Hindawi
International Journal of Differential Equations
Volume 2023, Article ID 7663326, 28 pages
https://doi.org/10.1155/2023/7663326

Research Article

Solving Nonlinear Partial Differential Equations of Special Kinds of 3rd Order Using Balance Method and Its Models

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Received 22 October 2021; Revised 27 November 2021; Accepted 7 February 2023; Published 21 February 2023

Academic Editor: Amar Nath Chatterjee

Most nonlinear partial differential equations have many applications in the physical world. Finding solutions to nonlinear partial differential equations is not easily solvable and hence different modified techniques are applied to get solutions to such nonlinear partial differential equations. Among them, we considered the modified Korteweg–de Vries third order using the balance method and constructing its models using certain parameters. The method is successfully implemented in solving the stated equations. We obtained kind one and two soliton solutions and their graphical models are shown using mathematical software-12. The obtained results lead to shallow wave models. A few illustrative examples were presented to demonstrate the applicability of the models. Furthermore, physical and geometrical interpretations are considered for different parameters to investigate the nature of soliton solutions to their models. Finally, the proposed method is a standard, effective, and easily computable method for solving the modified Korteweg–de Vries equations and determining its perspective models.

1. Introduction

It is significantly important in nonlinear phenomena to search for exact solutions to nonlinear partial differential equations (NLPDEs). Exact solutions play a vital role in understanding various qualitative and quantitative features of nonlinear phenomena. There are diverse classes of interesting exact solutions, traveling wave solutions, and soliton solutions, but it often needs specific mathematical techniques to construct exact solutions due to the nonlinearity present in their dynamical nature [1–3]. The NLPDEs are widely used as models to depict many important complex physical phenomena in a variety of fields of science and engineering. Some nonlinear partial differential equations can be written as follows:

Burger's equation is as follows:

$$u_t + uu_x = \alpha u_{xx}. \tag{1}$$

The modified Korteweg–de Vries (KdV) equation is as follows:

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2}$$

The Kadomtsev–Petriashvili (KP) equation is as follows:

$$\left(u_t + auu_x + bu_{xxx}\right)_x + u_{yy} = 0. \tag{3}$$

The NLPDEs are fundamentally important because lots of mathematical physics models are often described by such wave phenomena and the investigation of traveling wave solutions is becoming more and more attractive in nonlinear sciences nowadays. However, NLPDEs are very difficult to solve explicitly, specifically, or in detail. As a result, many powerful methods have been proposed and developed for finding analytical solutions to nonlinear problems.

Some of the researchers used to solve NLPDEs are the simple equation method [4], the tanh-function and balance methods [5], the inverse scattering transform method [6], Backland and transformation [7], and other mathematical procedures such as combinations of the equation, transformation procedure, bilinear method, integration, and so on. Of these methods, the balance method was based on the higher order of partial derivative terms, the highest nonlinear terms, and parameters of the intended technique to solve the proposed problem of the modified KdV equation.

The Kadomtsev–Petviashvili (KP) equation is a partial differential equation that describes nonlinear wave motion [8], which is usually written as

$$\left(u_t + 6uu_x + u_{xxx}\right)_x - \delta u_{yy} = 0. \tag{4}$$

$\delta = \pm 1$ that can be applied to mathematical physics as a way to model water waves of long wavelengths. It is a two-dimensional generalization of the one-dimensional KdV (2). Like the KdV equation, the KP equation is completely integrable.

KdV were among the scholars who derived the KdV equation and one of the most famous nonlinear PDEs that arise in a great number of physical situations. It was derived from fluid mechanics to describe shallow water waves in a rectangular channel.

$$u_t + uu_x + \beta u_{xxx} = 0. \tag{5}$$

The positive parameter β refers to a dispersive effect.

The generalized modified KdV equation is given by the following equation:

$$u_t + u_{xxx} + u^p u_x = 0. \tag{6}$$

$x \in \mathbb{R}$, where p is a positive parameter. Formulated in the moving frame $x = \xi - ct$, the generalized modified KdV equation reads as follows:

$$u_t - cu_x + u_{xxx} + u^p u_x = 0, \tag{7}$$

where c denotes the wave speed.

The inverse scattering transform method is a method introduced that yields a solution to the initial value problem for an NLPDE with the help of the solutions to the direct and inverse scattering problems for an associated linear ordinary differential equation. The balance method (BM) is a powerful method for finding exact or approximate solutions to given NLPDEs which was presented by Wang and other scholars in recent years and was improved by Fan with others to make it more straight forward and simple [9]. In addition to this, as El-Wakil and others [10], used the balance method and auto-Backlund transformation. We used balance method for solving nonlinear partial differential equations, the modified Korteweg–de Vries equation. It is well known the KdV equations describe the unidirectional propagation of shallow water waves and a number of generalizations.

Let us consider a general nonlinear PDE, say, in two variables,

$$P\left(u, u_x, u_t, u_{xx}, u_{xt}, \ldots.\right) = 0, \tag{8}$$

where P is a polynomial function of its arguments and the subscripts denote the partial derivatives. The balance method consists of the following steps:

Step 1. Suppose that the solution of (8) is in the form of the following equation:

$$u = a_{pd}\left((\ln \omega)_{p,d} + \sum_{k,j=0,k+j\neq 0,p+d}^{k=m,j=n} a_{kj}\left(\ln(\omega)_{k,j} + a_{00}\right)\right)^q, \tag{9}$$

where $u = u(x,t)$, $\omega = \omega(x,t)$, $((\ln \omega))_{k,j} = \partial^{k+j}(\ln \omega(x,t))/\partial x^k \partial t^j$, and $a_{k,j}\,(k = 0, 1, 2, \ldots .p, j = 0, 1, 2, \ldots.., d)$ balance coefficients are constants to be determined. By balancing the highest nonlinear terms and the higher order partial derivative terms in this expression; p, d, and q can be determined.

Step 2. Substituting (9) into (8) and arranging it at each order of ω yield an equation as follows:

$$\sum_{l=o} f_l \omega^l \tag{10}$$

where $f_l\,(l = 0, 1, 2, \ldots.)$ are differential-algebraic expressions of ω and a_{kj}. Setting $f_l = 0$ and using a compatible condition $(\omega_{xt} = \omega_{tx})$ yield a set of differential-algebraic equations (DAEs).

Step 3. Solving the set of DAEs, ω and $a_{kj}\,(k = 0, 1, 2, \ldots .p, j = 0, 1, 2, \ldots.., d)$ can be determined. By substituting ω, p, d, q, and a_{kj} into (9), the exact solutions of (8) can be obtained.

Nonlinear type partial differential equations can be solved by many different methods such as the Hirota method [11, 12]. Inverse scattering transform method and similarity reductions method [13]. Among those methods, the modified Korteweg–de Vries equation was solved by a few of them with their limitations as it is. The limitation of complexities to find the solutions are one core of the problem. This study aimed to find an alternative solution to the modified KdV equation which simplifies the complexity of solving NLPDEs showing the 1 and 2– soliton solutions of the modified KdV equation, by applying the balance method.

A long wave characterizes geophysical fluid dynamics in shallow waters and deep oceans [14, 15]. From another perspective, the KdV equation appeared for the first time in 1895 as a one-dimensional evolution equation describing the waves of an along surface gravity propagation in a water shallow canal [16]. It also appeared in a numeral of diverse physical phenomena such as hydromagnetic collision-free waves, ion acoustic waves, stratified waves interior, lattice dynamics, and physics of plasma [17]. By applying the differential transform technique, the approximate result of coupled KdV has been studied in [5]. One of the most attractive and surprising wave phenomena is the creation of solitary waves or solitons. An adequate theory for solitary waves was developed, in the form of a modified wave equation known as KdV [18–21]. The modified KdV equation has defined a wide variety of physical phenomena used to model the interaction and evolution of nonlinear waves [22–24]. Korteweg and de Vries pursued the work done by Rayleigh and including the effect of surface tension leading to the now famous KdV equation given as follows: $\partial\eta/\partial\tau + 3/2\sqrt{g/h}\,\partial/\partial\zeta\,(1/2\eta^2 + 2/3\alpha\eta + 1/3\delta\partial^2\eta/\partial\zeta^2) = 0$, where η is the surface elevation, of the wave, above the equilibrium level h, α is a constant related to the uniform motion of the liquid, with the unit of length, g is the gravitational acceleration and $\delta = 1/3h^3 - Th/\rho g$, with T is surface tension.

The permanent profile of a soliton solution of the KdV equation results from the equilibrium between two effects; the nonlinearity which is proportional to uu_x or $\eta\eta_\zeta$ and Dispersion which is proportional to u_{xxx} or $\eta_{\zeta\zeta\zeta}$. The nonlinearity tends to comprise, to constitute the wave and dispersions spread it out. The dispersive term is proportional to h^3, it decreases whereas, the nonlinearity term, is proportional to $1/\sqrt{h}$, increases leading to a large wave. The balance method has been developed for the analytical solutions of nonlinear partial differential equations. Compared to other methods, the balance method gives high accuracy and has a wide range of applications in many mathematical problems and many physical phenomena. In this paper, we used the balance method in combination with the Hirota bilinear equation method to solve the modified form of the KdV equation considered using the balance method and its respective models.

2. Mathematical Formulation and Main Results

2.1. Preliminaries

2.1.1. Bilinear Operators. Bilinear differential operator D mapping a pair of functions $D(f.g)$. Unlike usual linear differential operators like $(\partial/\partial x)^n$, which maps a single function f into a single function $\partial^n f/\partial x^n$. That is D: $C^\infty x C^\infty \longrightarrow C^\infty$: $(f.g)\mapsto D(f.g)$

Remark 1. For all integers $m, n \geq 0$,

$$[D_t^m D_x^n (f.g)](x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t) g\left(x', t'\right)\Big|x' = x, t' = t, \tag{11}$$

where f is a function of x and t, whereas g is a function of x' and t'; for which $f \neq g$.

