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Research Article

Stability Results for Nonlinear Implicit ϑ-Caputo Fractional Differential Equations with Fractional Integral Boundary Conditions

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This article examines the necessary conditions for the unique existence of solutions to nonlinear implicit ϑ-Caputo fractional differential equations accompanied by fractional order integral boundary conditions. The analysis draws upon Banach's contraction principle and Krasnoselskii's fixed point theorem. Furthermore, the circumstances leading to the attainment of Ulam–Hyers–Rassias forms of stability are established. An illustrative example is provided to demonstrate the derived findings.

1. Introduction

Fractional calculus, which belongs to the realm of mathematical analysis, has emerged as a potent instrument for representing intricate systems and phenomena characterized by memory and nonlocal behavior. Over the past few decades, it has attracted considerable attention from researchers and scientists because of its capacity to capture complex dynamic behaviors that elude traditional integer-order calculus. This mathematical framework has found extensive utility across a range of fields, including physics, engineering, biology, and economics, thereby enriching our comprehension of the underlying dynamics governing complex systems (refer to [1–7] and related works).

The foundation of fractional calculus hinges on the expansion of conventional derivatives and integrals into noninteger orders. Notably, fractional derivatives and integrals have proven indispensable in representing real-world phenomena marked by fractal geometry, anomalous diffusion, and long-range interactions. This transition from integer-order calculus to fractional calculus has laid the groundwork for groundbreaking contributions to the fields of science and engineering. You can explore this further in works such as [5, 8–13] and related references.

A notable advancement within the realm of fractional calculus is the ϑ-Caputo fractional derivative. Diverging from the established Riemann–Liouville and Caputo methodologies, the ϑ-Caputo derivative introduces a unique kernel incorporating the parameter ϑ. This distinctive characteristic sets it apart from classical derivative operators, offering a more adaptable instrument for characterizing intricate systems. Researchers have harnessed the potential of the ϑ-Caputo fractional derivative across various domains, including electromagnetics, fluid mechanics, signal processing, and beyond. Explore this development further in works such as [14–25] and related literature.

Furthermore, analyzing stability in fractional order differential equations has assumed paramount importance. A comprehensive comprehension of the stability characteristics of such equations proves vital in forecasting the long-term behavior of dynamic systems governed by fractional calculus. Researchers have extended classical stability concepts to the fractional domain, introducing concepts such as the Ulam–Hyers–Rassias stability types and their extensions. These advances have opened up new avenues for investigating the stability and resilience of fractional order systems under diverse conditions. Delve deeper into this topic through works such as [17, 26–34] and associated references.

In the subsequent discussions, we present a summary of recent research contributions in the field of fractional differential equations (FDEs) and their stability. We start with an overview of notable works.

In [35], Zada et al. studied the existence, uniqueness, and Hyers–Ulam stability results of the following implicit FDE with impulsive condition:

$$\begin{cases} {}^{c}\mathscr{D}^{r}x(t) = f\left(t, x(t), {}^{c}\mathscr{D}^{r}x(t), \int_{0}^{t}\dfrac{(t-\zeta)^{\sigma-1}}{\Gamma(\delta)}g(\zeta, x(\zeta)), {}^{c}\mathscr{D}^{r}x(\zeta)\right)d\zeta, \quad t \in I, t \neq t_i, 1 \leq i \leq m, \\ x(0) = h(x), \\ \Delta x(t_i) = I_i(x(t_i)), \quad i = 1, 2, \ldots, m, \end{cases} \tag{1}$$

where ${}^{c}\mathscr{D}^{r}$ is the Caputo fractional derivative of order $r \in (0,1)$, $I = [0,T]$, with $T > 0$, $\sigma > 0$. The functions $I_k\colon R \longrightarrow R$, $f, g\colon I \times R^2 \longrightarrow R$, and $h\colon X \longrightarrow R$ are continuous functions and $\Delta x(t_i) = x(t_i^{+}) - x(t_i^{-})$.

In [36], the authors considered a class of ψ-Hilfer nonlinear implicit fractional boundary value problems (FBVPs) describing the thermostat control model of the following form:

$$\begin{cases} {}^{H}\mathscr{D}_{0^+}^{\alpha,\rho;\psi}x(\varsigma) = f\left(\varsigma, x(\theta,\varsigma), \mathscr{I}_{0^+}^{q,\psi}x(\varepsilon,\varsigma)\right), \varsigma \in (0;T], \\ \sum_{i=1}^{m}\omega_i {}^{H}\mathscr{D}_{0^+}^{\beta_i,\rho;\psi}x(\xi_i) = A, \sum_{j=1}^{n}\lambda_j {}^{H}\mathscr{D}_{0^+}^{\mu_j,\rho;\psi}x(\sigma_j) + \sum_{k=1}^{r}\delta_k x(\eta_k) = B, \end{cases} \tag{2}$$

where ${}^{H}\mathscr{D}_{0^+}^{\nu,\rho;\psi}$ denotes the ψ-Hilfer fractional derivative operator of order $\nu = \{\alpha, \beta_i, \mu_j\}$, $\alpha \in (1,2]$, $\beta_i, \mu_j \in (0,1]$, A, B, ω_i, λ_j, $\delta_k \in R$, ξ_i, σ_j, $\eta_k \in (0,T)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, r$, $\rho \in [0,1]$, $\mathscr{I}_{0^+}^{q,\psi}$ is the ψ-Riemann–Liouville fractional integral of order $q > 0$, θ, $\varepsilon \in (0,1]$, $f \in C(J \times R^2, R)$, and $J = [0,T]$ with $T > 0$.

In [17], the authors explored the existence, uniqueness, and Ulam–Hyers type stability for the following nonlinear implicit ψ-Caputo fractional order integro-differential boundary value problem CIFDP:

$$\begin{cases} {}^{c}\mathscr{D}^{\alpha,\psi}y(t) = f\left(t, y(t), {}^{c}\mathscr{D}^{\beta,\psi}y(t), \int_{0}^{t}k(t,s){}^{c}\mathscr{D}^{\alpha,\psi}y(s)ds\right), t \in I, \\ y(0) = y_0, y(T) = y_T, \end{cases} \tag{3}$$

where ${}^{c}\mathscr{D}^{\alpha,\psi}$ is the ψ-Caputo fractional derivative of order $\alpha \in (0,1]$, $I = [0,T]$, $f\colon I \times R^3 \longrightarrow R$, $k\colon I \times R \longrightarrow R$, and y_0 and y_T are constant real numbers.

In [26], Al-Issa et al. developed existence and stability theorems for the implicit fractional order differential problem (ISDP):

$$\begin{aligned}
&\frac{d^2}{dt^2}y(\zeta) = f\left(\zeta, y(\zeta), {}^c\mathscr{D}^{\beta,\phi}y(\zeta), \int_0^\zeta k(\zeta,\sigma){}^c\mathscr{D}^{\alpha,\phi}y(\sigma)d\sigma\right),\\
&y(0) = \frac{1}{\Gamma(\gamma)}\int_0^1 \phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}h_1(\sigma, y(\sigma))d\sigma,\\
&y'(1) = \frac{1}{\Gamma(\gamma)}\int_0^1 \phi'(\sigma)(\phi(1)-\phi(v))^{\gamma-1}h_2(\sigma, y(\sigma))d\sigma,
\end{aligned} \tag{4}$$

where ${}^c\mathscr{D}^\delta$ is the Caputo fractional derivative of order $\delta \in \{\beta, \alpha\}$, with $1 < \beta < \alpha \le 2$, and $0 < \gamma < 1$. In addition, $f: I \times R^3 \longrightarrow R$ and $k: I \times R \longrightarrow R$ are continuous functions with $I = [0,1]$, and $\phi(\tau)$ is a nondecreasing function with $\phi'(\tau) \neq 0$ for all $\tau \in I$.

In [28], the authors investigated the existence and Ulam–Hyers stability of solutions for second-order differential equations with integral boundary conditions:

$$\frac{d^2}{dt^2}y(\zeta) = f\left(\zeta, y(\zeta), {}^c\mathscr{D}^\beta y(\zeta), \int_0^\zeta \theta(\zeta,s)k(\zeta,s){}^c\mathscr{D}^\alpha y(s)ds\right), \zeta \in (0,1), \tag{5}$$

with the following nonlocal boundary conditions:

$$\begin{aligned}
&y(0) = 0,\\
&y(1) = \frac{\lambda}{\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}L_1(s, y(s))ds + \frac{\mu}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}L_2(s, y(s))ds,\\
&y(0) = \frac{\lambda}{\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}L_1(s, y(s))ds,\\
&y(1) = \frac{\mu}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}L_2(s, y(s))ds,
\end{aligned} \tag{6}$$

where ${}^c\mathscr{D}^\delta$ is a Caputo fractional derivative of order $\delta \in \{\beta, \alpha\}$, with $1 < \beta < \alpha \le 2$, $0 < \gamma < 1$, $f: I \times R^3 \longrightarrow R$ and $\theta: I \times I \longrightarrow R$ are given functions, $L_i: I \times R \longrightarrow R$, $(i = 1, 2)$ is a continuous function, $\lambda, \mu \in R$, and $\eta, \tau \in [0,1]$.

