying coordinate's reduction methods [12]. So, an approach to the stability analysis problems and a direct control in the DAE formulation are required.

Not similar to the current approaches, our proposed approach helps in the establishment of stability to the DAE system class without explicitly calculating the reduced unconstrained systems. Yet, this method prevents additional time differentiation and algebraic manipulations which the classical approaches need for reducing the index to zero. In contrast, a process broad class where ordinary differential

equations decreased could be avoided by the model uncertainties or by nonlinearities [13, 14]. So, we need a direct approach to the stability analysis problems in the differential-algebraic formulation.

A singular bilinear system was studied as a special case of nonlinear descriptor systems [15]. A new set of sufficient conditions is derived via continuous state feedback that guarantees the global asymptotic stabilization of the closed-loop system for singular bilinear systems [16, 17].

This study discusses nonbilinear type of descriptor systems and techniques to decompose them into their differential-algebraic equations. Furthermore, it investigates the concept of the logarithmic norm approach and its capability to find robust stabilising controllers for uncertain descriptor bilinear systems.

This bilinear descriptor control system is designed to be stabilised by finding a robust controller using an exponential stabilization approach via a logarithmic norm and the finite-time stability concept. The theory and algorithms are the focus of this study concentrating on increased system stabilizable. This focus is on testing the problem. Therefore, the model can be used to solve several complex test problems.

2. Problem Description

Consider the index-k nonbilinear descriptor system (non-BDs)

$$E\dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)u(t)x(t) + f(x(t)), \tag{1}$$

where $x \in R^n, E \in R^{n\times n}$, with index-$k$, and $A, B \in R^{n\times n}$, δA and δB are perturbation matrices with $\|\delta A\| \le a, \|\delta B\| \le b$, $a, b \in Z^+$, $u(t)$ is the single input control, and finally $f(x(t))$ is a vector of nonlinear functions which represents the uncertainty of the system.

3. Remark

The set of the following nominal descriptor system:

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{2}$$

is solved for desired control input $u(t)$ and all desired coordinate trajectories. In addition, there is a unique solution guaranteed for regular descriptor systems.

4. Simple Algorithm to Regulate the Irregular Nominal Descriptor System

Step 1: consider index-k nominal descriptor system $E\dot{x}(t) = Ax(t) + Bu(t)$

Step 2: find the finite spectrum eigenvalues $\wp(E, A)$ and choose $r \notin \wp(E, A)$

Step 3: set $\widetilde{E} = (rE - A)^{-1}E, \widetilde{A} = (rE - A)^{-1}A, \widetilde{B} = (rE - A)^{-1}B$

Step 4: the transfer nominal descriptor system $\widetilde{E}\dot{x}(t) = \widetilde{A}x(t) + \widetilde{B}u(t)$ is regular.

5. An Overview of Dini Derivative and Finite-Time Stability

In recent years, many significant concepts of nonsmooth analyses have been extended from the Euclidean space to the Riemannian manifold setting to explore more optimisation problems.

Lipchitz and convex functions are crucial for the nonsmooth analysis of linear spaces. The Dini derivative is useful for analysing these functions.

Owing to the importance of the Dini derivative in the application, we discuss the Dini derivative and its properties on Riemannian manifolds, where the upper and lower Dini derivatives can be defined as follows:

$$\begin{aligned} D^+ f(x) &= \lim_{h\longrightarrow 0^+} \sup \frac{f(x+h) - f(x)}{h}, \\ D_+ f(x) &= \lim_{h\longrightarrow 0^+} \inf \frac{f(x+h) - f(x)}{h}. \end{aligned} \tag{3}$$

Here, f is the function defined on $I = [a, b]$.

The logarithmic norm (LN) is real-valued functional on operators. This can be derived from either inner products or vector norms. This could also be an induced operator norm [18]

$$\begin{aligned} \mathscr{L}(A) &= \mathscr{L} \\ &= \lim_{h\longrightarrow 0^+} \sup \frac{\|I - h.A\| - 1}{h}. \end{aligned} \tag{4}$$

Here, $\|\cdot\|$ is an induced matrix norm associated with the square matrix A. For further details, one can read [18–21].

The logarithmic norm name is sourced from the estimation of the solution norm logarithmic to the differential equation $x' = Ax$, but the Log maximal growth of $\|x\|$ is $\mathscr{L}(A)$.