2.1.2. Fractional Derivative. A fractional derivative is a derivative of any order, real or complex in applied mathematics or mathematical analysis.

2.1.3. The ELzaki Transform. The Elzaki transform is a function of the form $h(x,t)$ with respect to t

$$\psi[h(x,t)] = \psi(x,s) = s\int_0^\infty h(x,t)\exp\left(-\frac{t}{s}\right)\mathrm{d}t, \tag{12}$$

where s is the complex number, $t > 0$.

2.2. Main Results

2.2.1. Method of Solution. Considering the general nonlinear PDEs given in (8), let us consider the modified KdV equation that was chosen as an example to illustrate the balance method. Suppose that the solution of (5) is (9). Now considering u_{xxx}, uu_x balancing it in the (5), it is required that $pq + 3 = 2pq + 1$ (by letting $u = pq$, $u_x = 1$, $u_{xx} = 2$, and $u_{xxx} = 3$, where 1, 2, and 3 represent the order of derivatives for this case or condition). Again balancing u_t and uu_x in (9), it gave that $dq + 1 = 2dq + 1$, which implies $dq = 2dq$. Choosing $p = 2$, and then solving for the rest of the variables, we obtained that $q = 1$ and $d = 0$. That means from the equation $pq + 3 = 2pq + 1$, by simplifying it we found $pq = 2$. Since we chosen $p = 2$, then we have $q = 1$. Again from the second condition $dq + 1 = 2dq + 1$, we have $d = 0$.

Hence (9) $u = a_{pd}((\ln\omega)_{p,d} + \sum_{k,j=0,k+j\neq 0,p+d}^{k=p,j=d} a_{kj} (\ln(\omega)_{k,j} + a_{00}))^q, \ldots$

$$u = a_{20}(\ln\omega)_{xx} + a_{10}(\ln(\omega)_x + a_{00}). \tag{13}$$

Also, this equation can also be expressed as follows:

$$u = a_{20}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right) + a_{10}\frac{w_x}{w} + a_{00},$$

$$\begin{aligned}
u_x &= \frac{\partial}{\partial x}\left[a_{20}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right) + a_{10}\frac{w_x}{w} + a_{00}\right] \\
&= a_{20}\left[\left(\frac{w_{xx}w - w_{xx}w_x}{w^2}\right) - \left(\frac{2w_x w_{xx} w^2 - w_x^2 2ww_x}{w^4}\right)\right] + a_{10}\left(\frac{w_{xx}w - w_x w_x}{w^2}\right) \\
&= a_{20}\left(\frac{w_{xxx}}{w} - \frac{w_{xx}w_x}{w^2} - \frac{2w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right) + a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right)
\end{aligned}$$

$$= a_{20}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right) + a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right), \tag{14}$$

$$u_x = a_{20}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right) + a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right),$$

$$\begin{aligned} u_t &= \frac{\partial}{\partial_t}\left[a_{20}\left(\ln \omega\right)_{xx} + a_{10}\left(\ln \omega\right)_x + a_{00}\right] \\ &= \frac{\partial}{\partial_t}\left[a_{20}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right) + a_{10}\frac{w_x}{w} + a_{00}\right] \\ &= a_{20}\left[\left(\frac{w_{xxt}w - w_{xx}w_t}{w^2}\right) - \left(\frac{\left(2w_xw_{xt}w^2 - w_x^2w_t\right)2w}{w^4}\right)\right] + a_{10}\left(\frac{w_{xt}w - w_xw_t}{w^2}\right) \\ &= a_{20}\left(\frac{w_{xxt}}{w} - \frac{w_{xx}w_t}{w^2} - \frac{2w_xw_{xt}}{w^2} + \frac{2w_x^2w_t}{w^3}\right) + a_{10}\left(\frac{w_{xt}}{w} - \frac{w_xw_t}{w^2}\right) \\ &= a_{20}\left(\frac{w_{xxt}}{w} - \frac{w_{xx}w_t + 2w_xw_{xt}}{w^2} + \frac{2w_x^2w_t}{w^3}\right) + a_{10}\left(\frac{w_{xt}}{w} - \frac{w_xw_t}{w^2}\right), \end{aligned} \tag{15}$$

$$u_t = a_{20}\left(\frac{w_{xxt}}{w} - \frac{w_{xx}w_t + 2w_xw_{xt}}{w^2} + \frac{2w_x^2w_t}{w^3}\right) + a_{10}\left(\frac{w_{xt}}{w} - \frac{w_xw_t}{w^2}\right). \tag{16}$$

From equation (15), we get

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial_x}U_x = \frac{\partial}{\partial_x}\left[a_{20}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right) + a_{10}\left(\frac{w_{xx}}{w} - \frac{w_x^2}{w^2}\right)\right] \\ &= a_{20}\left[\frac{w_{xxxx}w - w_{xxx}w_x}{w^2} - 3\left(\frac{\left(w_{xxx}w_x + w_{xx}w_{xx}\right)w^2 - \left(w_{xx}w_x\right)2ww_x}{w^4}\right)\right. \\ &\quad \left. +2\left(\frac{3w_x^2w_{xx}.w^3 - w^3x3w^2w_x}{w^6}\right)\right] + a_{10}\left[\left(\frac{w_{xxx}w - w_{xx}w_x}{w^2}\right) - \left(\frac{2w_xw_{xx}w^2 - w_x^2 2w.w_x}{w^4}\right)\right] \\ &= a_{20}\left(\frac{w_{xxxx}}{w} - \frac{w_{xxx}w_x}{w^2} - \frac{3w_{xxx}w_x}{w^2} - \frac{3w^2xx}{w^2} + \frac{6w_{xx}w_x^2}{w^3} + \frac{6w_{xx}w_x^2}{w^3} - \frac{6w^4}{w^4}\right) + a_{10}\left(\frac{w_{xxx}}{w} - \frac{w_{xx}w_x}{w^2} - \frac{2w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right) \\ &= a_{20}\left(\frac{w_{xxxx}}{w} - \frac{4w_{xxx}w_x + 3w^2xx}{w^2} + \frac{12w_{xx}w_x^2}{w^3} - \frac{6w^4}{w^4}\right) + a_{10}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right), \end{aligned} \tag{17}$$

$$u_{xx} = a_{20}\left(\frac{w_{xxxx}}{w} - \frac{4w_{xxx}w_x + 3w^2xx}{w^2} + \frac{12w_{xx}w_x^2}{w^3} - \frac{6w_x^4}{w^4}\right) + a_{10}\left(\frac{w_{xxx}}{w} - \frac{3w_{xx}w_x}{w^2} + \frac{2w_x^3}{w^3}\right). \tag{18}$$

From (18), we get

$$
\begin{aligned}
u_{xxx} &= \frac{\partial}{\partial_x}\left[a_{20}\left(\frac{w_{xxxx}}{w}-\frac{4w_{xxx}w_x+3w^2xx}{w^2}+\frac{12w_{xx}w_x^2}{w^3}-\frac{6w^4}{w^4}\right)+a_{10}\left(\frac{w_{xxx}}{w}-\frac{3w_{xx}w_x}{w^2}+\frac{2w_x^3}{w^3}\right)\right] \\
&= a_{20}\left[\frac{w_{xxxxx}w-w_{xxxx}w_x}{w^2}-4\left(\frac{(w_{xxxx}w_x+w_{xxx}w_x)w^2-(w_{xxx}w_x)2w.w_x}{w^4}\right)-3\left(\frac{2w_{xx}w_{xxx}w^2-w_{xx}^2 2w.w_x}{w^4}\right)\right. \\
&\quad \left.+12\left(\frac{\left(w_{xxx}w_x^2+w_{xx}2w_xw_{xx}\right)w^3-\left(w_{xx}w_x^2 3w^2w_x\right)}{w^6}\right)-6\left(\frac{\left(4w_x^3w_{xx}w^4-w_x^4 4w^3w_x\right)}{w^8}\right)\right] \\
&\quad +a_{10}\frac{w_{xxxx}w-w_{xxx}w_x}{w^2}-3\left(\frac{(w_{xxx}w_x+w_{xx}w_{xx})w^2-(w_{xx}w_x2w.w_x)}{w^4}\right)+2\left(\frac{\left(3w_x^2w_{xx}w^3-w_x^3 3w^2w_x\right)}{w^6}\right)\Bigg] \\
&= a_{20}\left[\frac{w_{xxxxx}}{w}-\frac{w_{xxxx}w_x}{w^2}-4\frac{w_{xxxx}w_x}{w^2}-4\frac{w_{xxx}w_{xx}}{w^2}+8\frac{w_{xxx}w_x^2}{w^3}-6\frac{w_{xxx}w_{xx}}{w^2}+6\frac{w_{xx}^2w_x}{w^3}+12\frac{w_{xxx}w_x^2}{w^3}+24\frac{w_{xx}^2w_x}{w^3}\right. \\
&\quad \left.-36\frac{w_{xx}w_x^3}{w^4}-24\frac{w_{xx}w_x^3}{w^4}+24\frac{w_x^5}{w^5}\right]+a_{10}\left[\frac{w_{xxxx}}{w}-\frac{w_{xxx}w_x}{w^2}-3\frac{w_{xxx}w_x}{w^2}-3\frac{w_{xx}^2}{w^2}+6\frac{w_{xx}w_x^2}{w^3}+6\frac{w_{xx}w_x^2}{w^3}-6\frac{w_x^4}{w^4}\right] \\
&= a_{20}\left[\frac{w_{xxxxx}}{w}-5\frac{w_{xxxx}w_x}{w^2}-10\frac{w_{xxx}w_{xx}}{w^2}+20\frac{w_{xxx}w_x^2}{w^3}+30\frac{w_{xx}^2w_x}{w^3}-60\frac{w_{xx}w_x^3}{w^4}-24\frac{w_x^5}{w^5}\right] \\
&\quad +a_{10}\left[\frac{w_{xxxx}}{w}-4\frac{w_{xxx}w_x}{w^2}-3\frac{w_{xx}^2}{w^2}+12\frac{w_{xx}w_x^2}{w^3}-6\frac{w_x^4}{w^4}\right],
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
u_{xxx} = a_{20}&\left[\frac{w_{xxxxx}}{w}-5\frac{w_{xxxx}w_x}{w^2}-10\frac{w_{xxx}w_{xx}}{w^2}+20\frac{w_{xxx}w_x^2}{w^3}+30\frac{w_{xx}^2w_x}{w^3}-60\frac{w_{xx}w_x^3}{w^4}24\frac{w_x^5}{w^5}\right] \\
&+a_{10}\left[\frac{w_{xxxx}}{w}-4\frac{w_{xxx}w_x}{w^2}-3\frac{w_{xx}^2}{w^2}+12\frac{w_{xx}w_x^2}{w^3}-6\frac{w_x^4}{w^4}\right].
\end{aligned}
\tag{20}
$$