In light of the recent advancements in the field, this research article delves into the stability analysis of nonlinear implicit ϑ-Caputo fractional differential equations with fractional integral boundary conditions. We aim to contribute meaningfully to the expanding body of knowledge concerning the ϑ-Caputo derivative and its practical applications. Simultaneously, we aim to enrich our comprehension of the stability characteristics inherent in fractional-order differential equations. Our work builds upon the foundational work laid out by previous researchers and extends the utility of ϑ-Caputo derivatives to intricate boundary value problems, offering insights into the behavior of complex systems. Therefore, motivated by the preceding discussions, our study delves into the investigation of the existence and uniqueness of solutions for a nonlinear implicit ϑ-Caputo fractional order differential problem (ICFDP), characterized by the following equations:

$${}^c\mathscr{D}^{\alpha,\vartheta}y(\zeta) = \mathfrak{F}\left(\zeta, y(\zeta), {}^c\mathscr{D}^{\beta,\vartheta}y(\zeta), \int_0^\zeta k(\zeta,v){}^c\mathscr{D}^{\alpha,\vartheta}y(v)dv\right), \tag{7}$$

$$y(0) - y'(0) = \frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)(\vartheta(T)-\vartheta(v))^{\gamma-1}h_1(v, y(v))dv, \tag{8}$$

$$y(T) + y'(T) = \frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)(\vartheta(T)-\vartheta(v))^{\gamma-1}h_2(v, y(v))dv, \tag{9}$$

where $\vartheta(\zeta)$ is an increasing function with $\vartheta'(\zeta) \neq 0$ for all ζ in the interval $I = [0, T]$. The parameters α, β, and γ satisfy the conditions $1 < \beta < \alpha \leq 2$ and $0 < \gamma < 1$. The operator ${}^{c}\mathscr{D}^{\alpha,\vartheta}$ represents the ϑ-Caputo fractional derivative. Our primary findings, which are derived under specific assumptions, are established using the Banach and Krasnoselskii's fixed point theorems. Furthermore, our investigation encompasses the ϑ-Caputo fractional derivative, denoted as ${}^{c}\mathscr{D}^{\alpha,\vartheta}$. Additionally, we address the topics of Ulam–Hyers stability and the generalized Ulam–Hyers stability.

The article is structured as follows: We initiate our work with an introduction in Section 1. Following that, Section 2 covers notations, definitions, lemmas, and theorems that establish the fundamental basis for our study. In Section 3, we establish the existence and uniqueness of mild solutions for (ICFDPs) (7)–(9) by using the fixed point theorems of Banach and Krasnoselskii. Section 4 delves into Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability. Additionally, Section 5 provides an illustrative example to showcase the practical application of our main findings. Finally, we conclude the paper in Section 6.

2. Preliminaries

In the subsequent sections, we present various notations, definitions, lemmas, and theorems that hold significance in the progression of our findings within this article.

Definition 1 (see [14]). For any positive real number α, the left-sided ϑ-Riemann–Liouville fractional integral of order α for an integrable function $u\colon I \longrightarrow \mathbb{R}$ with respect to another function $\vartheta\colon I \longrightarrow \mathbb{R}$, which is a differentiable increasing function such that $\vartheta'(\zeta) \neq 0$ for all $\zeta \in I = [0, T]$, is defined as follows:

$$\mathscr{I}^{\alpha,\vartheta} u(\zeta) = \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} \vartheta'(v)(\vartheta(\zeta) - \vartheta(v))^{\alpha-1} u(v) dv, \quad (10)$$

where Γ represents the classical Euler gamma function.

Definition 2 (see [14]). Let n be a natural number, and let $\vartheta, u \in C^n(I, \mathbb{R})$ be two functions such that ϑ is an increasing function with $\vartheta'(\zeta) \neq 0$ for all $\zeta \in I$. In such cases, the left-sided ϑ-Caputo fractional derivative of a function u of order α is defined as follows:

$${}^{c}\mathscr{D}^{\alpha,\vartheta} u(\zeta) = \mathscr{I}^{n-\alpha,\vartheta}\left(\frac{1}{\vartheta'(\zeta)}\frac{d}{d\zeta}\right)^{n} u(\zeta) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\zeta} \vartheta'(v)(\vartheta(\zeta) - \vartheta(v))^{n-\alpha-1} u_{\vartheta}^{[n]}(v) dv, \quad (11)$$

where $u_{\vartheta}^{[n]}(\zeta) = ((1/\vartheta'(\zeta))(d/d\zeta))^n u(\zeta)$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

Remark 3. Let $\alpha > 0$, then the differential equation $({}^{c}\mathscr{D}_{a+}^{\alpha,\vartheta} h)(\zeta) = 0$ has the following solution:

$$h(\zeta) = c_0 + c_1(\vartheta(\zeta) - \vartheta(0)) + c_2(\vartheta(\zeta) - \vartheta(0))^2 + \cdots + c_{n-1}(\vartheta(\zeta) - \vartheta(0))^{n-1}, \quad (12)$$

where $c_i \in R, i = 0, 1, 2, \ldots, n-1, n = [\alpha] + 1$.

Lemma 4 (see [15]). *Let $\alpha, \beta \in R^+$ and $g(\zeta) \in L_1(I)$. Then, $\mathscr{I}_{a+}^{\alpha,\vartheta}\mathscr{I}_{a+}^{\beta,\vartheta} g(\zeta) = \mathscr{I}_{a+}^{\beta,\vartheta}\mathscr{I}_{a+}^{\alpha,\vartheta} g(\zeta) = \mathscr{I}_{a+}^{\alpha+\beta,\vartheta} g(\zeta)$ and $(\mathscr{I}_{a+}^{\alpha,\vartheta})^n g(\zeta) = \mathscr{I}_{a+}^{n\alpha,\vartheta} g(\zeta)$, where $n \in N$.*

Definition 5. A fixed point of a mapping $g\colon X \longrightarrow X$, where X is a given space, is a point $x \in X$ satisfying $x = g(x)$.

Theorem 6 (Banach fixed point theorem). *In a Banach space X, if C is a nonempty closed subset and T is a contraction mapping of C into itself, then T possesses a unique fixed point.*

Theorem 7 (Krasnoselskii's fixed point theorem). *For a Banach space X and a bounded closed convex subset S of X, given mappings Q_1 and Q_2 from S to X with the property $Q_1 x + Q_2 y \in S$ for all $x, y \in S$, if Q_1 is a contraction and Q_2 is completely continuous, then there exists a point $z \in S$ such that $Q_1 z + Q_2 z = z$.*

3. Existence and Uniqueness of the Solutions

In this section, we demonstrate the presence of mild solutions for the nonlinear implicit ϑ-Caputo fractional order differential problems (ICFDPs) (7)–(9), subjected to the following assumptions:

$(\mathscr{H}_1)$: The functions $h_i(t, x)$, where $i = 1, 2$, are continuous and possess Lipschitz continuity with constants $k_i \in [0, 1]$, obeying the following criterion:

$$\left|h_i(\zeta, u) - h_i(\zeta, v)\right| \leq k_i |u - v|. \quad (13)$$

$(\mathscr{H}_2)$: The function $\mathfrak{F}(t, u_1, u_2, u_3)$ is continuous, and there exists a positive function $\mu \in C(I, R^+)$ that satisfies the ensuing inequality:

$$\left|\mathfrak{F}(\zeta, u_1, u_2, u_3) - \mathfrak{F}(\zeta, v_1, v_2, v_3)\right| \le \mu(\zeta)\left(\left|u_1 - v_1\right| + \left|u_2 - v_2\right| + \left|u_3 - v_3\right|\right), \tag{14}$$

for all $\zeta \in I$ and $u_i, v_i \in R$ $(i = 1, 2, 3)$.

$(\mathscr{H}_3)$: The function $k(\zeta, v)$ is continuous over the domain $I \times I$, and there exists a positive constant K such that

$$\max_{\zeta, v \in [0,T]} |k(\zeta, v)| = K. \tag{15}$$

$(\mathscr{H}_4)$: The function $\Phi(t)$ is increasing and belongs to the class $C(I, R^+)$, with a positive constant $\lambda_\Phi > 0$ such that, for each $\zeta \in I$,

$$\mathscr{I}^\alpha \Phi(\zeta) \le \lambda_\Phi \Phi(\zeta). \tag{16}$$

$(\mathscr{H}_5)$: Positive functions $\phi(t)$ and $\chi_i(t)$ $(i = 1, 2, 3)$ exist in the class $C(I, R^+)$ such that

$$\left|\mathfrak{F}(\zeta, u_1, u_2, u_3)\right| \le \phi(\zeta) + \chi_1(\zeta)\left|u_1\right| + \chi_2(\zeta)\left|u_2\right| + \chi_3(\zeta)\left|u_3\right|, \quad \forall (\zeta, u_1, u_2, u_3) \in I \times R^3. \tag{17}$$

By fulfilling these assumptions, we establish the existence of a mild solution to (ICFDPs) (7)–(9).

Remark 8. Given assumption $(\mathscr{H}_1)$, it follows that

$$\left|h_i(\zeta, u)\right| - \left|h_i(\zeta, 0)\right| \le \left|h_i(\zeta, u) - h_i(\zeta, 0)\right| \le k_i|u - 0|, \tag{18}$$

which implies

$$\left|h_i(\zeta, u)\right| \le \left|h_i(\zeta, 0)\right| + k_1|u|, \tag{19}$$

and consequently,

$$\left|h_i(\zeta, u)\right| \le H_i + k_1|u|, \text{with } H_i = \sup_{\zeta \in I}\left|h_i(\zeta, 0)\right|. \tag{20}$$

Under assumption $(\mathscr{H}_2)$, we find that

$$\left|\mathfrak{F}(\zeta, u_1, u_2, u_3)\right| - |\mathfrak{F}(\zeta, 0, 0, 0)| \le \left|\mathfrak{F}(\zeta, u_1, u_2, u_3) - \mathfrak{F}(\zeta, 0, 0, 0)\right| \le \mu(\zeta)\left(\left|u_1\right| + \left|u_2\right| + \left|u_3\right|\right), \tag{21}$$

leading to the conclusion that

$$\left|\mathfrak{F}(\zeta, u_1, u_2, u_3)\right| \le \mu(\zeta)\left(\left|u_1\right| + \left|u_2\right| + \left|u_3\right|\right) + |\mathfrak{F}(\zeta, 0, 0, 0)|, \tag{22}$$

and furthermore,

$$\left|\mathfrak{F}(\zeta, u_1, u_2, u_3)\right| \le \mu(\zeta)\left(\left|u_1\right| + \left|u_2\right| + \left|u_3\right|\right) + F, \text{where } F = \sup_{\zeta \in I} |\mathfrak{F}(\zeta, 0, 0, 0)|. \tag{23}$$