The differential inequality $D_t^+ \mathrm{Log}\|x\| \le \mathscr{L}(A)$ expresses thus in which D_t^+ is upper right Dini derivative.

Logarithmic differentiation is a differential inequality that is expressed by: $D_t^+\|x\| \le \mathscr{L}(A).\|x\|$, which is the original idea that introduced LN to drive topological (norm) conditions on A guaranteeing a solution to the linear dynamical systems.

Definition 1. A system (2) is finite-time stable (FTS) with respect to $\{\gamma, \eta, , \varnothing(t)\}, \gamma < \eta$, if and only if $\forall x_0 \in \mathfrak{D} = \{\text{the class of consistent initial condi tion}\}$ satisfying $\|u(t)\| < \varnothing(t)$ and $\|x_0\| < \gamma \Longrightarrow \|x(t)\| < \eta, \forall t \in J$ [22, 23].

6. An Algorithm to Find Decomposed Nonbilinear Differential-Algebraic Equations

Consider index-k non-BDs (1) with rank $n_0$as

$$E\dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)u(t)x(t) + f(x(t)). \tag{5}$$

As noted earlier, a centred issue of the descriptor system, which might be regular or irregular, is crucial yet falls outside the scope of this work. Therefore, the regularity of the system was guaranteed.

Step 1: Because the nominal system (2) is regular, there exist two nonsingular matrices: S, T such that $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = T^{-1}x$ with $x_1 \in R^{n_0}, x_2 \in R^{n-n_0}$, and then (1) can be written as

$$ET\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = (A+\delta A)T\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + (B+\delta B)Tu(t)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + f(x(t)). \tag{6}$$

Step 2: Multiply (3) by S to get

$$\mathrm{SET}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = S(A+\delta A)T\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + S(B+\delta B)Tu(t)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Sf(x(t)). \tag{7}$$

Step 3: Notice that

$$\begin{aligned} \mathrm{SET} &= \mathrm{diag}\left(I_{n_0}, 0\right), \\ \mathrm{SAT} &= \mathrm{diag}\left(A_1, I_{n-n_0}\right), \\ S\delta AT &= \begin{bmatrix} \delta A_1 & \delta A_2 \\ \delta A_3 & \delta A_4 \end{bmatrix}, \\ \mathrm{SBT} &= \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \\ S\delta BT &= \begin{bmatrix} \delta B_1 & \delta B_2 \\ \delta B_3 & \delta B_4 \end{bmatrix}, \\ Sf(x(t)) &= \left[\, f_1(x_1(t), x_2(t)) \quad f_2(x_1(t), x_2(t)) \,\right]^T. \end{aligned} \tag{8}$$

Step 4: Equation (7) becomes

$$\begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1+\delta A_1 & \delta A_2 \\ \delta A_3 & I_{n-n_0}+\delta A_4 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1+\delta B_1 & B_2+\delta B_2 \\ B_3+\delta B_3 & B_4+\delta B_4 \end{bmatrix}u(t)\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} f_1(x_1(t), x_2(t)) \\ f_2(x_1(t), x_2(t)) \end{bmatrix}. \tag{9}$$

Formula (9) can be further rewritten as

$$\dot{x}_1(t) = (A_1+\delta A_1)x_1(t) + \delta A_2 x_2(t) + (B_1+\delta B_1)u(t)x_1(t) + (B_2+\delta B_2)u(t)x_2(t) + f_1(x_1(t), x_2(t)), \tag{10}$$

$$0 = \delta A_3 x_1(t) + \left(I_{n-n_0}+\delta A_4\right)x_2(t) + (B_3+\delta B_3)u(t)x_1(t) + (B_4+\delta B_4)u(t)x_2(t) + f_2(x_1(t), x_2(t)). \tag{11}$$

7. Assumption

In this section, we assume that

(1) $S\delta AT = \begin{bmatrix} \delta A_1 & 0 \\ 0 & 0 \end{bmatrix}, SBT = \begin{bmatrix} B_1 & 0 \\ B_3 & 0 \end{bmatrix}$, and $S\delta BT = \begin{bmatrix} \delta B_1 & 0 \\ 0 & 0 \end{bmatrix}$ as a special case with $\|\delta A_1\| \le a_1$ and $\|\delta B_1\| \le b_1$, for some a_1, b_1 positive constants

(2) $Sf(x(t)) = \left[\, f_1(x_1(t), x_2(t)) \quad 0 \,\right]^T$, and there exist positive constants k such that $\|f_1(x_1(t), x_2(t))\| \le k\|x_1(t)\|$

(3) $\|u\| \le \varnothing(t)$.