Now substituting equations (14), (15), (16), and (20) into (5), we have;

$$
\begin{aligned}
&a_{20}\left(\frac{w_{xxt}}{w}-\frac{w_{xx}w_t+2w_xw_{xt}}{w^2}+\frac{2w_x^2w_t}{w^3}\right)+a_{10}\left(\frac{w_{xt}}{w}-\frac{w_xw_t}{w^2}\right) \\
&+\left[a_{20}\left(\frac{w_{xx}}{w}-\frac{w_x^2}{w^2}\right)+a_{10}\frac{w_x}{w}+a_{00}\right]\left[a_{20}\left(\frac{w_{xxx}}{w}-\frac{3w_{xx}w_x}{w^2}+\frac{2w_x^3}{w^3}\right)+\left(a_{10}\frac{w_{xx}}{w}-\frac{w_x^2}{w^2}\right)\right]+ \\
&\beta\left[a_{20}\left[\frac{w_{xxxxx}}{w}-5\frac{w_{xxxx}w_x}{w^2}-10\frac{w_{xxx}w_{xx}}{w^2}+20\frac{w_{xxx}w_x^2}{w^3}+30\frac{w_{xx}^2w_x}{w^3}-60\frac{w_{xx}w_x^3}{w^4}-24\frac{w_x^5}{w^5}\right]\right. \\
&\left.+a_{10}\left[\frac{w_{xxxx}}{w}-4\frac{w_{xxx}w_x}{w^2}-3\frac{w_{xx}^2}{w^2}+12\frac{w_{xx}w_x^2}{w^3}-6\frac{w_x^4}{w^4}\right]\right]=0
\end{aligned}
$$

$$a_{20}\frac{w_{xxt}}{w}-a_{20}\left(\frac{w_{xx}w_t+2w_xw_{xt}}{w^2}\right)+a_{20}\frac{2w_x^2w_t}{w^3}+a_{10}\frac{w_{xt}}{w}-a_{10}\frac{w_xw_t}{w^2}+$$
$$\left[a_{20}\frac{w_{xx}}{w}-a_{20}\frac{w_x^2}{w^2}+a_{10}\frac{w_x}{w}+a_{00}\right]\left[a_{20}\frac{w_{xxx}}{w}-a_{20}\frac{3w_{xx}w_x}{w^2}+a_{20}\frac{2w_x^3}{w^3}+a_{10}\frac{w_{xx}}{w}-a_{10}\frac{w_x^2}{w^2}\right]+$$
$$\beta\left[a_{20}\frac{w_{xxxxx}}{w}-a_{20}5\frac{w_{xxxx}w_x}{w^2}-a_{20}10\frac{w_{xxx}w_{xx}}{w^2}+a_{20}20\frac{w_{xxx}w_x^2}{w^3}+a_{20}30\frac{w_{xx}^2w_x}{w^3}-a_{20}60\frac{w_{xx}w_x^3}{w^4}\right.$$
$$\left.-a_{20}24\frac{w_x^5}{w^5}+a_{10}\frac{w_{xxxx}}{w}-a_{10}4\frac{w_{xxx}w_x}{w^2}-a_{10}3\frac{w_{xx}^2}{w^2}+a_{10}12\frac{w_{xx}w_x^2}{w^3}-a_{10}6\frac{w_x^4}{w^4}\right]=0$$

$$a_{20}\frac{w_{xxt}}{w}-a_{20}\left(\frac{w_{xx}w_t+2w_xw_{xt}}{w^2}\right)+a_{20}\frac{2w_x^2w_t}{w^3}+a_{10}\frac{w_{xt}}{w}-a_{10}\frac{w_xw_t}{w^2}+$$
$$\left[a_{20}\frac{w_{xx}}{w}\right]a_{20}\frac{w_{xxx}}{w}-a_{20}\frac{3w_{xx}w_x}{w^2}+a_{20}\frac{2w_x^3}{w^3}+a_{10}\frac{w_{xx}}{w}-a_{10}\frac{w_x^2}{w^2}$$
$$\left[-a_{20}\frac{w_x^2}{w^2}\right]\left[a_{20}\frac{w_{xxx}}{w}-a_{20}\frac{3w_{xx}w_x}{w^2}+a_{20}\frac{2w_x^3}{w^3}+a_{10}\frac{w_{xx}}{w}-a_{10}\frac{w_x^2}{w^2}\right]+$$
$$\left[a_{10}\frac{w_x}{w}\right]\left[a_{20}\frac{w_{xxx}}{w}\right]-\left[a_{20}\frac{3w_{xx}w_x}{w^2}+a_{20}\frac{2w_x^3}{w^3}+a_{10}\frac{w_{xx}}{w}-a_{10}\frac{w_x^2}{w^2}\right]+$$
$$\left[a_{00}\right]\left[a_{20}\frac{w_{xxx}}{w}-a_{20}\frac{3w_{xx}w_x}{w^2}+a_{20}\frac{2w_x^3}{w^3}+a_{10}\frac{w_{xx}}{w}-a_{10}\frac{w_x^2}{w^2}\right]+$$
$$\beta\begin{bmatrix}a_{20}\frac{w_{xxxxx}}{w}-a_{20}5\frac{w_{xxxx}w_x}{w^2}-a_{20}10\frac{w_{xxx}w_{xx}}{w^2}+a_{20}20\frac{w_{xxx}w_x^2}{w^3}+a_{20}30\frac{w_{xx}^2w_x}{w^3}\\-a_{20}60\frac{w_{xx}w_x^3}{w^4}-a_{20}24\frac{w_x^5}{w^5}+a_{10}\frac{w_{xxxx}}{w}-a_{10}4\frac{w_{xxx}w_x}{w^2}-a_{10}3\frac{w_{xx}^2}{w^2}+a_{10}12\frac{w_{xx}w_x^2}{w^3}-a_{10}6\frac{w_x^4}{w^4}\end{bmatrix}=0$$

$$a_{20}\frac{w_{xxt}}{w}-a_{20}\left(\frac{w_{xx}w_t+2w_xw_{xt}}{w^2}\right)+a_{20}\frac{2w_x^2w_t}{w^3}+a_{10}\frac{w_{xt}}{w}-a_{10}\frac{w_xw_t}{w^2}+$$
$$\left[a_{20}\frac{w_{xx}}{w}a_{20}\frac{w_{xxx}}{w}-a_{20}\frac{w_{xx}}{w}a_{20}\frac{3w_{xx}w_x}{w^2}+a_{20}\frac{w_{xx}}{w}a_{20}\frac{2w_x^3}{w^3}+a_{20}\frac{w_{xx}}{w}a_{10}\frac{w_{xx}}{w}-a_{20}\frac{w_{xx}}{w}a_{10}\frac{w_x^2}{w^2}\right]-$$
$$\left[-a_{20}\frac{w_x^2}{w^2}a_{20}\frac{w_{xxx}}{w}-a_{20}\frac{w_x^2}{w^2}a_{20}\frac{3w_{xx}w_x}{w^2}+a_{20}\frac{w_x^2}{w^2}a_{20}\frac{2w_x^3}{w^3}+a_{20}\frac{w_x^2}{w^2}a_{10}\frac{w_{xx}}{w}-a_{20}\frac{w_x^2}{w^2}a_{10}\frac{w_x^2}{w^2}\right]-$$
$$\left[a_{10}\frac{w_x}{w}a_{20}\frac{w_{xxx}}{w}-a_{10}\frac{w_x}{w}a_{20}\frac{3w_{xx}w_x}{w^2}+a_{10}\frac{w_x}{w}a_{20}\frac{2w_x^3}{w^3}+a_{10}\frac{w_x}{w}a_{10}\frac{w_{xx}}{w}-a_{10}\frac{w_x}{w}a_{10}\frac{w_x^2}{w^2}\right]+$$
$$\left[a_{00}a_{20}\frac{w_{xxx}}{w}-a_{00}a_{20}\frac{3w_{xx}w_x}{w^2}+a_{00}a_{20}\frac{2w_x^3}{w^3}+a_{00}a_{10}\frac{w_{xx}}{w}-a_{00}a_{10}\frac{w_x^2}{w^2}\right]+$$
$$\beta\begin{bmatrix}a_{20}\frac{w_{xxxxx}}{w}-a_{20}5\frac{w_{xxxx}w_x}{w^2}-a_{20}10\frac{w_{xxx}w_{xx}}{w^2}+a_{20}20\frac{w_{xxx}w_x^2}{w^3}+a_{20}30\frac{w_{xx}^2w_x}{w^3}-a_{20}60\frac{w_{xx}w_x^3}{w^4}\\-a_{20}24\frac{w_x^5}{w^5}+a_{10}\frac{w_{xxxx}}{w}-a_{10}4\frac{w_{xxx}w_x}{w^2}-a_{10}3\frac{w_{xx}^2}{w^2}+a_{10}12\frac{w_{xx}w_x^2}{w^3}-a_{10}6\frac{w_x^4}{w^4}\end{bmatrix}=0$$