Lemma 9. *The mild solution of (ICFDPs) (7)–(9) is the solution of the following Volterra integral equation:*

$$y(\zeta) = h(\zeta, y(\zeta)) + \int_0^T \vartheta'(v) G(\zeta, v) u(v) dv, \tag{24}$$

where u is the solution of the following functional integral equation:

$$u(\zeta) = \mathfrak{F}\left(\zeta, h(\zeta) + \frac{1}{\Gamma(\alpha)}\int_0^T \vartheta'(v) G(\zeta, v) u(v) dv, \mathscr{I}^{\alpha-\beta,\vartheta} u(\zeta), \int_0^\zeta k(\zeta, v) u(v) dv\right), \tag{25}$$

where $G(\zeta, v)$ is the Green's function defined by

$$G(\zeta, v)=\begin{cases}\dfrac{\vartheta'(\zeta)-\varphi(\zeta)}{\varphi(T)}\left(\dfrac{\vartheta'(\zeta)}{\Gamma(\alpha-1)}\right)(\vartheta(T)-\vartheta(v))^{\alpha-2}+\dfrac{\vartheta'(\zeta)-\varphi(\zeta)}{\varphi(T)\Gamma(\alpha)}(\vartheta(T)-\vartheta(v))^{\alpha-1}+\dfrac{1}{\Gamma(\alpha)}(\vartheta(\zeta)-\vartheta(v))^{\alpha-1}, & \text{if } 0\le v\le\zeta\le T,\\[2ex] \dfrac{\vartheta'(\zeta)-\varphi(\zeta)}{\varphi(T)}\left(\dfrac{\vartheta'(\zeta)}{\Gamma(\alpha-1)}\right)(\vartheta(T)-\vartheta(v))^{\alpha-2}+\dfrac{\vartheta'(\zeta)-\varphi(\zeta)}{\varphi(T)\Gamma(\alpha)}(\vartheta(T)-\vartheta(v))^{\alpha-1}, & \text{if } 0\le\zeta\le v\le \mathrm{T},\end{cases} \tag{26}$$

with

$$\begin{aligned}G_0&=\max\{|G(\zeta, v)|,(\zeta, v)\in I\times I\},\\ \varphi(\zeta)&=\vartheta'(\zeta)+\vartheta(\zeta)+\vartheta'(0)-\vartheta(0),\end{aligned} \tag{27}$$

and

$$\begin{aligned}h(\zeta, y(\zeta))={}&\left(1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right)\frac{1}{\Gamma(\gamma)}\int_0^T\vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}h_1(v, y(v))dv\\ &+\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\frac{1}{\Gamma(\gamma)}\int_0^T\vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}h_2(v, y(v))dv.\end{aligned} \tag{28}$$

Proof. Let ${}^c\mathscr{D}^{\alpha,\vartheta}y(\zeta)=u(\zeta)$ in equation (7), where

$$\begin{aligned}u(\zeta)&=\mathfrak{F}\left(\zeta, y(\zeta),\mathscr{I}^{\alpha-\beta,\vartheta}u(\zeta),\int_0^\zeta k(\zeta, v)u(v)dv\right),\\ y(\zeta)&=a_0+a_1(\vartheta(\zeta)-\vartheta(0))+\frac{1}{\Gamma(\alpha)}\int_0^\zeta\vartheta'(v)(\vartheta(\zeta)-\vartheta(v))^{\alpha-1}u(v)dv.\end{aligned} \tag{29}$$

From equations (8) and (9), we can obtain

$$a_0-a_1\vartheta'(0)=\frac{1}{\Gamma(\gamma)}\int_0^T\vartheta'(v)(\vartheta(T)-\vartheta(v))^{\gamma-1}h_1(v, y(v))dv \tag{30}$$

$$\begin{aligned}&a_0+a_1\Big(\vartheta'(T)+\vartheta(T)-\vartheta(0)\Big)+\frac{1}{\Gamma(\alpha)}\int_0^T\vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\alpha-1}u(v)dv\\ &\quad+\frac{\vartheta'(T)}{\Gamma(\alpha-1)}\int_0^T\vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\alpha-2}u(v)dv\\ &\quad=\frac{1}{\Gamma(\gamma)}\int_0^T\vartheta'(v)(\vartheta(T)-\vartheta(v))^{\gamma-1}h_2(v, y(v))dv.\end{aligned} \tag{31}$$

Solving equations (30) and (31), and if $\varphi(\zeta) = \vartheta'(\zeta) + \vartheta(\zeta) + \vartheta'(0) - \vartheta(0)$, then it is obtained that

$$
\begin{aligned}
a_0 &= \left(1 - \frac{\vartheta'(0)}{\varphi(T)}\right)\frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1} h_1(v, y(v))dv \\
&\quad + \frac{\vartheta'(0)}{\varphi(T)\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1} h_2(v, y(v))dv \\
&\quad - \frac{\vartheta'(0)}{\varphi(T)\Gamma(\alpha)}\int_0^T \vartheta'(v)(\vartheta(T) - \vartheta(v))^{\alpha-1} u(v)dv \\
&\quad - \frac{\vartheta'(T)\vartheta'(0)}{\varphi(T)\Gamma(\alpha-1)}\int_0^T \vartheta'(v)(\vartheta(T) - \vartheta(v))^{\alpha-2} u(v)dv, \\
a_1 &= \frac{1}{\varphi(T)\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1}(h_2(v, y(v)) - h_1(v, y(v)))dv \\
&\quad - \frac{1}{\varphi(T)\Gamma(\alpha)}\int_0^T \vartheta'(v)(\vartheta(T) - \vartheta(v))^{\alpha-1} u(v)dv \\
&\quad - \frac{\vartheta'(T)}{\varphi(T)\Gamma(\alpha-1)}\int_0^T \vartheta'(v)(\vartheta(T) - \vartheta(v))^{\alpha-2} u(v)dv.
\end{aligned}
\tag{32}
$$

Then, the solution of (ICFDPs) (7)–(9) is given by

$$
\begin{aligned}
y(\zeta) &= \left(\frac{\varphi(T) - \vartheta'(0) - \vartheta(\zeta) + \vartheta(0)}{\varphi(T)}\right)\frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1} h_1(v, y(v))dv \\
&\quad + \left(\frac{\vartheta(\zeta) + \vartheta'(0) - \vartheta(0)}{\varphi(T)}\right)\frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1} h_2(v, y(v))dv \\
&\quad - \left(\frac{\vartheta(\zeta) + \vartheta'(0) - \vartheta(0)}{\varphi(T)\Gamma(\alpha)}\right)\int_0^T \vartheta'(v)(\vartheta(T) - \vartheta(v))^{\alpha-1} u(v)dv \\
&\quad - \vartheta'(T)\left(\frac{\vartheta(\zeta) + \vartheta'(0) - \vartheta(0)}{\varphi(T)\Gamma(\alpha-1)}\right)\int_0^T \vartheta'(v)(\vartheta(T) - \vartheta(v))^{\alpha-2} u(v)dv \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^\zeta \vartheta'(v)(\vartheta(\zeta) - \vartheta(v))^{\alpha-1} u(v)dv \\
&= \left(\frac{\varphi(T) - \varphi(\zeta) + \vartheta'(\zeta)}{\varphi(T)}\right)\frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1} h_1(v, y(v))dv \\
&\quad + \left(\frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right)\frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1} h_2(v, y(v))dv
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)\Gamma(\alpha)}\int_0^T \vartheta'(v)(\vartheta(T)-\vartheta(v))^{\alpha-1}u(v)dv \\
&-\vartheta'(T)\left(\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)\Gamma(\alpha-1)}\right)\int_0^T \vartheta'(v)(\vartheta(T)-\vartheta(v))^{\alpha-2}u(v)dv \\
&+\frac{1}{\Gamma(\alpha)}\int_0^\zeta \vartheta'(v)(\vartheta(\zeta)-\vartheta(v))^{\alpha-1}u(v)dv.
\end{aligned}
\tag{33}
$$

Using the fact that $\int_0^T = \int_0^\zeta + \int_\zeta^T$, we get equation (24), and the proof is complete. □

Definition 10. A mild solution of the nonlinear implicit ϑ-Caputo fractional order differential problems (ICFDPs) (7)–(9) refers to a function $u \in C(I,\mathbb{R})$ that fulfills the integral equation (24). In this context, u represents the solution to the following functional integral equation:

$$
u(\zeta) = \mathfrak{F}\left(\zeta, h(\zeta) + \frac{1}{\Gamma(\alpha)}\int_0^T \vartheta'(v)G(\zeta,v)u(v)dv, \mathcal{I}^{\alpha-\beta,\vartheta}u(\zeta), \int_0^\zeta k(\zeta,v)u(v)dv\right),
\tag{34}
$$

for all $\zeta \in I$.

Lemma 11. *The function h: $I\times R \longrightarrow R$ satisfies the following Lipschitz condition:*

$$
|h(\zeta,x) - h(\zeta,y)| \leq c\|x-y\|.
\tag{35}
$$

Proof. For arbitrary $u, v \in X$ and for each $\zeta \in I$, we have

$$
\begin{aligned}
&|h(\zeta,x(\zeta)) - h(\zeta,y(\zeta))| \\
&\leq \frac{1}{\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}\left|h_1(v,x(v)) - h_1(v,y(v))\right|dv \\
&\quad + \frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}\left|h_2(v,x(v)) - h_2(v,y(v))\right|dv \\
&\quad + \frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}\left|h_1(v,y(v)) - h_1(v,x(v))\right|dv \\
&\leq \frac{(\vartheta(T)-\vartheta(0))^\gamma}{\Gamma(\gamma+1)}\left\|h_1(v,x) - h_1(v,y)\right\| \\
&\quad + \frac{\left(\varphi(\zeta)-\vartheta'(\zeta)\right)(\vartheta(T)-\vartheta(0))^\gamma}{\varphi(T)\Gamma(\gamma+1)}\left\|h_2(v,x) - h_2(v,y)\right\| \\
&\quad + \frac{\left(\varphi(\zeta)-\vartheta'(\zeta)\right)(\vartheta(T)-\vartheta(0))^\gamma}{\varphi(T)\Gamma(\gamma+1)}\left\|h_1(v,y) - h_1(v,x)\right\| \\
&\leq \frac{(\vartheta(T)-\vartheta(0))^\gamma}{\Gamma(\gamma+1)}\left[\left\|h_1(v,x) - h_1(v,y)\right\| + \frac{\left(\varphi(\zeta)-\vartheta'(\zeta)\right)}{\varphi(T)}\left(\left\|h_2(v,x) - h_2(v,y)\right\| + \left\|h_1(v,y) - h_1(v,x)\right\|\right)\right]
\end{aligned}
$$

$$\begin{aligned}&\leq \frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[k_1\|x-y\|+\frac{\left(\varphi(\zeta)-\vartheta'(\zeta)\right)}{\varphi(T)}\left(k_2\|x-y\|+k_1\|y-x\|\right)\right]\\&\leq \frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[k_1+\frac{\left(\varphi(\zeta)-\vartheta'(\zeta)\right)}{\varphi(T)}\left(k_2+k_1\right)\right]\|x-y\|.\end{aligned}\tag{36}$$

Thus,

$$\|h(\zeta,x)-h(\zeta,y)\|\leq c\|x-y\|,\tag{37}$$

where $c=((\vartheta(T)-\vartheta(0))^{\gamma}/\Gamma(\gamma+1))[k_1+((\varphi(\zeta)-\vartheta'(\zeta))/\varphi(T))(k_2+k_1)]$.