Then, (10) and (11) can be formulated as

$$\dot{x}_1(t) = (A_1+\delta A_1)x_1(t) + (B_1+\delta B_1)u(t)x_1(t) + f_1(x_1(t), x_2(t)), \tag{12}$$

$$0 = x_2(t) + B_3 u(t) x_1(t). \tag{13}$$

The system defined on the space of consistent initial condition (C.I.C) is as follows:

$$\mathfrak{D} = \left\{(x_1(t), x_2(t)) \,\middle|\, x_2^0 = -B_3 u(0) x_1^0 \text{ and } \left\|x_1^0\right\| \le \gamma\right\}. \tag{14}$$

8. Finite-Time Stability Technique for Stabilizability of Nonbilinear Differential-Algebraic Equations

In the following, we recall equations (12) and (13) with C.I.C (14):

$$\begin{aligned} \dot{x}_1(t) &= (A_1+\delta A_1)x_1(t) + (B_1+\delta B_1)u(t)x_1(t) + f_1(x_1(t), x_2(t)), \\ x_2(t) &= -B_3 u(t) x_1(t), \end{aligned} \tag{15}$$

with $t \in J, J = \{t \mid 0 \le t \le \tau\}$.

By using the concept of Dini derivative for state $x_1(t)$, one can get

$$D_t^+\|x_1(t)\| \le \lim_{h\longrightarrow 0^+}\sup\frac{\|x_1(t+h)-x_1(t)\|}{h},$$
$$D_t^+\|x_1(t)\| \le \lim_{h\longrightarrow 0^+}\frac{\|x_1((t)+h.\dot{x}_1(t))-x_1(t)\|}{h} \tag{16}$$
$$\le \lim_{h\longrightarrow 0^+}\frac{\|I-h.(A_1+\delta A_1)\|-1}{h}\|x_1(t)\|+\|(B_1+\delta B_1)u(t)x_1(t)+f_1(x_1(t),x_2(t))\|.$$

By taking $\mathscr{L}(A_1+\delta A_1)=\mathscr{L}=\lim_{h\longrightarrow 0^+}(\|I-h.(A_1+\delta A_1)\|-1/h)$,

$$D_t^+\|x_1(t)\| \le \mathscr{L}.\|x_1(t)\| +\|(B_1+\delta B_1)u(t)x_1(t)+f_1(x_1(t),x_2(t))\|. \tag{17}$$

By multiplying both sides of (17) by $e^{-\mathscr{L}t}$ and integrating from 0 to t, one can obtain

$$\|x_1(t)\|e^{-\mathscr{L}t}-\|x_1(0)\| \le \int_0^t\|(B_1+\delta B_1)u(s)x_1(s)+f(x_1(s),x_2(s))\|e^{-\mathscr{L}s}\mathrm{d}s,$$
$$\|x_1(t)\| \le e^{\mathscr{L}t}\|x_1(0)\|+e^{\mathscr{L}t}\int_0^t\|(B_1+\delta B_1)u(s)x_1(s)+f_1(x_1(s),x_2(s))\|e^{-\mathscr{L}s}\mathrm{d}s. \tag{18}$$

Since the nominal system is regular and from (17) and by choosing δA_1 in a away such that $e^{\mathscr{L}(A_1+\delta A_1)t}\le(\eta/\gamma), \forall\gamma<\eta$ and η,γ are positive integers, then

$$\|x_1(t)\| \le \gamma\frac{\eta}{\gamma}+\int_0^t\frac{\eta^2}{\gamma^2}\left[\left(\|B_1\|+b_1\right)\varnothing(s)\|x_1(s)\|+k\|x_1(s)\|\right]\mathrm{d}s,$$
$$\|x_1(t)\| \le \eta+\int_0^t\frac{\eta^2}{\gamma^2}\left[\left(\|B_1\|+b_1\right)\varnothing(s)+k\right]\|x_1(s)\|\mathrm{d}s. \tag{19}$$

Using the generalised Bellman–Gronwall lemma [24, 25], (19) became

$$\|x_1(t)\| \le \eta.\exp\left(\int_0^t\frac{\eta^2}{\gamma^2}\left[\left(\|B_1\|+b_1\right)\varnothing(s)+k\right]\right)\mathrm{d}s, \tag{20}$$

whereas when applying equation (20) to (13), we obtain that

$$\|x_2(t)\| \le -\|B_3\|.\varnothing(t).\eta.\exp\left(\int_0^t\frac{\eta^2}{\gamma^2}\left[\left(\|B_1\|+b_1\right)\varnothing(s)+k\right]\right)\mathrm{d}s. \tag{21}$$

Now, $\|(x_1(t),x_2(t))\| \le [1-\|B_3\|.\varnothing(t)].\eta.\exp\left(\int_0^t(\eta^2/\gamma^2)[(\|B_1\|+b_1)\varnothing(s)+k]\right)\mathrm{d}s$.