$$a_{20}\frac{w_{xxt}}{w}-a_{20}\left(\frac{w_{xx}w_t+2w_xw_{xt}}{w^2}\right)+a_{20}\frac{2w_x^2w_t}{w^3}+a_{10}\frac{w_{xt}}{w}-a_{10}\frac{w_xw_t}{w^2}$$
$$+\left[a_{20}^2\frac{w_{xx}w_{xxx}}{w^2}-a_{20}^2\frac{3w_{xx}^2w_x}{w^3}+a_{20}^2\frac{2w_x^3w_{xx}}{w^4}+a_{20}a_{10}\frac{w_{xx}^2}{w^2}-a_{20}a_{10}\frac{w_x^2w_{xx}}{w^3}\right]$$

$$-\left[a_{20}^2\frac{w_x^2 w_{xxx}}{w^3}+a_{20}^2\frac{3w_{xx}w_x^3}{w^4}-a_{20}^2\frac{2w_x^5}{w^5}-a_{20}a_{10}\frac{w_{xx}w_x^2}{w^3}+a_{20}a_{10}\frac{w_x^4}{w^4}\right]$$
$$+\left[a_{10}a_{20}\frac{w_x w_{xxx}}{w^2}-a_{10}a_{20}\frac{3w_{xx}w_x^2}{w^3}+a_{10}a_{20}\frac{2w_x^4}{w^4}+a_{10}^2\frac{w_x w_{xx}}{w^2}-a_{10}^2\frac{w_x^3}{w^3}\right]$$
$$+\left[a_{00}a_{20}\frac{w_{xxx}}{w}-a_{00}a_{20}\frac{3w_{xx}w_x}{w^2}+a_{00}a_{20}\frac{2w_x^3}{w^3}+a_{00}a_{10}\frac{w_{xx}}{w}-a_{00}a_{10}\frac{w_x^2}{w^2}\right]+$$
$$\beta\left[\begin{array}{c}a_{20}\frac{w_{xxxxx}}{w}-a_{20}5\frac{w_{xxxx}w_x}{w^2}-a_{20}10\frac{w_{xxx}w_{xx}}{w^2}+a_{20}20\frac{w_{xxx}w_x^2}{w^3}+a_{20}30\frac{w_{xx}^2w_x}{w^3}-a_{20}60\frac{w_{xx}w_x^3}{w^4}-a_{20}24\frac{w_x^5}{w^5}\\ +a_{10}\frac{w_{xxxx}}{w}-a_{10}4\frac{w_{xxx}w_x}{w^2}-a_{10}3\frac{w_{xx}^2}{w^2}+a_{10}12\frac{w_{xx}w_x^2}{w^3}-a_{10}6\frac{w_x^4}{w^4}\end{array}\right]=0$$

$$\begin{aligned}
&a_{20}\frac{w_{xxt}}{w}-a_{20}\left(\frac{w_{xx}w_t+2w_xw_{xt}}{w^2}\right)+a_{20}\frac{2w_x^2w_t}{w^3}+a_{10}\frac{w_{xt}}{w}-a_{10}\frac{w_xw_t}{w^2}+a_{20}^2\frac{w_{xx}w_{xxx}}{w^2}-a_{20}^2\frac{3w_{xx}^2w_x}{w^3}+a_{20}^2\frac{2w_x^3w_{xx}}{w^4}\\
&+a_{20}a_{10}\frac{w_{xx}^2}{w^2}-a_{20}a_{10}\frac{w_x^2w_{xx}}{w^3}-a_{20}^2\frac{w_x^2w_{xxx}}{w^3}+a_{20}^2\frac{3w_{xx}w_x^3}{w^4}-a_{20}^2\frac{2w_x^5}{w^5}-a_{20}a_{10}\frac{w_{xx}w_x^2}{w^3}+a_{20}a_{10}\frac{w_x^4}{w^4}+a_{10}a_{20}\frac{w_xw_{xxx}}{w^2}\\
&-a_{10}a_{20}\frac{3w_{xx}w_x^2}{w^3}+a_{10}a_{20}\frac{2w_x^4}{w^4}+a_{10}^2\frac{w_xw_{xx}}{w^2}-a_{10}^2\frac{w_x^3}{w^3}+a_{00}a_{20}\frac{w_{xxx}}{w}-a_{00}a_{20}\frac{3w_{xx}w_x}{w^2}+a_{00}a_{20}\frac{2w_x^3}{w^3}+a_{00}a_{10}\frac{w_{xx}}{w}\\
&-a_{00}a_{10}\frac{w_x^2}{w^2}+\beta a_{20}\frac{w_{xxxxx}}{w}-\beta a_{20}5\frac{w_{xxxx}w_x}{w^2}-\beta a_{20}10\frac{w_{xxx}w_{xx}}{w^2}+\beta a_{20}20\frac{w_{xxx}w_x^2}{w^3}+\beta a_{20}30\frac{w_{xx}^2w_x}{w^3}-\beta a_{20}60\frac{w_{xx}w_x^3}{w^4}\\
&-\beta a_{20}24\frac{w_x^5}{w^5}+\beta a_{10}\frac{w_{xxxx}}{w}-\beta a_{10}4\frac{w_{xxx}w_x}{w^2}-\beta a_{10}3\frac{w_{xx}^2}{w^2}+\beta a_{10}12\frac{w_{xx}w_x^2}{w^3}-\beta a_{10}6\frac{w_x^4}{w^4}=0
\end{aligned}\qquad(21)$$

Now equating the coefficients of w_x^5/w^5 and w_x^4/w^4 to zero on the left-hand side, we have $a_{20}=12\beta$ and $a_{10}=0$. This is to mean that; equating $a_{20}\beta 24w_x^5/w^5=0$, $a_{10}\beta 6w_x^4/w^4=0$, $a_{10}a_{20}2w_x^4/w^4=0$, $-2a_{20}^2w_x^5/w^5=0$, and $a_{10}a_{20}w_x^4/w^4=0$. Now considering the two equations $a_{20}\beta 24w_x^5/w^5=0$ and $-2a_{20}^2w_x^5/w^5=0$, it is possible to solve for a_{20}. Since $\beta\neq 0$ and $w\neq 0$ $24a_{20}\beta=0$ and $-2a_{20}^2=0$, we solved for the reset a_{20} as $\begin{cases}24a_{20}\beta=0\\ -2a_{20}^2=0\end{cases}$

Adding these two equations together and solving for a_{20}, as $24a_{20}\beta-2a_{20}^2=0$ $2a_{20}[12\beta-a_{20}]=0$. From this, we can get $a_{20}=12\beta$

Again considering the equations; $-6\beta a_{10}w_x^4/w^4=0, 2a_{10}a_{20}w_x^4/w^4=0$, and $a_{10}a_{20}w_x^4/w^4=0$.

And adding them together using a system of equations as $\begin{cases}-6\beta a_{10}=0,\\ 2a_{10}a_{20}=0,\\ a_{10}a_{20}=0.\end{cases}$ we have $3a_{20}a_{10}-6\beta a_{10}=0$. $3a_{20}a_{10}=6\beta a_{10}$, Simplifying it we get, $a_{20}a_{10}=2\beta a_{10}$. Substituting $a_{20}=12\beta$ into this equation we get, $12\beta a_{10}=2a_{10}$. Which give us $10\beta a_{10}=0$, $\beta\neq 0$. And hence it implies that $a_{10}=0$

Substituting $a_{20}=12\beta$ and $a_{10}=0$ into the following equation: $u=a_{20}\ln(\omega)_{xx}+a_{10}\ln(\omega)_x+a_{00}$, we get $u=12\,(\ln\omega)_{xx}+0\ln(\omega)_x+a_{00}=12(\ln\omega)_{xx}+a_{00}=12\beta(w_{xx}/w-w_x^2/w^2)+a_{00}$, where a_{00} is any arbitrary constant.

$$u=12(\ln\omega)_{xx}+a_{00}=12\beta\left(\frac{w_{xx}}{w}-\frac{w_x^2}{w^2}\right)+a_{00}.\qquad(22)$$

Putting (22) into (5), we have the following results, where, $a_{20}=12\beta$ and $a_{10}=0$.

$$\begin{aligned}
&12\beta\frac{w_{xxt}}{w}-12\beta\frac{w_{xx}w_t}{w^2}-12\beta\frac{2w_xw_{xt}}{w^2}+12\beta\frac{2w_x^2w_t}{w^3}+0\left(\frac{w_{xt}}{w}\right)-0\left(\frac{w_xw_t}{w^2}\right)+(12\beta)^2\frac{w_{xx}w_{xxx}}{w^2}-(12\beta)^2\frac{3w_{xx}^2w_x}{w^3}\\
&+(12\beta)^2\frac{2w_x^3w_{xx}}{w^4}+0\left(12\beta\frac{w_{xx}^2}{w^2}\right)-0\left(12\beta\frac{w_x^2w_{xx}}{w^3}\right)-(12\beta)^2\frac{w_x^2w_{xxx}}{w^3}+(12\beta)^2\frac{3w_{xx}w_x^3}{w^4}-(12\beta)^2\frac{2w_x^5}{w^5}-0\left(12\beta\frac{w_{xx}w_x^2}{w^3}\right)\\
&+0\left(12\beta\frac{w_x^4}{w^4}\right)+0\left(12\beta\frac{w_xw_{xxx}}{w^2}\right)-0\left(12\beta\frac{3w_{xx}w_x^2}{w^3}\right)+0\left(12\beta\frac{2w_x^4}{w^4}\right)+0\left(\frac{w_xw_{xx}}{w^2}\right)-0\left(\frac{w_x^3}{w^3}\right)+
\end{aligned}$$