Our first result is based on Banach's fixed point theorem to obtain the existence of a unique solution of (ICFDPs) (7)–(9). □

Theorem 12. *Suppose that assumptions $(\mathscr{H}_1)$-$(\mathscr{H}_3)$ hold, and c is the Lipschitz constant as defined in Lemma 11 with*

$$c+\frac{G_0\|\mu\|T}{1-\|\mu\|\left(\mathrm{KT}+\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)}<1,\tag{38}$$

such that $\|\mu\|(KT+((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)))<1$, then (ICFDPs) (7)–(9) have a unique mild solution on I.

Proof. Transform (ICFDPs) (7)–(9) into a fixed point problem. Define the operator $\mathscr{A}: C^2(I,R)\longrightarrow C^2(I,R)$ by

$$\mathscr{A}y(\zeta)=h(\zeta,y(\zeta))+\int_0^T\vartheta'(v)G(\zeta,v)\nu(v)dv,\tag{39}$$

where $\nu\in C(I,R)$ satisfies the following implicit functional equation:

$$\nu(\zeta)=\mathfrak{F}\left(\zeta,y(\zeta),\mathscr{I}^{\alpha-\beta,\vartheta}\nu(\zeta),\int_0^{\zeta}k(\zeta,v)\nu(v)dv\right),\tag{40}$$

where G and h are the functions defined by equations (26) and (28), respectively. We define the ball $\mathfrak{B}_r$ with radius r as follows:

$$\mathfrak{B}_r=\{y\in C(I,R):\ \|y\|\leq r\},\tag{41}$$

where

$$\begin{aligned}r&\geq\frac{F\omega+((\vartheta(T)-\vartheta(0))^{\gamma}/\Gamma(\gamma+1))\left[\left|1-\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|H_1+\left|\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|H_2\right]}{1-\|\mu\|\omega-((\vartheta(T)-\vartheta(0))^{\gamma}/\Gamma(\gamma+1))\left[\left|1-\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|k_1+\left|\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|k_2\right]},\\\omega&=\frac{G_0(\vartheta(T)-\vartheta(0))}{1-\|\mu\|\left(\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)+KT\right)}.\end{aligned}\tag{42}$$

First, we show that the operator $\mathscr{A}$ is well-defined, i.e., we show that $\mathscr{A}(\mathfrak{B}_r)\subset\mathfrak{B}_r$, where

$$\mathfrak{B}_r=\{y\in C(I,R):\ \|y\|\leq r\}.\tag{43}$$

Let $y\in\mathfrak{B}_r$. In the following, we show that $\mathscr{A}y\in\mathfrak{B}_r$ for each $\zeta\in I$ as follows:

$$\begin{aligned}|\mathscr{A}y(\zeta)|&=\left|h(\zeta,y(\zeta))+\int_0^T\vartheta'(v)G(\zeta,v)\nu(v)dv\right|\\&\leq|h(\zeta,y(\zeta))|+\int_0^T\vartheta'(v)|G(\zeta,v)||\nu(v)|dv,\end{aligned}\tag{44}$$

where $\nu(\zeta)=\mathfrak{F}(\zeta,y(\zeta),\mathscr{I}^{\alpha-\beta,\vartheta}\nu(\zeta),\int_0^{\zeta}k(\zeta,v)\nu(v)dv)$, such that

$$\begin{aligned}|\nu(\zeta)|&=\left|\mathfrak{F}\left(\zeta,y(\zeta),\mathscr{I}^{\alpha-\beta,\vartheta}\nu(\zeta),\int_0^{\zeta}k(\zeta,v)\nu(v)dv\right)\right|\\&\leq\mu(\zeta)\left(|y(\zeta)|+\int_0^{\zeta}\vartheta'(v)\frac{(\vartheta(\zeta)-\vartheta(v))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|\nu(v)|dv+\int_0^{\zeta}|k(\zeta,v)||\nu(v)|dv\right)+F.\end{aligned}\tag{45}$$

Taking supremum for $\zeta\in I$, we get

$$\|v\| \leq \|\mu\|\left(\|y\| + \frac{(\vartheta(T)-\vartheta(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|v\| + K\|v\|T \right) + F. \tag{46}$$

Then,

$$\begin{aligned}
\|v\| &\leq \frac{r\|\mu\| + F}{1-\|\mu\|\left(\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)+\mathrm{KT}\right)},\\
|h(\zeta, y(\zeta))| &\leq \frac{1}{\Gamma(\gamma)}\left(1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right)\int_0^T \vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}\left|h_1(v, y(v))\right|dv\\
&\quad + \frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)\Gamma(\gamma)}\int_0^T \vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}\left|h_2(v, y(v))\right|dv\\
&\leq \frac{1}{\Gamma(\gamma)}\left|1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|\int_0^T \vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}\left[H_1+k_1|y(v)|\right]dv\\
&\quad + \left|\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)\Gamma(\gamma)}\right|\int_0^T \vartheta'(v)[\vartheta(T)-\vartheta(v)]^{\gamma-1}\left[H_2+k_2|y(v)|\right]dv\\
&\leq \left|1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right| \frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[H_1+k_1\|y(v)\|\right]\\
&\quad + \left|\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right| \frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[H_2+k_2\|y(v)\|\right]\\
&\leq \frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[\left|1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|H_1 + \left|\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|H_2\right]\\
&\quad + \frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[\left|1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|k_1 + \left|\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|k_2\right]r.
\end{aligned} \tag{47}$$

Thus, equation (44) implies that, for each $\zeta \in I$,

$$\begin{aligned}
|Ay(\zeta)| &\leq \frac{(r\mu(\zeta)+F)G_0}{1-\mu(\zeta)\left(\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)+\mathrm{KT}\right)}(\vartheta(T)-\vartheta(0))\\
&\quad + \frac{(\vartheta(1)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[\left|1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|H_1 + \left|\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|H_2\right]\\
&\quad + \frac{(\vartheta(1)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[\left|1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|k_1 + \left|\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right|k_2\right]r\\
&\leq r.
\end{aligned} \tag{48}$$

If we take supremum for all $\zeta \in I$, we get

$$r \geq \frac{\left(F\left((G_0(\vartheta(T)-\vartheta(0)))/\left(1-\|\mu\|\left(\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)+\mathrm{KT}\right)\right)\right)+((\vartheta(T)-\vartheta(0))^{\gamma}/\Gamma(\gamma+1))\left[\left|1-\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|H_1+\left|\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|H_2\right]\right)}{\left(1-\|\mu\|\left((G_0(\vartheta(T)-\vartheta(0)))/\left(1-\|\mu\|\left(\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)+\mathrm{KT}\right)\right)\right)-((\vartheta(T)-\vartheta(0))^{\gamma}/\Gamma(\gamma+1))\left[\left|1-\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|k_1+\left|\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|k_2\right]\right)}. \tag{49}$$

Hence, $\mathscr{A}(\mathfrak{B}_r) \subset \mathfrak{B}_r$.

Second, we show that the operator $\mathscr{A}$ is a contraction.

Let $x, y \in C(I, R)$. Then, for any $\zeta \in I$, we have

$$\mathscr{A}x(\zeta) - \mathscr{A}y(\zeta) = h(\zeta, x(\zeta)) + \int_0^T \vartheta'(v)G(\zeta, v)u(v)dv - h(\zeta, y(\zeta)) + \int_0^T \vartheta'(v)G(\zeta, v)v(v)dv, \tag{50}$$

where $u, v \in C(I, R)$ such that

$$\begin{aligned} u(\zeta) &= \mathfrak{F}\left(\zeta, x(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}u(\zeta), \int_0^\zeta k(\zeta, v)u(v)dv\right), \\ v(\zeta) &= \mathfrak{F}\left(\zeta, y(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}v(\zeta), \int_0^\zeta k(\zeta, v)v(v)dv\right). \end{aligned} \tag{51}$$

Then, for any $\zeta \in I$, we have

$$|\mathscr{A}x(\zeta) - \mathscr{A}y(\zeta)| \leq |h(\zeta, x(\zeta)) - h(\zeta, y(\zeta))| + \int_0^T \vartheta'(v)G(\zeta, v)|u(v) - v(v)|dv. \tag{52}$$

But, by assumption $(\mathscr{H}_2)$, we have

$$\begin{aligned} &|u(\zeta) - v(\zeta)| \\ &= \left|\mathfrak{F}\left(\zeta, x(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}u(\zeta), \int_0^\zeta k(\zeta, v)u(v)dv\right) - \mathfrak{F}\left(\zeta, y(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}v(\zeta), \int_0^\zeta k(\zeta, v)v(v)dv\right)\right| \\ &\leq \mu(\zeta)\left(\begin{array}{c} |x(\zeta) - y(\zeta)| + \displaystyle\int_0^\zeta \vartheta'(v)\frac{(\vartheta(\zeta) - \vartheta(v))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|u(v) - v(v)|dv \\ + \displaystyle\int_0^\zeta k(\zeta, v)|u(v) - v(v)|dv \end{array}\right). \end{aligned} \tag{53}$$

Taking the supremum for all $\zeta \in I$, we get

$$\|u - v\| \leq \|\mu\|\left(\|x - y\| + \frac{(\vartheta(T) - \vartheta(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|u - v\| + K\|u - v\|T\right). \tag{54}$$

Thus,

$$\|u-v\| \leq \frac{\|\mu\|}{1-\|\mu\|\left(KT+\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)}\|x-y\|. \tag{55}$$

Now, return to equation (52), and by Lemma 11, we have

$$\begin{aligned}|\mathscr{A}x(\zeta)-\mathscr{A}y(\zeta)| &\leq c\|x-y\|+\frac{G_0\|\mu\|T}{1-\|\mu\|\left(\mathrm{KT}+\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)}\|x-y\| \\ &\leq \left(c+\frac{G_0\|\mu\|T}{1-\|\mu\|\left(\mathrm{KT}+\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)}\right)\|x-y\|,\end{aligned} \tag{56}$$

and taking supremum for $\zeta \in I$, we get

$$\|\mathscr{A}x-\mathscr{A}y\| \leq \left(c+\frac{G_0\|\mu\|T}{\left(1-\|\mu\|\left(\mathrm{KT}+\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)\right)}\right)\|x-y\|. \tag{57}$$

Now, if $c+(G_0\|\mu\|T/(1-\|\mu\|(K\,T+(((\vartheta(T)-\vartheta(0))^{\alpha-\beta})/(\Gamma(\alpha-\beta+1))))))<1$, then operator $\mathscr{A}$ is a contraction.