The aim is to have the control u stabilizing the dynamics $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$.

Hence, dynamics (20) and (21) introduce stabilizable dynamics $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$ which is FTS with respect to $\{\gamma,\eta,\varnothing(t)\}$.

Because $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=T^{-1}x$, then x is finite-time stable with respect to $\{\gamma,\eta,\varnothing(t)\}$.

9. An Application Point of View

Figure 1 shows the mechanical system, which consists of one-mass oscillators linked by a dashpot part, described in its final form as follows:

$$E\dot{x}(t) = (A + \delta A)x(t) + (B + \delta B)u(t)x(t) + f(x(t)),$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0.1 & 0.1 & -0.3 & 0.4 & -0.1 \\ 0.2 & 0.1 & 0.2 & 0.1 & 0.2 \\ -0.2 & 0.3 & 0.2 & 0.1 & -0.2 \\ 0.1 & -0.1 & 0.2 & 0.1 & -0.1 \\ 0.1 & 0.1 & -0.1 & 0.2 & -0.1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$+ \left[\begin{bmatrix} -1 & -1 & 0 & 1 & 2 \\ -1 & 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & 2 & 1 \\ -2 & 1 & 2 & 4 & 0 \\ -1 & 3 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0.7 & 0.7 & 0.2 & -0.3 & 0.1 \\ 0.2 & 0 & 0.2 & 0.2 & 0.1 \\ -0.2 & 0.1 & -0.3 & 0.2 & 0.1 \\ 0.4 & -0.1 & 0 & 0.2 & 0 \\ 0.1 & 0.1 & 0.1 & -0.1 & 0 \end{bmatrix} \right] \sin(t) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ t^3 - 1 \\ 2t^2 + 1 \\ 0 \end{bmatrix}. \tag{22}$$

(1) Notice that $rank(E) = 4 = n_0, n - n_0 = 1$.

(2) Because the nominal system is irregular, suppose $\exists 1 \notin$ {set of all finite spectrum eigenvalues}, such that $\widetilde{E} = (E - A)^{-1}E =$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{A} = (E - A)^{-1}A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{\delta A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{\delta B} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{f} = \begin{bmatrix} 0 \\ 0 \\ t^3 - 1 \\ 2t^2 + 1 \\ 0 \end{bmatrix}. \tag{23}$$

(3) There are two nonsingular matrices S, T such that $\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{21} \end{bmatrix} = T^{-1}x$, and

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{14} \end{bmatrix} = \left(\begin{bmatrix} 2.5 & -0.01 & -0.02 & 0.3 \\ 0.01 & 0.1 & -0.7 & -0.1 \\ 0.1 & 0.7 & -0.08 & 0.08 \\ -0.2 & 0.1 & -0.002 & -0.5 \end{bmatrix} + \begin{bmatrix} 1.1 & -0.4 & -0.9 & 0.4 \\ 0.05 & 0.1 & 0.2 & 0.07 \\ 0.06 & 0.1 & 0.1 & -0.04 \\ 0.05 & 0.04 & -0.02 & 0.06 \end{bmatrix} \right) \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix}$$
$$+ \left(\begin{bmatrix} 3.1 & 0 & -2.9 & 0 \\ 0.5 & 0.2 & 0.3 & 0.3 \\ -0.7 & -1.1 & 0.3 & 0.3 \\ 0.2 & 0.3 & -0.7 & -0.7 \end{bmatrix} \begin{bmatrix} 0 & -0.2 & 0.9 & -0.07 \\ -0.4 & -0.02 & -0.2 & 0.2 \\ 0.04 & 0.2 & -0.03 & -0.1 \\ 0.009 & -0.03 & -0.7 & -0.01 \end{bmatrix} \right) \cos t \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix}$$
$$+ \begin{bmatrix} 0.7 \cos x_{13}(t) + 0.3 \sin x_{11}(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

$$0 = x_{21} + \begin{bmatrix} -1.3 & 1.8 & -0.6 & -0.6 \end{bmatrix} \cos t \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix}. \tag{25}$$