$$
\begin{aligned}
& a_{00}12\beta\frac{w_{xxx}}{w}-a_{00}12\beta\frac{3w_{xx}w_x}{w^2}+a_{00}12\beta\frac{2w_x^3}{w^3}+0\left(a_{00}\frac{w_{xx}}{w}\right)-0\left(a_{00}\frac{w_x^2}{w^2}\right)+\beta 12\beta\frac{w_{xxxxx}}{w}-12\left(\beta\right)^2 5\frac{w_{xxxx}w_x}{w^2} \\
& -12\left(\beta\right)^2 10\frac{w_{xxx}w_{xx}}{w^2}+12\left(\beta\right)^2 20\frac{w_{xxx}w_x^2}{w^3}+12\left(\beta\right)^2 30\frac{w_{xx}^2w_x}{w^3}-12\left(\beta\right)^2 60\frac{w_{xx}w_x^3}{w^4}-12\left(\beta\right)^2 24\frac{w_x^5}{w^5} \\
& +0\left(\beta\frac{w_{xxxx}}{w}\right)-0\left(4\beta\frac{w_{xxx}w_x}{w^2}\right)-0\left(3\beta\frac{w_{xx}^2}{w^2}\right)+0\left(12\beta\frac{w_{xx}w_x^2}{w^3}\right)-0\left(6\beta\frac{w_x^4}{w^4}\right)=0 \\
& 12\beta\left(\frac{w_{xxt}}{w}-\frac{w_{xx}w_t}{w^2}-\frac{2w_xw_{xt}}{w^2}+\frac{2w_x^2w_t}{w^3}\right)+144\left(\beta\right)^2\frac{w_{xx}w_{xxx}}{w^2}-144\left(\beta\right)^2\frac{3w_{xx}^2w_x}{w^3}+144\left(\beta\right)^2\frac{2w_x^3w_{xx}}{w^4} \\
& +144\left(\beta\right)^2\frac{w_x^2w_{xxx}}{w^3}+144\left(\beta\right)^2\frac{3w_{xx}w_x^3}{w^4}-144\left(\beta\right)^2\frac{2w_x^5}{w^5}-a_{00}12\beta\frac{w_{xxx}}{w} \\
& -a_{00}36\beta\frac{w_{xx}w_x}{w^2}+a_{00}24\beta\frac{w_x^3}{w^3}+\beta 12\beta\frac{w_{xxxxx}}{w}-60\left(\beta\right)^2\frac{w_{xxxx}w_x}{w^2}-120\left(\beta\right)^2\frac{w_{xxx}w_{xx}}{w^2} \\
& +240\left(\beta\right)^2\frac{w_{xxx}w_x^2}{w^3}+360\left(\beta\right)^2\frac{w_{xx}^2w_x}{w^3}-720\left(\beta\right)^2\frac{w_{xx}w_x^3}{w^4}-288\left(\beta\right)^2\frac{w_x^5}{w^5}=0 \\
& 12\beta\left(\frac{w_{xxt}}{w}-\frac{w_{xx}w_t}{w^2}-\frac{2w_xw_{xt}}{w^2}+\frac{2w_x^2w_t}{w^3}\right)+12\beta\left(12\beta\frac{w_{xx}w_{xxx}}{w^2}\right)-12\beta\left(36\beta\frac{w_{xx}^2w_x}{w^3}\right)+12\beta\left(24\beta\frac{w_x^3w_{xx}}{w^4}\right) \\
& +12\beta\left(12\beta\frac{w_x^2w_{xxx}}{w^3}\right)+12\beta\left(36\beta\frac{w_{xx}w_x^3}{w^4}\right)-12\beta\left(24\beta\frac{w_x^5}{w^5}\right)+12\beta\left(a_{00}\frac{w_{xxx}}{w}-a_{00}3\frac{w_{xx}w_x}{w^2}+a_{00}2\frac{w_x^3}{w^3}\right) \\
& +12\beta\left(\beta\frac{w_{xxxxx}}{w}-5\beta\frac{w_{xxxx}w_x}{w^2}-10\beta\frac{w_{xxx}w_{xx}}{w^2}+20\beta\frac{w_{xxx}w_x^2}{w^3}+30\beta\frac{w_{xx}^2w_x}{w^3}-60\beta\frac{w_{xx}w_x^3}{w^4}-24\beta\frac{w_x^5}{w^5}\right)=0 \\
& 12\beta\left(\frac{w_{xxt}}{w}-\frac{w_{xx}w_t}{w^2}-\frac{2w_xw_{xt}}{w^2}+\frac{2w_x^2w_t}{w^3}\right)+12\beta\left(a_{00}\frac{w_{xxx}}{w}-a_{00}3\frac{w_{xx}w_x}{w^2}+a_{00}2\frac{w_x^3}{w^3}\right)+12\beta\left[12\beta\frac{w_{xx}w_{xxx}}{w^2}-36\beta\frac{w_{xx}^2w_x}{w^3}\right. \\
& +24\beta\frac{w_x^3w_{xx}}{w^4}+12\beta\frac{w_x^2w_{xxx}}{w^3}+36\beta\frac{w_{xx}w_x^3}{w^4}-24\beta\frac{w_x^5}{w^5}+\beta\frac{w_{xxxxx}}{w}-5\beta\frac{w_{xxxx}w_x}{w^2}-10\beta\frac{w_{xxx}w_{xx}}{w^2} \\
& \left.+20\beta\frac{w_{xxx}w_x^2}{w^3}+30\beta\frac{w_{xx}^2w_x}{w^3}-60\beta\frac{w_{xx}w_x^3}{w^4}-24\beta\frac{w_x^5}{w^5}\right]=0 \\
& 12\beta\left(\frac{w_{xxt}}{w}-\frac{w_{xx}w_t}{w^2}-\frac{2w_xw_{xt}}{w^2}+\frac{2w_x^2w_t}{w^3}\right)+12\beta\left(a_{00}\frac{w_{xxx}}{w}-a_{00}3\frac{w_{xx}w_x}{w^2}+a_{00}2\frac{w_x^3}{w^3}\right)+12\beta\left[12\beta\frac{w_{xx}w_{xxx}}{w^2}-10\beta\frac{w_{xxx}w_{xx}}{w^2}\right. \\
& -36\beta\frac{w_{xx}^2w_x}{w^3}+30\beta\frac{w_{xx}^2w_x}{w^3}+24\beta\frac{w_x^3w_{xx}}{w^4}+36\beta\frac{w_{xx}w_x^3}{w^4}+-60\beta\frac{w_{xx}w_x^3}{w^4}-12\beta\frac{w_x^2w_{xxx}}{w^3}+20\beta\frac{w_{xxx}w_x^2}{w^3}-24\beta\frac{w_x^5}{w^5} \\
& \left.+24\beta\frac{w_x^5}{w^5}+\beta\frac{w_{xxxxx}}{w}-5\beta\frac{w_{xxxx}w_x}{w^2}\right]=0 \\
& 12\beta\left(\frac{w_{xxt}}{w}-\frac{w_{xx}w_t}{w^2}-\frac{2w_xw_{xt}}{w^2}+\frac{2w_x^2w_t}{w^3}\right)+12\beta\left(a_{00}\frac{w_{xxx}}{w}-a_{00}3\frac{w_{xx}w_x}{w^2}+a_{00}2\frac{w_x^3}{w^3}\right)+12\beta\left[2\beta\frac{w_{xx}w_{xxx}}{w^2}\right. \\
& \left.-6\beta\frac{w_{xx}^2w_x}{w^3}+60\beta\frac{w_x^3w_{xx}}{w^4}-60\beta\frac{w_{xx}w_x^3}{w^4}+8\beta\frac{w_x^2w_{xxx}}{w^3}+\beta\frac{w_{xxxxx}}{w}-5\beta\frac{w_{xxxx}w_x}{w^2}\right]=0 \\
& 12\beta\left(\frac{w_{xxt}}{w}-\frac{w_{xx}w_t}{w^2}-\frac{2w_xw_{xt}}{w^2}+\frac{2w_x^2w_t}{w^3}\right)+12\beta\left(a_{00}\frac{w_{xxx}}{w}-a_{00}3\frac{w_{xx}w_x}{w^2}+a_{00}2\frac{w_x^3}{w^3}\right) \\
& +12\beta\left[\beta\left[\frac{w_{xxxxx}}{w}+\frac{2w_{xxx}w_{xx}-5w_{xxxx}w_x}{w^2}+\frac{8w_{xxx}w_x^2-6w_{xx}^2w_x}{w^3}\right]\right]=0 \\
& 12\left(\beta K_1+K_2+K_3\right)=0,
\end{aligned}
\tag{23}
$$

where

$$K_1 = \frac{w_{xxt}}{w} - \frac{w_{xx}w_t}{w^2} - \frac{2w_x w_{xt}}{w^2} + \frac{2w_x^2 w_t}{w^3},$$
$$K_2 = a_{00}\frac{w_{xxx}}{w} - a_{00}3\frac{w_{xx}w_x}{w^2} + a_{00}2\frac{w_x^3}{w^3}, \tag{24}$$
$$K_3 = \beta\left[\frac{w_{xxxxx}}{w} + \frac{2w_{xxx}w_{xx} - 5w_{xxxx}w_x}{w^2} + \frac{8w_{xxx}w_x^2 - 6w_{xx}^2 w_x}{w^3}\right],$$

$$12\left(\beta K_1 + K_2 + K_3\right) = 0. \tag{25}$$

Simplifying (25) and integrating once with respect to x, we get

$$\frac{\partial}{\partial x}\left[\frac{\left(w_{xt}w - w_x w_t\right) + \beta\left(w_{xxxx}w - 4w_x w_{xxx} + 3w_{xx}^2\right) + a_{00}\left(w_{xx}w - w_x^2\right)}{w^2}\right] = 0. \tag{26}$$

Equations (20) and (26) are identical with

$$\frac{\left(w_{xt}w - w_x w_t\right) + \beta\left(w_{xxxx}w - 4w_x w_{xxx} + 3w_{xx}^2\right) + a_{00}\left(w_{xx}w - w_x^2\right)}{w^2} - C(t)w^2 = 0, \tag{27}$$

where $C(t)$ is an arbitrary function of t, and a_{00} is an arbitrary constant. Especially, taking $C(t)$ as zero in equation (27), we get the bilinear equation of (5) as follows:

$$\frac{\left(w_{xt}w - w_x w_t\right) + \beta\left(w_{xxxx}w - 4w_x w_{xxx} + 3w_{xx}^2\right) + a_{00}\left(w_{xx}w - w_x^2\right)}{w^2} = 0. \tag{28}$$

Equation (28) can be written concisely in terms of D -operator as

$$\left(D_x D_t + \beta D_X^4 + a_{00}D_x^2\right)(w, w) = 0, \tag{29}$$

where $D_x^m D_t^n (a.b) = (\partial x - \partial x')^m (\partial t - \partial t')^n a(x,t) b(x', t')|$ $x = x', t = t'$.