Therefore, by Banach's contraction principle, we deduce that $\mathscr{A}$ has a unique fixed point $x \in C(I,R)$, which is a mild solution of (ICFDPs) (7)–(9) on $[0,T]$.

In the following, we present our second existence result for the mild solution of (ICFDPs) (7)–(9) based on the Krasnoselskii's fixed point theorem [1]. □

Theorem 13. *Assume that the assumptions* $(\mathscr{H}_1)-(\mathscr{H}_2)$ *and* $(\mathscr{H}_5)$ *hold. If*

$$\frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[\left(1-\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}\right)k_1+\frac{\varphi(\zeta)-\vartheta'(\zeta)}{\varphi(T)}k_2\right]-\frac{G_0T\|\chi_1\|}{1-\kappa}<1, \tag{58}$$

where $\kappa=\|\chi_2\|((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1))+\|\chi_3\|KT$, *then (ICFDPs) (7)–(9) have at least one mild solution on* I.

Proof. Let the operator be $\mathscr{A}$ defined in (39). Define the closed disk

$$\wp_\rho=\{y \in C(I,R)\colon \|y\| \leq \rho\}, \tag{59}$$

with

$$\rho \geq \frac{\left(((\vartheta(T)-\vartheta(0))^{\gamma}/\Gamma(\gamma+1))\left[\left|1-\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|H_1+\left|\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|H_2\right]+(G_0T\|\phi\|/(1-\kappa))\right)}{\left(1-((\vartheta(T)-\vartheta(0))^{\gamma}/\Gamma(\gamma+1))\left[\left|1-\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|k_1+\left|\left(\varphi(T)-\vartheta'(T)/\varphi(T)\right)\right|k_2\right]-\left(\left(G_0T\|\chi_1\|/(1-\kappa)\right)\right)\right)}. \tag{60}$$

In addition, define the operators $\mathscr{A}_1$ and $\mathscr{A}_2$ on $\wp_\rho$ by

$$\begin{aligned}\mathscr{A}_1 y(\zeta) &= h(\zeta, y(\zeta)),\\ \mathscr{A}_2 y(\zeta) &= \int_0^T \vartheta'(v) G(\zeta, v) \nu(v) dv.\end{aligned} \tag{61}$$

Taking into account that $\mathscr{A}_1$ and $\mathscr{A}_2$ are defined on $\wp_\rho$, and for any $y \in C(I, R)$, we have

$$\mathscr{A} y(\zeta) = \mathscr{A}_1 y(\zeta) + \mathscr{A}_2 y(\zeta), \quad \zeta \in I. \tag{62}$$

The proof is divided into several steps:

Step 1: $\mathscr{A}$ is well defined.

Let $y_1, y_2 \in \wp_\rho$. Then, for any $\zeta \in I$, we have

$$\begin{aligned}\left|\mathscr{A}_1 y_1(\zeta) + \mathscr{A}_2 y_2(\zeta)\right| &\le \left|\mathscr{A}_1 y_1(\zeta)\right| + \left|\mathscr{A}_2 y_2(\zeta)\right|\\ &\le \left|h(\zeta, y_1(\zeta))\right| + \int_0^T \vartheta'(v) |G(\zeta, v)| |\nu(v)| dv,\end{aligned} \tag{63}$$

where $\nu(\zeta) = \mathfrak{F}(\zeta, y_2(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}\nu(\zeta), \int_0^\zeta k(\zeta, v)\nu(v)dv)$ such that

$$\begin{aligned}|\nu(\zeta)| &= \left|\mathfrak{F}\left(\zeta, y_2(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}\nu(\zeta), \int_0^\zeta k(\zeta, v)\nu(v)dv\right)\right|\\ &\le \phi(\zeta) + \chi_1(\zeta)\left|y_2(\zeta)\right| + \chi_2(\zeta)\int_0^\zeta \vartheta'(v)\left(\frac{(\vartheta(\zeta) - \vartheta(v))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right)|\nu(v)|dv\\ &\quad + \chi_3(\zeta)\int_0^\zeta |k(\zeta, v)||\nu(v)|dv.\end{aligned} \tag{64}$$

Taking supremum for all $\zeta \in I$, we have

$$\begin{aligned}\|\nu\| &\le \|\phi\| + \|\chi_1\|\|y_2\|\\ &\quad + \|\chi_2\|\left(\frac{(\vartheta(T) - \vartheta(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|\nu\| + \|\chi_3\|K\|\nu\|T.\end{aligned} \tag{65}$$

Thus,

$$\|\nu\| \le \frac{\|\phi\| + \|\chi_1\|\rho}{1 - \left(\|\chi_2\|\left((\vartheta(T) - \vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right) + \|\chi_3\|KT\right)}. \tag{66}$$

Moreover,

$$\begin{aligned}\left|h(\zeta, y_1(\zeta))\right| &\le \frac{1}{\Gamma(\gamma)}\left|1 - \frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right| \int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1}\left|h_1(v, y_1(v))\right|dv\\ &\quad + \left|\frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)\Gamma(\gamma)}\right| \int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1}\left|h_2(v, y_1(v))\right|dv\\ &\le \frac{1}{\Gamma(\gamma)}\left|1 - \frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right| \int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1}\left[H_1 + k_1\left|y_1(v)\right|\right]dv\\ &\quad + \left|\frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)\Gamma(\gamma)}\right| \int_0^T \vartheta'(v)[\vartheta(T) - \vartheta(v)]^{\gamma-1}\left[H_2 + k_2\left|y_1(v)\right|\right]dv\\ &\le \left|1 - \frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right| \frac{(\vartheta(T) - \vartheta(0))^\gamma}{\Gamma(\gamma+1)}\left[H_1 + k_1\left\|y_1(v)\right\|\right]\\ &\quad + \left|\frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right| \frac{(\vartheta(T) - \vartheta(0))^\gamma}{\Gamma(\gamma+1)}\left[H_2 + k_2\left\|y_1(v)\right\|\right]\\ &\le \frac{(\vartheta(T) - \vartheta(0))^\gamma}{\Gamma(\gamma+1)}\left[\left|1 - \frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right|H_1 + \left|\frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right|H_2\right]\\ &\quad + \frac{(\vartheta(T) - \vartheta(0))^\gamma}{\Gamma(\gamma+1)}\left[\left|1 - \frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right|k_1 + \left|\frac{\varphi(\zeta) - \vartheta'(\zeta)}{\varphi(T)}\right|k_2\right]\rho.\end{aligned} \tag{67}$$

Thus, equation (63) implies that, for each $\zeta \in I$, we obtain that

$$\begin{aligned}\left|\mathscr{A}_1 y_1(\zeta)+\mathscr{A}_2 y_2(\zeta)\right| &\leq \frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[\left|1-\frac{\varphi(\zeta)-\vartheta^{'}(\zeta)}{\varphi(T)}\right|H_1+\left|\frac{\varphi(\zeta)-\vartheta^{'}(\zeta)}{\varphi(T)}\right|H_2\right]\\ &+\frac{(\vartheta(T)-\vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[\left|1-\frac{\varphi(\zeta)-\vartheta^{'}(\zeta)}{\varphi(T)}\right|k_1+\left|\frac{\varphi(\zeta)-\vartheta^{'}(\zeta)}{\varphi(T)}\right|k_2\right]\rho\\ &+\frac{G_0 T\left(\|\phi\|+\|\chi_1\|\rho\right)}{1-\kappa},\\ &\leq \rho.\end{aligned} \tag{68}$$

Taking supremum over $\zeta \in I$, we have

$$\left\|\mathscr{A}_1 y_1+\mathscr{A}_2 y_2\right\| \leq \rho. \tag{69}$$

This proves that $\mathscr{A}_1 y_1+\mathscr{A}_2 y_2 \in \wp_\rho$ for every $y_1, y_2 \in \wp_\rho$, where ρ is given in equation (60).

Step 2: The operator $\mathscr{A}_1$ demonstrates contraction behavior within $\wp_\rho$. It is clear from Lemma 11 that operator $\mathscr{A}_1$ is a contraction mapping for $c<1$.

Step 3: The operator $\mathscr{A}_2$ exhibits complete continuity (both compactness and continuity) on $\wp_\rho$.