Multiply (24) by a^{-1} where $a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{14} \end{bmatrix} = \left(\begin{bmatrix} 2.5 & -0.01 & -0.02 & 0.3 \\ 0.05 & 0.5 & -0.3 & -0.5 \\ 0.1 & 0.7 & -0.08 & 0.08 \\ -0.2 & 0.1 & -0.002 & -0.5 \end{bmatrix} + \begin{bmatrix} 1.1 & -0.4 & -0.9 & 0.4 \\ 0.25 & 0.5 & 1 & 0.035 \\ 0.06 & 0.1 & 0.1 & -0.04 \\ 0.05 & 0.04 & -0.02 & 0.06 \end{bmatrix} \right) \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix}$$
$$+ \left(\begin{bmatrix} 3.1 & 0 & -2.9 & 0 \\ 2.1 & 1 & 1.5 & 1.5 \\ -0.7 & -1.1 & 0.3 & 0.3 \\ 0.2 & 0.3 & -0.7 & -0.7 \end{bmatrix} \begin{bmatrix} 0 & -0.2 & 0.9 & -0.07 \\ -0.2 & -0.1 & -0.1 & 0.1 \\ 0.04 & 0.2 & -0.03 & -0.1 \\ 0.009 & -0.03 & -0.7 & -0.01 \end{bmatrix} \right) \cos t \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix} \tag{26}$$
$$+ \begin{bmatrix} 0.7 \cos x_{13}(t) + 0.3 \sin x_{11}(t) \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$0 = x_{21} + \begin{bmatrix} -1.3 & 1.8 & -0.6 & -0.6 \end{bmatrix} \cos t \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \end{bmatrix}. \tag{27}$$

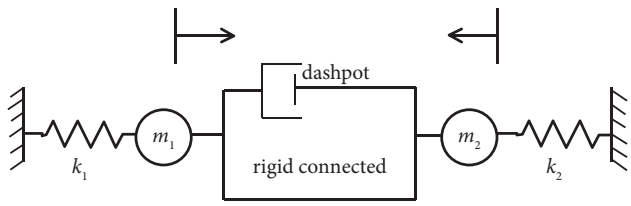

Figure 1: Representations of two connected one-mass oscillators.

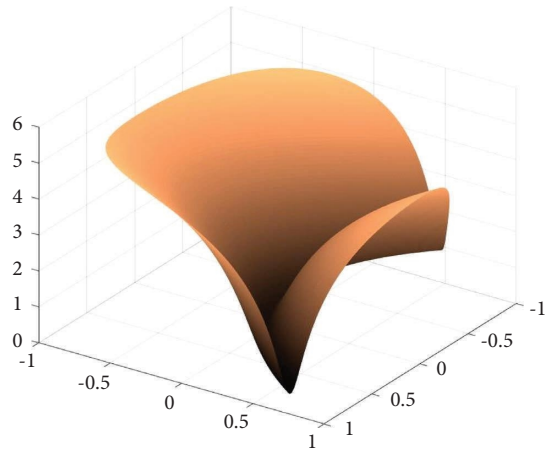

Figure 2: Stability space.

From (27), one can find the space of consistent initial conditions.

$$\mathfrak{D} = \left\{ (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}) \,\middle|\, x_{21} = -\begin{bmatrix} -1.3 & 1.8 & -0.6 & -0.6 \end{bmatrix} \cos t \begin{bmatrix} x_{11}^0 \\ x_{12}^0 \\ x_{13}^0 \\ x_{14}^0 \end{bmatrix} \right\}. \tag{28}$$

By setting $(x_{11}, x_{12}, x_{13}, x_{14}) = (0.5, 0, 0, 0)$,

$$\mathfrak{D} = \{(x_{11}, x_{12}, x_{13}, x_{14}, x_{21}) = (0.5, 0, 0, 0, 0.65)\}. \tag{29}$$