Applying Hirota's method, the bilinear equation of (5) can be written as

$$\left(D_x D_t + \beta D_X^4\right)(w, w) = 0. \tag{30}$$

Equation (30) is obtained by setting $a_{00} = 0$ in equation (29). Obviously, equation (30) is a special case of equation (29). Therefore, a more general bilinear equation of the modified KdV equation is obtained by using Hirota's method.

Let us consider the modified KdV equation $u_t + 6uu_x + u_{xxx} = 0$ by using Hirota's method.

Substituting

$$u = w_x, \tag{31}$$

which gives the following new equation: $w_{xt} + 6w_x w_{xx} + w_{xxxx} = 0$. Now integrating both sides with respect to x, we have;

$$\int \left(\left(w_{xt} + 6w_x w_{xx} + w_{xxxx}\right)d_x = 0\right),$$
$$\int w_{xt}d_x + \int 6w_x w_{xx}d_x + \int w_{xxxx}d_x = g(t), \tag{32}$$
$$w_t + 6\frac{w_x^2}{2} + w_{xxx} = C = g(t).$$

where $C = g(t)$ is constant of integration. Likely we can get, without loss of generality, the integration "constant" with respect to x, $g(t)$ can be observed by a redefinition of w that does not change the KdV field, $u = w_x w_{\text{old}} = w_{\text{new}} + \int_{t_0}^{t} dt' g(t')$, dt' is the time derivative.

Using the new w, we have

$$w_t + 3w_x^2 + w_{xxx} = 0. \tag{33}$$

Now considering the 1-soliton solution of the modified KdV equation, let

$$u = 2\mu^2 \operatorname{sech}^2\left[\mu\left(x - x_0 - 4\mu^2 t\right)\right]. \tag{34}$$

with

$$\mu = \frac{\sqrt{v}}{2} \geq 0, \tag{35}$$

where v is the velocity of the wave propagating.

It can be written as $u = w_x$, with

$$w = 2\mu^2 \tanh\left[\mu\left(x - x_0 - 4\mu^2 t\right)\right]. \tag{36}$$

In fact, we can integrate the right-hand side of equation (36) once again, using $\tanh y = d/d_y \log\cosh y$. Therefore, equation (34) can be written as follows:

$$u = 2\frac{\partial^2}{\partial x^2}\operatorname{logcosh}\left[\mu\left(x - x_0 - 4\mu^2 t\right)\right]. \tag{37}$$

Let

$$X = x - x_0 - 4\mu^2 t. \tag{38}$$

Then, $u = 2\partial^2/\partial X^2 \log\left(\exp^{(-\mu X)}\left(1 + \exp^{(2\mu X)}\right)/2\right)$, where $\cosh y = \exp^{(-\mu X)}\left(1 + \exp^{(2\mu X)}\right)/2$

$$\begin{aligned}
&= 2\frac{\partial^2}{\partial x^2}\left(-\mu x - \log 2 + \log\left(1 + \exp^{(2\mu X)}\right)\right)\\
&= -2\frac{\partial^2}{\partial x^2}\mu x - 2\frac{\partial^2}{\partial x^2}\log 2 + 2\frac{\partial^2}{\partial x^2}\log\left(1 + \exp^{(2\mu X)}\right)\\
&= 2\frac{\partial^2}{\partial X^2}\log\left(1 + \exp^{2\mu X}\right).
\end{aligned} \tag{39}$$

That is, $u(x,t) = 2\partial^2/\partial x^2 \log\left(1 + \exp^{(2\mu)(x - x_0 - 4\mu^2 t)}\right) = 2\partial^2/\partial x^2 \log\left(1 + \exp^{(2\mu)(X)}\right)$

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\log\left(1 + \exp^{(2\mu)(x - x_0 - 4\mu^2 t)}\right) = 2\frac{\partial^2}{\partial x^2}\log\left(1 + \exp^{(2\mu)(X)}\right), \tag{40}$$

which is the 1-soliton solution of the modified KdV equation, that we use in the following:

Considering equations (31) and (33), the KdV equation in a bilinear form, and rewriting it in a quadratic form, we have the following equation. Inspired by equation (34), let us substitute

$$w = 2\frac{\partial}{\partial x}\log f = 2\frac{f_x}{f} \Leftrightarrow u = 2\frac{\partial^2}{\partial x^2}\log f, \tag{41}$$

where $f = 1 + \exp^{(2\mu)(x - x_0 - 4\mu^2 t)}$. Substituting in equation (33), we get

$$\begin{aligned}
\frac{1}{2}w_t &= \frac{f_{xt} - f_x f_t}{f^2},\\
\frac{1}{2}w_x &= \frac{f_{xx} - f_x^2}{f^2},\\
\frac{1}{2}w_{xx} &= \frac{f_{xxx}}{f} - 3\frac{f_x f_{xx}}{f^2} + 2\frac{f_x^3}{f^3},\\
\frac{1}{2}w_{xxx} &= \frac{f_{xxxx}}{f} - 4\frac{f_x f_{xxx}}{f^2} - 3\frac{f_{xx}^2}{f^2} + 12\frac{f_{xx} f_x^2}{f^3} - 6\frac{f_x^4}{f^4}.
\end{aligned} \tag{42}$$

$$\frac{1}{2}w_{xxx} = \frac{f_{xxxx}}{f} - 4\frac{f_x f_{xxx}}{f^2} - 3\frac{f_{xx}^2}{f^2} + 12\frac{f_{xx} f_x^2}{f^3} - 6\frac{f_x^4}{f^4}, \tag{43}$$

Equation (43) is quadratic in f.

Thus, equation (33) for w becomes

$$\begin{aligned}
&w_t + 3w_x^2 + w_{xxx} = 0,\\
&= 2\left(\frac{f_{xt} - f_x f_t}{f^2}\right) + 3(2)^2\left(\frac{f_{xx}}{f} - \frac{f_x^2}{f^2}\right)^2 + 2\frac{f_{xxxx}}{f} - 4(2)\frac{f_x f_{xxx}}{f^2} - 3(2)\frac{f_{xx}^2}{f^2} + 12(2)\frac{f_{xx} f_x^2}{f^3} - 6(2)\frac{f_x^4}{f^4}
\end{aligned}$$

$$
\begin{aligned}
&= 2\left(\frac{f_{xt}-f_xf_t}{f^2}\right)+12\left(\frac{f_{xx}^2}{f^2}-2\frac{f_{xx}f_x^2}{f^3}+\frac{f_x^4}{f^4}\right)+2\frac{f_{xxxx}}{f}-8\frac{f_xf_{xxx}}{f^2}-6\frac{f_{xx}^2}{f^2}+24\frac{f_{xx}f_x^2}{f^3}-12\frac{f_x^4}{f^4}\\
&= 2\left(\frac{f_{xt}-f_xf_t}{f^2}\right)+12\frac{f_{xx}^2}{f^2}-24\frac{f_{xx}f_x^2}{f^3}+12\frac{f_x^4}{f^4}+2\frac{f_{xxxx}}{f}-8\frac{f_xf_{xxx}}{f^2}-6\frac{f_{xx}^2}{f^2}+24\frac{f_{xx}f_x^2}{f^3}-12\frac{f_x^4}{f^4}\\
&\frac{1}{2}\left(=2\left(\frac{f_{xt}-f_xf_t}{f^2}\right)+2\frac{f_{xxxx}}{f}-8\frac{f_xf_{xxx}}{f^2}+6\frac{f_{xx}^2}{f^2}\right)\\
&= \frac{f_{xt}-f_xf_t}{f^2}+\frac{f_{xxxx}}{f}-4\frac{f_xf_{xxx}}{f^2}+3\frac{f_{xx}^2}{f^2}.
\end{aligned}
\tag{44}
$$

Multiplying by f^2 throughout, we have

$$ff_{xt}-f_xf_t+ff_{xxxx}-4f_xf_{xxx}+3f_{xx}^2=0 \tag{45}$$

This is the bilinear form of the modified KdV equation. Despite the fact that some nontrivial cancellations took place it looks more complicated than the original problem, however, its special form makes it possible to solve using bilinear operators.