First, we establish the continuity of the operator $\mathscr{A}_2$. Assume that $\{y_n \in N\}$ is a sequence such that $y_n \longrightarrow y$ as $n \longrightarrow \infty$ in $C(I,R)$. Then, for every $\zeta \in I$, the following relation holds:

$$\left|\mathscr{A}_2 y_n(\zeta)-\mathscr{A}_2 y(\zeta)\right| \leq \left|h(\zeta, y_n(\zeta))-h(\zeta, y(\zeta))\right|+\int_0^T \vartheta^{'}(v)\left|G(\zeta,v)\right|\left|v_n(v)-v(v)\right|dv, \tag{70}$$

where $v_n, v \in C(I,R)$, such that

$$\begin{aligned} v_n(\zeta) &= \mathfrak{F}\left(\zeta, y_n(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}v_n(\zeta), \int_0^\zeta k(\zeta,v)v_n(v)dv\right),\\ v(\zeta) &= \mathfrak{F}\left(\zeta, y(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}v(\zeta), \int_0^\zeta k(\zeta,v)v(v)dv\right).\end{aligned} \tag{71}$$

By Lemma 11, we have

$$\left|h(\zeta, y_n(\zeta))-h(\zeta, y(\zeta))\right| \leq c\left\|y_n-y\right\|, \tag{72}$$

where $c<1$ is a Lipschitz constant. Moreover, by the assumption $(\mathscr{H}_2)$, we have

$$\begin{aligned}&\left|v_n(\zeta)-v(\zeta)\right|\\ &=\left|\mathfrak{F}\left(\zeta, y_n(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}v_n(\zeta), \int_0^\zeta k(\zeta,v)v_n(v)dv\right)-\mathfrak{F}\left(\zeta, y(\zeta), \mathscr{I}^{\alpha-\beta,\vartheta}v(\zeta), \int_0^\zeta k(\zeta,v)v(v)dv\right)\right|\\ &\leq \mu(\zeta)\left(\left|y_n(\zeta)-y(\zeta)\right|+\int_0^\zeta \vartheta^{'}(v)\frac{(\vartheta(\zeta)-\vartheta(v))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|v_n(v)-v(v)\right|dv+\int_0^\zeta k(\zeta,v)\left|v_n(v)-v(v)\right|dv\right)\\ &\leq \|\mu\|\left(\left\|y_n-y\right\|+\frac{(\vartheta(T)-\vartheta(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\left\|v_n-v\right\|+\mathrm{KT}\left\|v_n-v\right\|\right).\end{aligned} \tag{73}$$

Taking supremum for all $\zeta \in I$, we get

$$\left\|v_n - v\right\| \leq \frac{\|\mu\|}{1-\left(\mathrm{KT}+\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)}\left\|y_n - y\right\|. \tag{74}$$

Since $y_n \longrightarrow y$, then we get $v_n(\zeta) \longrightarrow v(\zeta)$ as $n \longrightarrow \infty$ for each $\zeta \in I$. Consider $\varepsilon > 0$ such that for any $\zeta \in I$, we have $|v_n(\zeta)| \leq (\varepsilon/2)$ and $|v(\zeta)| \leq (\varepsilon/2)$. Thus,

$$\begin{aligned}\left|G(\zeta, \upsilon)\right|\left|v_n(\upsilon) - v(\upsilon)\right| &\leq \left|G(\zeta, \upsilon)\right|\left[\left|v_n(\upsilon)\right| + \left|v(\upsilon)\right|\right],\\ &\leq \varepsilon\left|G(\zeta, \upsilon)\right|.\end{aligned} \tag{75}$$

For each $\zeta \in I$, the function $\upsilon \longrightarrow \varepsilon|G(\zeta, \upsilon)|$ is integrable on I. Then, applying Lebesgue dominated convergence theorem and equation (70), we deduce that

$$\begin{aligned}\left|\mathscr{A}_2 y_n(\zeta) - \mathscr{A}_2 y(\zeta)\right| &\leq c\left\|y_n - y\right\| + \int_0^T \vartheta'(\upsilon)|G(\zeta, \upsilon)|\left|v_n(\upsilon) - v(\upsilon)\right|d\upsilon\\ &\leq c\left\|y_n - y\right\| + \varepsilon|G(\zeta, \upsilon)|(\vartheta(T) - \vartheta(0)).\end{aligned} \tag{76}$$

Hence, $\|\mathscr{A}_2 y_n(\zeta) - \mathscr{A}_2 y(\zeta)\| \longrightarrow 0$ as $n \longrightarrow \infty$, and consequently $\mathscr{A}_2$ is continuous.

Second, due to the definition of ρ, it is easy to verify that $\mathscr{A}_2$ satisfies that

$$\left\|\mathscr{A}_2 y\right\| \leq \frac{(r\mu(\zeta) + F)G_0}{1-\|\mu\|\left(\left((\vartheta(T)-\vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)+\mathrm{KT}\right)}(\vartheta(T) - \vartheta(0)) \leq \rho. \tag{77}$$

This proves that $\mathscr{A}_2$ is uniformly bounded on $\wp_\rho$.

Third, we prove that $\mathscr{A}_2$ maps bounded sets into equicontinuous sets of $C(I, R)$, i.e., $\wp_\rho$ is equicontinuous.

Now, suppose that for every $\epsilon > 0$, there exist $\delta > 0$, and $\zeta_1, \zeta_2 \in I$ such that $\zeta_1 < \zeta_2$ and $|\zeta_2 - \zeta_1| < \delta$. Then,

$$\begin{aligned}\left|\mathscr{A}_2 y(\zeta_2) - \mathscr{A}_2 y(\zeta_1)\right| &\leq \left|h(\zeta, y(\zeta_2)) - h(\zeta, y(\zeta_1))\right| + \int_0^T \vartheta'(\upsilon)\left|G(\zeta_2, \upsilon) - G(\zeta_1, \upsilon)\right||v(\upsilon)|d\upsilon\\ &\leq c\left\|y(\zeta_2) - y(\zeta_1)\right\| + \|v\|\int_0^T \vartheta'(\upsilon)\left|G(\zeta_2, \upsilon) - G(\zeta_1, \upsilon)\right|d\upsilon\\ &\leq c\left\|y(\zeta_2) - y(\zeta_1)\right\| + \frac{\|\phi\| + \|\chi_1\|\rho}{(1-\kappa)}\int_0^T \vartheta'(\upsilon)\left|G(\zeta_2, \upsilon) - G(\zeta_1, \upsilon)\right|d\upsilon.\end{aligned} \tag{78}$$

It is clear that as $\zeta_1 \longrightarrow \zeta_2$, the right-hand side of the above inequality tends to zero. Consequently,

$$\left|\mathscr{A}_2 y(\zeta_2) - \mathscr{A}_2 y(\zeta_1)\right| \longrightarrow 0, \quad \forall\left|\zeta_2 - \zeta_1\right| \longrightarrow 0. \tag{79}$$

Hence, the equicontinuity of $\{\mathscr{A} y\}$ holds on $\wp_\rho$ along with the compactness of operator $\mathscr{A}$ that is established by the Arzela–Ascoli theorem. This leads to the inference that $\mathscr{A}: C(I, R) \longrightarrow C(I, R)$ maintains both continuity and compactness.

Notably, all prerequisites essential for Krasnoselskii's fixed point theorem are satisfied. This shows that the operator $\mathscr{A}_1 + \mathscr{A}_2$ has a fixed point on $\wp_\rho$. Therefore, (ICFDPs) (7)–(9) have a mild solution on I. This concludes the proof. □

4. Ulam Stability of the Solutions

Consider now the Ulam stability for (ICFDPs) (7)–(9). Let $\epsilon > 0$ and $\Phi: I \longrightarrow R^+$ be a continuous function. We investigate the following inequalities:

$$\left| {}^{c}\mathscr{D}^{\alpha,\vartheta} y(\zeta) - \mathfrak{F}\left(\zeta, y(\zeta), {}^{c}\mathscr{D}^{\beta,\vartheta} y(\zeta), \int_0^\zeta k(\zeta,v) {}^{c}\mathscr{D}^{\alpha,\vartheta} y(v) dv\right)\right| \leq \epsilon, \quad \zeta \in I, \tag{80}$$

$$\left| {}^{c}\mathscr{D}^{\alpha,\vartheta} y(\zeta) - \mathfrak{F}\left(\zeta, y(\zeta), {}^{c}\mathscr{D}^{\beta,\vartheta} y(\zeta), \int_0^\zeta k(\zeta,v) {}^{c}\mathscr{D}^{\alpha,\vartheta} y(v) dv\right)\right| \leq \Phi(\zeta), \quad \zeta \in I, \tag{81}$$

$$\left| {}^{c}\mathscr{D}^{\alpha,\vartheta} y(\zeta) - \mathfrak{F}\left(\zeta, y(\zeta), {}^{c}\mathscr{D}^{\beta,\vartheta} y(\zeta), \int_0^\zeta k(\zeta,v) {}^{c}\mathscr{D}^{\alpha,\vartheta} y(v) dv\right)\right| \leq \epsilon\Phi(\zeta), \quad \zeta \in I. \tag{82}$$

Definition 14. (ICFDPs) (7)–(9) are considered Ulam–Hyers stable if there exists a positive real number $c_{\mathfrak{F}} > 0$ such that, for every $\epsilon > 0$ and for any solution $y \in C(I, R)$ satisfying inequality (80), there exists a solution $x \in C(I, R)$ of (ICFDPs) (7)–(9) with the following property:

$$|y(\zeta) - x(\zeta)| \leq \epsilon\, c_{\mathfrak{F}}, \quad \text{for all } \zeta \in I. \tag{83}$$

Definition 15. (ICFDPs) (7)–(9) are said to be generalized Ulam–Hyers stable if there exists a function $c_{\mathfrak{F}} \in C(R, R)$ with $c_{\mathfrak{F}}(0) = 0$, such that for each $\epsilon > 0$ and each solution $y \in C(I, R)$ satisfying inequality (80), there exists a solution $x \in C(I, R)$ of (ICFDPs) (7)–(9) with

$$|y(\zeta) - x(\zeta)| \leq c_{\mathfrak{F}}(\epsilon), \quad \text{for all } \zeta \in I. \tag{84}$$

Definition 16. (ICFDPs) (7)–(9) are considered Ulam–Hyers–Rassias stable with respect to Φ if there exists a positive real number $c_{\mathfrak{F},\Phi} > 0$ such that, for every $\epsilon > 0$ and for any solution $y \in C(I, R)$ satisfying inequality (81), there exists a solution $x \in C(I, R)$ of the systems (7)–(9) with the following property:

$$|y(\zeta) - x(\zeta)| \leq \epsilon\, c_{\mathfrak{F},\Phi}\, \Phi(\zeta), \quad \text{for all } \zeta \in I. \tag{85}$$