Now, getting the LN of $(A + \delta A)$, $\mathscr{L}(A_1 + \delta A_1) = \mathscr{L} = \lim_{h \longrightarrow 0^+} (\|I - h.(A_1 + \delta A_1)\| - 1/h) = \lim_{h \longrightarrow 0^+}$

$$\left(\left\| I - h. \begin{bmatrix} 3.6 & -0.41 & -0.092 & 0.7 \\ 0.075 & 1 & 0.7 & -0.15 \\ 0.16 & 0.8 & 0.02 & 0.04 \\ -0.15 & 0.14 & -0.022 & -0.44 \end{bmatrix} \right\| - 1/h \right),$$

then set $\eta = 4, \gamma = 3$ and $\|B_1\| = 7, \|\delta B_1\| < 2$

$$\|x_1(t)\| \leq 4.\exp\left(\int_0^t \frac{16}{9} [(7+2).1 + 1] \right) ds = 4e^{17t}. \tag{30}$$

From (27),

$$\|x_2(t)\| \leq 17.2e^{17t},$$

$$\|(x_1(t), x_2(t))\| \leq 21.2e^{17t} < \alpha \Longrightarrow t < \ln\left(\frac{(\alpha/17.2)}{17} \right). \tag{31}$$

The stability region and the state of the oscillator dynamic model are shown in Figures 2 and 3.

10. Result and Discussion

This work shows the manipulation of the features of the solution manifold of nonbilinear. This bilinear descriptor control system is designed to be stabilised by finding a robust controller using an exponential stabilization

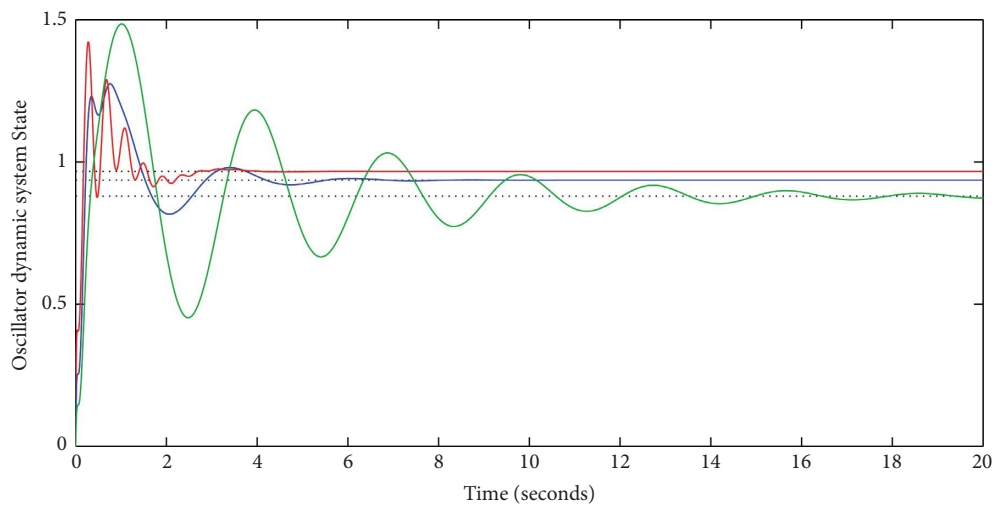

Figure 3: State response of the oscillator dynamic system that can be modelled with differential-algebraic equations.

approach via a logarithmic norm and the finite-time stability concept.

In contrast, the proposed approach helps in finding the decomposition of nonbilinear differential-algebraic equations. On the other hand, such a modified functional (logarithmic norm) can be defined by a suitable selection of the Dini derivative as a bound in finite-time stability.

An example of two connected one-mass oscillators, the analysis of the nonlinear mechanical system motives, has been utilized for the validation of the technique. The stability region and the state of the oscillator dynamic model are shown in Figures 2 and 3.

11. Conclusions

This study first reviewed the classical theory, concerned with matrix bounds in which we define the logarithmic norm. Then, this work developed a general theory to establish the logarithmic norms in a nonbilinear functional analysis framework. To answer the question of "how to apply the notation of logarithmic norms to nonlinear DAEs," we modified the definition of the logarithmic norm and introduced it as a more general concept through the extension of the functional to nonbilinear maps and unbounded operators.

In this study, we have shown that the new notation of logarithmic norm can be used to cover nonbilinear DAEs; this analysis is possible by mixing this concept with finite-time stability. The theory also reveals details about how perturbations in the algebraic equations affect the stability of the entire system in the nonbilinear case and provides quantitate bounds of perturbation nonbilinear DAEs.

Data Availability

The data used in the article can be made available upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

I express my deep gratitude to Mustansiriyah University (https://www.uomustansiriyah.edu.iq), Baghdad, Iraq, for its moral support in the present work.

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