Claim 1. $\partial/\partial x\partial/\partial t\,(1/2f^2)=\partial/\partial x f f_t = ff_{xt}+f_xf_t$

To express equation (45), we use the following expressions:

$$
\begin{aligned}
[D_t(f.g)](x,t) &= \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)f(x,t)g\left(x',t'\right)\Big|x'=x,t'=t\\
&= f_t(x,t)g\left(x',t\right)-f(x,t)g_{t'}\left(x',t'\right)x'=x,t'=t\\
&= f_t(x,t)g(x,t)-f(x,t)g_t(x,t)\\
&\quad [D_t(f.g)](x,t)=f_t(x,t)g(x,t)-f(x,t)g_t(x,t)\\
&\quad D_t(f.f)(x,t)=0,\\
&\quad D_x(f.f)(x,t)=0 \text{ for } f=g\\
D_tD_x(f.g) &= \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)f(x,t)g\left(x',t'\right)\Big|x'=x,t'=t\\
&= \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)f_x(x,t)g\left(x',t'\right)-f(x,t)g_{x'}\left(x',t'\right)\Big|x'=x,t'=t\\
&= f_{xt}g-f_tg_x-f_xg_t-fg_{xt}\\
&\quad f_{xt}g-f_tg_x-f_xg_t+fg_{xt}\\
D_tD_x(f.f) &= f_{xt}f-f_tf_x-f_xf_t+ff_{xt}\\
&= 2\left(ff_{xt}-f_tf_x\right)\\
D_tD_x(f.f) &= 2\left(ff_{xt}-f_tf_x\right)
\end{aligned}
\tag{46}
$$

Like the first two terms in (45).

$$
\begin{aligned}
D_x^2(f.g) &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^2 f_x(x,t) g\left(x',t'\,\middle|\, x' = x, t' = t\right. \\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) f_x(x,t) g\left(x',t'\right) - f(x,t) g_{x'}\left(x',t'\right)\Big| x' = x, t' = t \\
&= f_{xx} g - f_x g_x - f_x g_x + f g_{xx} \\
&= f_{xx} g - 2 f_x g_x + f g_{xx}, \\
D_x^2(f.g) &= f_{xx} g - 2 f_x g_x + f g_{xx},
\end{aligned}
\tag{47}
$$

$D_x^2(f.f) = f_{xx} f - 2 f_x f_x + f f_{xx} = 2(f f_{xx} - f_x^2)$ all functions of x, t.

Again this looks a bit like the differentiation of a product. But, we have $D_x^2(f,f) = 2 f f_{xx} - 2 f_x^2$ even though $D_x(f.f)](x,t) = 0$. So importantly the operators are not associative since $D_x^2(f.f) \neq D_x[D_x(f.f)]$.

In fact, the right-hand side is meaningless, because $D_x(f.f)$ is a single function, but the outer D_x needs to act on a pair of functions.

$$
\begin{aligned}
D_x^4(f.g) &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^4 f_x(x,t) g\left(x',t'\middle| x' = x, t' = t\right. \\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^3 \left(f_x(x,t) g\left(x',t'\right) - f(x,t) g_{x'}\left(x',t'\right)\right| x' = x, t' = t \\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^2 \left(f_{xx} g - 2 f_x g_{x'} + f g_{x'x'}\right)\Big| x' = x, t' = t \\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) \left(f_{xxx} g - 3 f_{xx} g_{x'} + 3 f_x g_{x'x'} - f g_{x'x'x'}\right)\Big| x' = x, t' = t \\
&= \left(f_{xxxx} g - 4 f_{xxx} g_{x'} + 6 f_{xx} g_{x'x'} - 4 f_x g_{x'x'x'} + f g_{x'x'x'x'}\right)\Big| x' = x, t' = t \\
&= f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{xxxx} \\
D_x^4(f.g) &= f_{xxxx} g - 4 f_{xxx} g_x + 6 f_{xx} g_{xx} - 4 f_x g_{xxx} + f g_{xxxx}.
\end{aligned}
\tag{48}
$$

Again it looks just like a differentiated product of f and g but with significant changes on the odd terms. $D_x^4(f.f) = f_{xxxx} f - 4 f_{xxx} f_x + 6 f_{xx} f_{xx} - 4 f_x f_{xxx} + f f_{xxxx} = 2 f_{xxxx} f - 8 f_{xxx} f_x + 6 f_{xx}^2$.

$$D_x^4(f.f) = 2 f_{xxxx} f - 8 f_{xxx} f_x + 6 f_{xx}^2 \tag{49}$$

Here, we have to notice that (49) is like $\partial_x^4(fg)$, but we have an alternative signs;

$$D_x^4(f.f) = 2\left(f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2\right). \tag{50}$$

So, the KdV equation is in a quadratic form in equation (45) and can be recast as $(D_t D_x + D_x^4)(f.f) = 0$ is a bilinear form of the KdV equation.

$$
\begin{gathered}
\left(D_t D_x + D_x^4\right)(f.f) = 0, \\
\left(D_t D_x + D_x^4\right) = D_x\left(D_t + D_x^3\right).
\end{gathered}
\tag{51}
$$

(1) Solutions of the Bilinear KdV Equation. For example (1) and considering $u = 2\partial_x^2 \log f$, we have two cases.

Case 1. For $f = 1$, then $u = 0$, which is a vacuum.

For $f = 1 + \exp^{(2\mu)(x - x_0 - 4\mu^2 t)}$, then $u = 1 -$ soliton solution. Since equation (51) is a bilinear equation, then this suggests that multisoliton solutions might be the sums of exponents of linear functions of x and t.

(2). The 1-Soliton Solution. In Hirota form the multisoliton solutions are sums of exponential of linear expressions in x and t.

For the one-soliton solution try

$$f = e^0 + e^\theta. \tag{52}$$

where $= ax + bt + c, \Rightarrow x = ax, c = -x_0$, and $-4\mu^2 = t$

Theorem 1. *For any θ_1 and θ_2, we have $D_t^m D_x^n (e^{\theta_1}.e^{\theta_2}) = (b_1 - b_2)^m (a_1 - a_2)^n (e^{\theta_1+\theta_2})$, where $m, n \geq 0$.*

Proof. By induction.

Let if $\theta_i = a_i x + b_i t + c_i$, $(i = 1, 2)$. Then,

$$\begin{aligned} D_t^m D_x^n \left(e^{\theta_1}.e^{\theta_2}\right) &= (b_1 - b_2)^m (a_1 - a_2)^n \left(e^{\theta_1+\theta_2}\right) \\ &= (b_1 - b_2)^m (a_1 - a_2)^n \left(e^{\theta_1 (x,t)+\theta_2 (x',t')}\right)\Big| x' = x, t' = t \end{aligned} \tag{53}$$

Then,

$$\begin{aligned} D_t^m D_x^{n+1} \left(e^{\theta_1}.e^{\theta_2}\right) &= (b_1 - b_2)^m (a_1 - a_2)^n \left(\frac{\partial}{\partial_x} - \frac{\partial}{\partial x'}\right)\left(e^{\theta_1 (x,t)+\theta_2 (x',t')}\right)\Big| x' = x, t' = t \\ &= (b_1 - b_2)^m (a_1 - a_2)^n (a_1 - a_2)\left(e^{\theta_1 (x,t)+\theta_2 (x',t')}\right)\Big| x' = x, t' = t \\ &= (b_1 - b_2)^m (a_1 - a_2)^{n+1} \left(e^{\theta_1+\theta_2}\right) \\ D_t^m D_x^{n+1} \left(e^{\theta_1}.e^{\theta_2}\right) &= (b_1 - b_2)^m (a_1 - a_2)^{n+1} \left(e^{\theta_1+\theta_2}\right). \end{aligned} \tag{54}$$

and similarly for D_t operator.

Finally, the equation is obviously true for $n = 1, m = 0$ and $n = 0, m = 1$ so by induction must be true for all n and m. In particular, we find

$$D_t^m D_x^n \left(e^\theta.e^\theta\right) = 0. \tag{55}$$

Unless $m = n = 0$

$$\begin{aligned} D_t^m D_x^n \left(e^\theta.1\right) &= b^m a^n e^\theta \\ D_t^m D_x^n \left(1.e^\theta\right) &= (-1)^{m+n} b^m a^n \left(1, e^\theta\right). \end{aligned} \tag{56}$$

Then, the bilinear form of the KdV equation for $f = 1 + e^\theta$ is

$$\begin{aligned} &\left(D_t D_x + D_x^4\right)\left(1 + e^\theta, 1 + e^\theta\right) = 0 \\ &= \left(D_t D_x + D_x^4\right)\left[(1.1) + \left(1.e^\theta\right) + \left(e^\theta.1\right) + \left(e^\theta.e^\theta\right)\right] \\ &= \left(D_t D_x + D_x^4\right)\left[\left(e^\theta.1\right) + \left(1.e^\theta\right)\right] \\ &= 2bae^\theta + 2a^4 e^\theta \\ &2a\left(ba + a^3\right)e^\theta = 0. \end{aligned} \tag{57}$$

Since $e^\theta \neq 0$, we have two opportunities. These two cases or opportunities are as follows:

(1) $a = 0$. If $a = 0$, then f is independent of x. So $u = 0$, (which is a boundary condition).

(2) $b = -a^3$. If $b = -a^3$, then $f = 1 + e^{ax-a^3t+c}$ and hence $u = 2\partial_x^2 \log f$, $f = 1 + e^{ax-a^3t+c}$, which is $u = 2\partial_x^2 \log\left(1 + e^{ax-a^3t+c}\right)$

$$\begin{aligned} &= \frac{a^2}{2}\operatorname{sech}^2\left[\frac{1}{2}\left(ax - a^3 t + c\right)\right], \\ u &= \frac{a^2}{2}\operatorname{sech}^2\left[\frac{1}{2}\left(ax - a^3 t + c\right)\right]. \end{aligned} \tag{58}$$

This is the one-soliton solution with $v = a^2$.

Figure 1 shows the disturbance of the water surface having kink-shaped traveling waves with amplitude and antibell soliton. The follow of this wave is to the front.

Figure 2 shows the propagation of waves having kink shaped and like some periodic soliton. And follow the wave from back to front.

Figure 3 shows surface waves of long wave length having kink-shaped and bright periodic soliton. And the wave is from left to right.

Figure 4 surface waves of long wave length having kink shaped and bright periodic soliton. And the wave is from left to right. This is the other form of Figure 3, but it is different by the value of x, that is why it is broken from the above.