Definition 17. (ICFDPs) (7)–(9) are said to possess generalized Ulam–Hyers–Rassias stability with respect to the function Φ if there exists a positive real constant $c_{\mathfrak{F},\Phi} > 0$ such that for every solution $y \in C(I, R)$ of the inequality 23, there is a solution $x \in C(I, R)$ of the (ICFDPs) (7)–(9) satisfying the following condition:

$$|y(\zeta) - x(\zeta)| \leq c_{\mathfrak{F},\Phi}\, \Phi(\zeta), \quad \text{for all } \zeta \in I. \tag{86}$$

4.1. Ulam–Hyers Stability. In the following, we study the Ulam–Hyers stability for (ICFDPs) (7)–(9).

Theorem 18. *Assume that the assumptions of Theorem 12 are satisfied. Then, (ICFDPs) (7)–(9) are Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C(I, R)$ be a function which satisfies inequality (80), i.e.,

$$\left| {}^{c}\mathscr{D}^{\alpha,\vartheta} z(\zeta) - \mathfrak{F}\left(\zeta, y(\zeta), {}^{c}\mathscr{D}^{\beta,\vartheta} z(\zeta), \int_0^\zeta k(\zeta,v) {}^{c}\mathscr{D}^{\alpha,\vartheta} z(v) dv\right)\right| \leq \epsilon, \quad \text{for all } \zeta \in I, \tag{87}$$

and let $y \in C(I, R)$ be the unique solution of (ICFDPs) (7)–(9) which by Lemma 9 is equivalent to the following fractional order integral equation:

$$y(\zeta) = h(\zeta, y(\zeta)) + \int_0^T \vartheta'(v) G(\zeta, v) u(v) dv, \tag{88}$$

where u is the solution of the following functional integral equation:

$$u(\zeta) = \mathfrak{F}\left(\zeta, h(\zeta) + \frac{1}{\Gamma(\alpha)} \int_0^T \vartheta'(v) G(\zeta, v) u(v) dv, \mathscr{I}^{\alpha-\beta,\vartheta} u(\zeta), \int_0^\zeta k(\zeta, v) u(v) dv\right). \tag{89}$$

Applying $\mathscr{I}^{\alpha,\vartheta}$ on both sides of equation (80), we get

$$\left|z(\zeta) - h(\zeta, z(\zeta)) - \int_0^T \vartheta'(v) G(\zeta, v) v(v) dv\right| \leq \frac{\epsilon(\vartheta(T) - \vartheta(0))^\alpha}{\Gamma(\alpha+1)}, \tag{90}$$

where $v \in C(I, R)$, such that

$$\|u - v\| \leq \frac{\|\mu\|}{1 - \|\mu\|\left(\mathrm{KT} + \left((\vartheta(T) - \vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)} \|z - y\|. \tag{91}$$

This implies that for each $\zeta \in I$, we have

$$\begin{aligned} |z(\zeta) - y(\zeta)| &= \left|z(\zeta) - h(\zeta, y(\zeta)) - \int_0^T \vartheta'(v) G(\zeta, v) u(v) dv\right| \\ &\leq \left|z(\zeta) - h(\zeta, z(\zeta)) - \int_0^T \vartheta'(v) G(\zeta, v) v(v) dv + h(\zeta, z(\zeta))\right. \\ &\quad \left. + \int_0^T \vartheta'(v) G(\zeta, v) v(v) dv - h(\zeta, y(\zeta)) - \int_0^T \vartheta'(v) G(\zeta, v) u(v) dv\right| \\ &\leq \frac{\epsilon(\vartheta(T) - \vartheta(0))^\alpha}{\Gamma(\alpha+1)} + |h(\zeta, z(\zeta)) - h(\zeta, y(\zeta))| + \int_0^T \vartheta'(v) G(\zeta, v) |v(v) - u(v)| dv \\ &\leq \frac{\epsilon(\vartheta(T) - \vartheta(0))^\alpha}{\Gamma(\alpha+1)} + c|z(\zeta) - y(\zeta)| + \int_0^T \vartheta'(v) G(\zeta, v) |v(v) - u(v)| dv \\ &\leq \frac{\epsilon(\vartheta(T) - \vartheta(0))^\alpha}{\Gamma(\alpha+1)} + c\|z - y\| + \frac{G_0\|\mu\|(\vartheta(T) - \vartheta(0))}{1 - \|\mu\|\left(\mathrm{KT} + \left((\vartheta(T) - \vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)} \|z - y\|. \end{aligned} \tag{92}$$

Thus, if we take supremum for all $\zeta \in I$, we get

$$\|z - y\| \leq \frac{\epsilon(\vartheta(T) - \vartheta(0))^\alpha}{\Gamma(\alpha+1)} + c\|z - y\| + \frac{G_0\|\mu\|(\vartheta(T) - \vartheta(0))}{1 - \|\mu\|\left(\mathrm{KT} + \left((\vartheta(T) - \vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)\right)} \|z - y\|. \tag{93}$$

Then, for $\zeta = 1 - c - (G_0\|\mu\|(\vartheta(T) - \vartheta(0))/(1 - \|\mu\| (\mathrm{KT} + ((\vartheta(T) - \vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)))))$, we get the following equation:

$$\|z - y\| \leq \left(\frac{(\vartheta(T) - \vartheta(0))^\alpha}{\Gamma(\alpha+1)} \zeta^{-1}\right)\epsilon = c_{\mathfrak{F}}\epsilon. \tag{94}$$

Therefore, (ICFDPs) (7)–(9) are Ulam–Hyers stable. This completes the proof. □

Remark 19. If we put $\Phi(\epsilon) = c_{\mathfrak{F}}\epsilon$, then $\Phi(0) = 0$, which yields that (ICFDPs) (7)–(9) are generalized Ulam–Hyers stable.

4.2. Ulam–Hyers–Rassias Stability. Now, we show that (ICFDPs) (7)–(9) satisfy the Ulam–Hyers–Rassias stable type.

Theorem 20. *Assume that assumptions $(\mathscr{H}_2)$–$(\mathscr{H}_4)$ hold. Then, (ICFDPs) (7)–(9) are Ulam–Hyers–Rassias stable with respect to Φ.*

Proof. Let $z \in C(I, R)$ be a mild solution of inequation (82), i.e.,

$$\left|{}^c\mathscr{D}^{\alpha,\vartheta} z(\zeta) - \mathfrak{F}\left(\zeta, y(\zeta), {}^c\mathscr{D}^{\beta,\vartheta} z(\zeta), \int_0^\zeta k(\zeta, v) {}^c\mathscr{D}^{\alpha,\vartheta} z(v) dv\right)\right| \leq \epsilon\, \Phi, \quad \text{for all } \zeta \in I, \tag{95}$$

and assume that y is a solution of (ICFDPs) (7)–(9), such that

$$y(\zeta) = h(\zeta, y(\zeta)) + \int_0^T \vartheta'(v) G(\zeta, v) u(v) dv, \tag{96}$$

where $u \in C(I, R)$ satisfies the following integral equation:

$$u(\zeta) = \mathfrak{F}\left(\zeta, y(\zeta), \mathcal{I}^{\alpha-\beta,\vartheta} u(\zeta), \int_0^\zeta k(\zeta, v) u(v) dv\right). \tag{97}$$

Operating by $I^{\alpha,\vartheta}$ on both sides of inequality (82) and then integrating, we get

$$\begin{aligned} \left| z(\zeta) - h(\zeta, z(\zeta)) - \int_0^T \vartheta'(v) G(\zeta, v) v(v) dv \right| &\leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^\zeta \vartheta'(v) (\vartheta(\zeta) - \vartheta(v))^{\alpha-1} \Phi(v) dv \\ &\leq \epsilon \lambda_\Phi \Phi(\zeta), \end{aligned} \tag{98}$$

where $v \in C(I, R)$ such that

$$v(\zeta) = \mathfrak{F}\left(\zeta, z(\zeta), \mathcal{I}^{\alpha-\beta,\vartheta} v(\zeta), \int_0^\zeta k(\zeta, v) v(v) dv\right). \tag{99}$$

But,

$$\|u - v\| \leq \frac{\|\mu\|}{1 - \|\mu\| \left(\mathrm{KT} + \left((\vartheta(T) - \vartheta(0))^{\alpha-\beta} / \Gamma(\alpha - \beta + 1) \right) \right)} \|z - y\|. \tag{100}$$

Hence, in a similar manner as above, we have for each $\zeta \in I$,

$$\begin{aligned} |z(\zeta) - y(\zeta)| &= \left| z(\zeta) - h(\zeta, y(\zeta)) - \int_0^T \vartheta'(v) G(\zeta, v) u(v) dv \right| \\ &\leq \left| z(\zeta) - h(\zeta, z(\zeta)) - \int_0^T \vartheta'(v) G(\zeta, v) v(v) dv \right| + |h(\zeta, z(\zeta)) - h(\zeta, y(\zeta))| \\ &\quad + \int_0^T \vartheta'(v) G(\zeta, v) |v(v) - u(v)| dv \\ &\leq \epsilon \lambda_\Phi \Phi(\zeta) + c \|z - y\| + \frac{G_0 \|\mu\| (\vartheta(T) - \vartheta(0))}{1 - \|\mu\| \left(\mathrm{KT} + \left((\vartheta(T) - \vartheta(0))^{\alpha-\beta} / \Gamma(\alpha - \beta + 1) \right) \right)} \|z - y\|. \end{aligned} \tag{101}$$

Taking supremum for all $\zeta \in I$, we get

$$\|z - y\| \leq \epsilon \lambda_\Phi \Phi(\zeta) + \left(c + \frac{G_0 \|\mu\| (\vartheta(T) - \vartheta(0))}{1 - \|\mu\| \left(\mathrm{KT} + \left((\vartheta(T) - \vartheta(0))^{\alpha-\beta} / \Gamma(\alpha - \beta + 1) \right) \right)} \right) \|z - y\|. \tag{102}$$

If we take $\zeta = 1 - c - (G_0 \|\mu\| (\vartheta(T) - \vartheta(0)) / (1 - \|\mu\| (\mathrm{KT} + ((\vartheta(T) - \vartheta(0))^{\alpha-\beta} / \Gamma(\alpha - \beta + 1)))))$, then

$$\|z - y\| \leq \frac{\lambda_\Phi \Phi(\zeta)}{\zeta} \epsilon. \tag{103}$$

Therefore, (ICFDPs) (7)–(9) are Ulam–Hyers–Rassias stable with respect to Φ and with a real constant $c_{\mathfrak{F},\Phi} = (\lambda_\Phi / \zeta)$. This completes the proof. □

5. Special Cases and Example

The results we just established concerning the existence of a solution and its stability also hold for special cases. These fractional derivative classes are created by selecting an appropriate value for $\vartheta(\zeta)$ and taking into account the value of β.

In particular, we can deduce some existence results from our approach in the following discussion:

(i) When $\vartheta(\zeta) = \zeta$, then the obtained outcomes in the current paper incorporate the investigation of the following implicit fractional order differential problem (ICFDP):

$$\begin{aligned}
{}^{c}\mathscr{D}^{\alpha} y(\zeta) &= \mathfrak{F}\left(\zeta, y(\zeta), {}^{c}\mathscr{D}^{\beta} y(\zeta), \int_0^{\zeta} k(\zeta, v)\, {}^{c}\mathscr{D}^{\alpha} y(v) dv\right),\\
y(0) - y'(0) &= \frac{1}{\Gamma(\gamma)} \int_0^T (T - v)^{\gamma - 1} h_1(v, y(v)) dv,\\
y(T) + y'(T) &= \frac{1}{\Gamma(\gamma)} \int_0^T (T - v)^{\gamma - 1} h_2(v, y(v)) dv.
\end{aligned} \tag{104}$$

(ii) Also, if $\vartheta(\zeta) = \zeta$, $\mathfrak{F}(\zeta, x, y, z) = \mathfrak{F}(\zeta, x, y)$, $T = 1$, and $\beta = \alpha$, then we have the following implicit fractional-order differential equation which generalized the results studied in [37, 38]:

$$\begin{aligned}
{}^{c}\mathscr{D}^{\alpha} y(\zeta) &= \mathfrak{F}(\zeta, y(\zeta), {}^{c}\mathscr{D}^{\alpha} y(\zeta)),\\
y(0) - y'(0) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - v)^{\gamma - 1} h_1(v, y(v)) dv,\\
y(1) + y'(1) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - v)^{\gamma - 1} h_2(v, y(v)) dv.
\end{aligned} \tag{105}$$

(iii) Putting $\mathfrak{F}(\zeta, x, y, z) = a(\zeta) + x(\zeta).z(\zeta)$ and $T = 1$ in equation (106), we have the following quadratic implicit fractional differential equations with fractional integral boundary conditions:

$$\begin{aligned}
{}^{c}\mathscr{D}^{\alpha} y(\zeta) &= a(\zeta) + y(\zeta) \int_0^{\zeta} k(\zeta, v)\, {}^{c}\mathscr{D}^{\alpha} y(v) dv,\\
y(0) - y'(0) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - v)^{\gamma - 1} h_1(v, y(v)) dv,\\
y(1) + y'(1) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - v)^{\gamma - 1} h_2(v, y(v)) dv,
\end{aligned} \tag{106}$$

where $a \in L_1(I)$.

Example 1. Consider the following ICFDP:

$$ {}^{c}\mathscr{D}^{(7/5)} = \frac{2e^{-\zeta}}{69 + e^{-\zeta}} \left(3 + \frac{|y(\zeta)|}{1 + |y(\zeta)} + \frac{\left|{}^{c}\mathscr{D}^{(6/5)} y(\zeta)\right|}{1 + \left|{}^{c}\mathscr{D}^{(6/5)} y(\zeta)\right|} + \frac{\left|\int_0^1 e^{(\zeta + v)c}\mathscr{D}^{(7/5)} y(\zeta)\right|}{1 + \left|\int_0^1 e^{(\zeta + v)c}\mathscr{D}^{(7/5)} y(\zeta)\right|}\right), \tag{107}$$

with the following boundary conditions:

$$\begin{aligned}
y(0) - y'(0) &= \frac{1}{\Gamma(1/3)} \int_0^1 \vartheta'(v) (\vartheta(1) - \vartheta(v))^{-(2/3)} \frac{\cos(y(v))}{7} dv,\\
y(1) + y'(1) &= \frac{1}{\Gamma(1/3)} \int_0^1 \vartheta'(v) (\vartheta(1) - \vartheta(v))^{-(2/3)} \frac{\ln(y(v))}{9} dv,
\end{aligned} \tag{108}$$

such that

$$\begin{aligned}
\vartheta(\zeta) &= e^{2\zeta} + 1,\\
\vartheta(\zeta) &= 1 + 3e^{2\zeta}.
\end{aligned} \tag{109}$$

Setting

$$\mathfrak{F}(\zeta, u, v, w) = \frac{2e^{-\zeta}}{69 + e^{-\zeta}} \left(3 + \frac{|u(\zeta)|}{1 + |u(\zeta)|} + \frac{|v(\zeta)|}{1 + |v(\zeta)|} + \frac{|w(\zeta)|}{1 + |w(\zeta)|}\right), \tag{110}$$

it is clear that the function $\mathfrak{F}$ is jointly continuous. In fact, for any $u_1.v_1, w_1, u_2, v_2, w_2 \in R$, and for every $\zeta \in [0,1]$, we have

$$\begin{aligned}\left|\mathfrak{F}(\zeta, u_1, v_1, w_1) - \mathfrak{F}(\zeta, u_2, v_2, w_2)\right| &\le \frac{2e^{-\zeta}}{69+e^{-\zeta}}\left(\left|u_1-u_2\right| + \left|v_1-v_2\right| + \left|w_1-w_2\right|\right)\\ &\le \frac{2}{70}\left(\left|u_1-u_2\right| + \left|v_1-v_2\right| + \left|w_1-w_2\right|\right).\end{aligned} \tag{111}$$

Hence, the condition $(\mathscr{H}_2)$ holds with $\mu(\zeta) = (2e^{-\zeta}/(69+e^{-\zeta}))$ and $\|\mu\| = (2/70)$. On the other hand, we have

$$\begin{aligned}\left|h_1(\zeta, x(\zeta)) - h_1(\zeta, y(\zeta))\right| &\le \left|\frac{\cos x(v)}{7} - \frac{\cos y(v)}{7}\right| \le \frac{1}{7}|x(\zeta) - y(\zeta)|,\\ \left|h_2(\zeta, x(\zeta)) - h_2(\zeta, y(\zeta))\right| &\le \left|\frac{\ln(x(v))}{9} - \frac{\ln(y(v))}{9}\right| \le \frac{1}{9}|x(\zeta) - y(\zeta)|.\end{aligned} \tag{112}$$

Hence, the assumption $(\mathscr{H}_1)$ is satisfied with $k_1 = (1/7)$ and $k_2 = (1/9)$. This implies that if $T = 1$,

$$\begin{aligned}|h(\zeta, x(\zeta)) - h(\zeta, y(\zeta))| &\le \frac{(\vartheta(1) - \vartheta(0))^{\gamma}}{\Gamma(\gamma+1)}\left[k_1 + \frac{\left(\vartheta(\zeta) - \vartheta'(\zeta)\right)}{\vartheta(T)}(k_2 + k_1)\right]\|x - y\|\\ &\le 0.456403\|x - y\|.\end{aligned} \tag{113}$$

Hence, $h(\zeta, y(\zeta))$ is Lipschitz with constant $c = 0.456403 < 1$.

In addition, the Green's function is as follows:

$$G(\zeta, v) = \begin{cases} \dfrac{2e^{2\zeta}\left(-e^{2\zeta}-1\right)}{\left(3e^{2T}+1\right)\Gamma(2/5)\left(e^{2T}-e^{2v}\right)^{3/5}} + \dfrac{\left(-e^{2\zeta}-1\right)\left(e^{2T}-e^{2v}\right)^{2/5}}{\left(3e^{2T}+1\right)\Gamma(7/5)} + \dfrac{\left(e^{2\zeta}-e^{2v}\right)^{2/5}}{\Gamma(7/5)}, & \text{if } 0 \le v \le \zeta \le T,\\[2ex] \dfrac{\left(-e^{2\zeta}-1\right)\left(e^{2T}-e^{2v}\right)^{2/5}}{\left(3e^{2T}+1\right)\Gamma(7/5)} + \dfrac{2e^{2\zeta}\left(-e^{2\zeta}-1\right)}{\left(3e^{2T}+1\right)\Gamma(2/5)\left(e^{2T}-e^{2v}\right)^{3/5}}, & \text{if } 0 \le \zeta \le v \le \mathrm{T}.\end{cases} \tag{114}$$

Then, straightforward calculations with $T = 1$, $\alpha = (7/5)$, $\beta = (6/5)$, $c = 0.456403$, $\mu = (2/7)$, $K = e^2$, and $G_0 < 0.716747$ yield the following conditions:

$$\begin{gathered}\|\mu\|\left(K\,T + \frac{(\vartheta(T) - \vartheta(0))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right) < 1,\\ c + \frac{G_0\|\mu\|T}{1 - \|\mu\|\left(K\,T + (\vartheta(T) - \vartheta(0))^{\alpha-\beta}/\Gamma(\alpha-\beta+1)\right)} < 0.456403 + 0.0282471 = 0.484\,65 < 1.\end{gathered} \tag{115}$$

It follows from Theorem 12 that (ICFDPs) (7) and (8) have a unique mild solution on $I = [0, 1]$.

6. Conclusion

In our research, we achieved several key outcomes. Firstly, we established a connection between (ICFDPs) (7)–(9) and Volterra integration equation (24). Next, utilizing Banach's contraction principle and Krasnoselskii's fixed point theorem, we successfully demonstrated the existence and uniqueness of mild solutions for boundary value problems of implicit fractional order differential equations. Additionally, we verified Ulam–Hayers stability and other related stability types for (ICFDPs) (7)–(9). Notably, we presented a practical numerical example highlighting our findings' applicability. Furthermore, we emphasized the significance of our results, noting that different variations of $\vartheta(\zeta)$ and diverse values for β in (ICFDPs) (7)–(9) lead to various implicit fractional-order differential equations. In conclusion, our work represents a significant advancement in the field of qualitative analysis of fractional differential equations, introducing a generalized nonlocal boundary condition that investigates Ulam–Hyers stability within the framework of ϑ-Caputo fractional derivatives. Future work will delve into exploring coupled systems in greater depth.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors have contributed equally to this paper. The authors reviewed the results and approved the final version of the manuscript.

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