(3) Multisoliton Solutions. □

Case 2. For the power series in an auxiliary with parameter ε

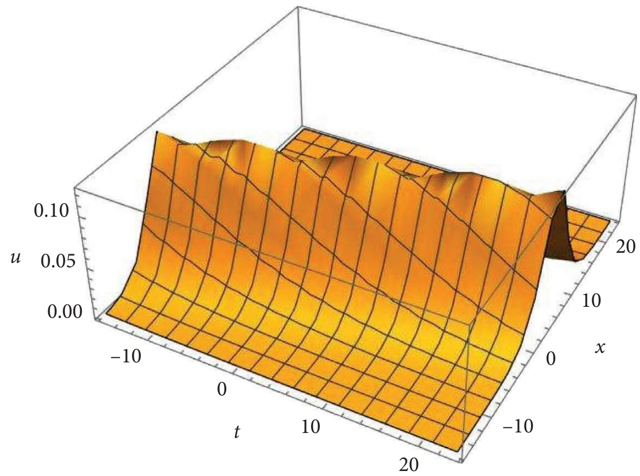

Figure 1: The result of one-soliton solution for $a = 1/2$, $c = 0.1$.

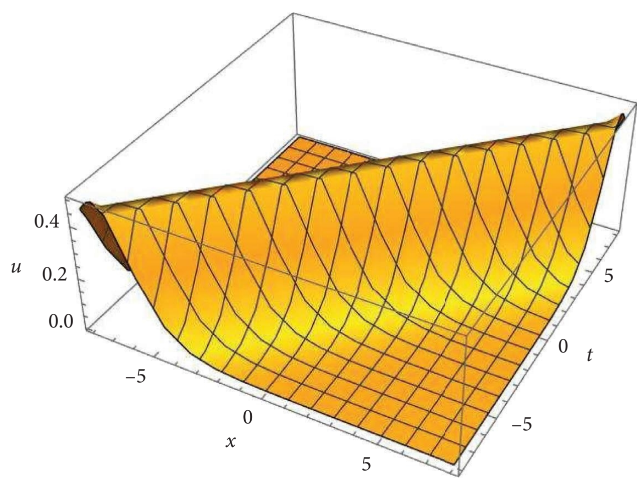

Figure 2: The result of one-soliton solution for $a = 0.6$, $c = 0.2$.

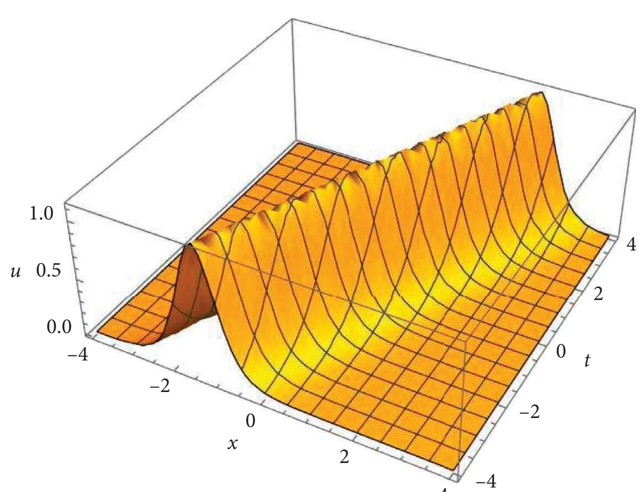

Figure 3: The result of one-soliton solution for $a = 0.7, c = 0.3$.

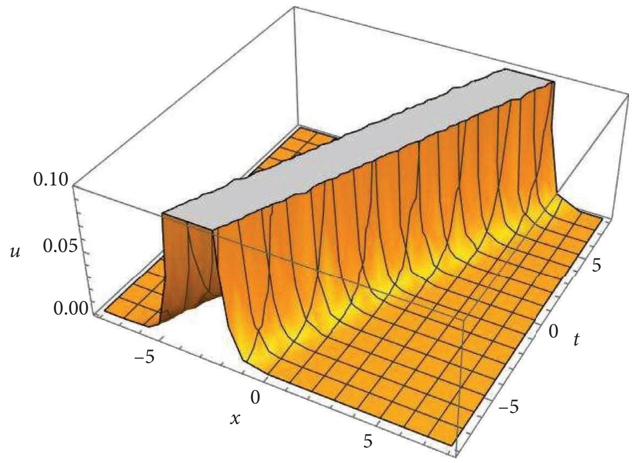

Figure 4: The result of one-soliton solution for $a = 0.7$, $c = 0.3$.

$$f(x,t) = \sum_{n=0}^{\infty} \varepsilon^n f_n(x,t), \tag{59}$$

With $f_0 = 1$ in which the series will terminate at some power of ε to give exact results rather than an infinite series in ε. So, we can take ε finite $\varepsilon = 1$.

To begin with, we put $f(x,t) = 1 + \sum_{n=0}^{\infty} \varepsilon^n f_n(x,t)$ and collect powers of ε. We start with 1 as the ε^0 term as it appeared in the one-soliton solution. Defining

$$B = \left(D_t D_x + D_x^4\right). \tag{60}$$

We want to find solutions such that

$$B(f.f) = 0. \tag{61}$$

Substituting (59) into (60), we have $0 = B(\sum_{n_1=0}^{\infty} \varepsilon^{n_1} f_{n_1}, \sum_{n_2=0}^{\infty} \varepsilon^{n_2} f_{n_2})$, with $f_0 = 1 = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \varepsilon^{n_1+n_2} B(f_{n_1}, f_{n_2})$ and gathering terms of the same degree $n = n_1 + n_2 in\varepsilon$ and also expanding as for the one soliton case we have

$$\begin{aligned} B(f.f) &= B(1.1) + \varepsilon\left(B(1.f)_1 + B(f_1.1)\right) + \varepsilon^2\left(B(1.f_2) + B(f_1.f_1) + B(f_2.1)\right) = 0, \\ &= \sum_{n=0}^{\infty} \varepsilon^n \sum_{m=0}^{n} B(f_{n-m}.f_m). \end{aligned} \tag{62}$$

so we have to solve an infinite set of equations order by order in ε

$$\sum_{m=0}^{n} B(f_{n-m}.f_m) = 0, \tag{63}$$

for all $n = 1, 2, \ldots$.

Equation (63) can be written as

$$B(f_n, 1) + B(1, f_n) = -\sum_{m=1}^{n-1} B(f_{n-m}.f_m). \tag{64}$$

(Expression involving in $f_1, f_2, \ldots, f_{n-1}$).

Now we can solve (63) which is equivalent to (64) recursively to determine the coefficients f_n. That is we begin with f_1 and solve iteratively to get $f_{2,3,..}$ hoping that it terminates at some point. To do this we need the following:

Theorem 2. *For any given f, we have; $D_t^m D_x^n(f.1) = (-1)^{n+m} D_t^m D_x^n(1.f)$, where $m, n \geq 0$*

Proof. By considering to interchange primed and unprimed values as a simple label; i.e.,

$$\begin{aligned} D_t^m D_x^n(f.g) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t) g\left(x', t'\right) \Big| x' = x, t' = t \\ &= (-1)^{m+n} \left(\frac{\partial}{\partial t'} - \frac{\partial}{\partial t}\right)^m \left(\frac{\partial}{\partial x'} - \frac{\partial}{\partial x}\right)^n g\left(x', t'\right) f(x,t) \Big| x' = x, t' = t \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{m+n}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n g(x,t)f\left(x',t'\right)\Big| x' = x, t' = t\\
&= (-1)^{m+n} D_t^m D_x^n (g.f)\\
D_t^m D_x^n (f.1) &= (-1)^{n+m} D_t^m D_x^n (1.f) = \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n} f\\
D_t^m D_x^n (f.1) &= \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n} f,
\end{aligned}
\tag{65}
$$

where f is any function of x and t

Thus, (70) became

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)f_n = -\frac{1}{2}\sum_{m=1}^{n-1} B(f_{n-m}.f_m). \tag{66}$$

For $n = 1$, it reduced to

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial_x^3}\right)f_1 = 0 \Rightarrow \left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)f_1 = 0. \tag{67}$$

Beginning with f_1 we may iterate(repeat) to find all the f_n and note that at any order we are now solving linear partial differential equations. A simple particular solution to the last equation would be

$$
\begin{aligned}
f_1 &= \sum_{i=1}^{N} \varepsilon^{\left(aix - a^3 it + ci\right)} = \sum_{i=1}^{N} \varepsilon^{\theta i},\\
f_1 &= \varepsilon^{\left(aix - a^3 it + ci\right)}.
\end{aligned}
\tag{68}
$$

Using (63) for f_2, we have $\partial/\partial x(\partial/\partial t + \partial^3/\partial x^3)f_2 = -1/2B(f_1.f_1) = 0$, where this result follows (55). Here is the fact, with f_1, the expansion of (59), terminated at order N or (the $f_{3\ldots\infty}$ all vanish trivially). The series has terminated with

$$f = 1 + \varepsilon f_1. \tag{69}$$

All higher-order equations with $n > N$ are solved by $f_n = 0$ for $n > N$. We can always absorb ε into the constant c and we recognize the single soliton solution. Our hope is to find other solutions where the series terminates in this way, so that the solution is exact. Consider beginning with

$$f_1 = \sum_{i=1}^{N} \varepsilon^{\left(aix - a^3 it + ci\right)}. \tag{70}$$

$f = 1 + \varepsilon^1 f_1 + \varepsilon^2 f_2 + \ldots \varepsilon^N f_N$. We have already seen the single soliton solution. Let's verify that we recover the two soliton solutions. Take for $N = 2$,

$$f_1 = e^{\theta_1} + e^{\theta_2}. \tag{71}$$

where $\theta_{1,2}$ are as above chosen to satisfy the f_1 equation, and $\theta_i = a_i x - a_i^3 t + c_i$. The f_2 equation became

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)f_n &= -\frac{1}{2}B(f_1.f_1),\\
&= -\frac{1}{2}B\left(e^{\theta_1} + e^{\theta_2}.e^{\theta_1} + e^{\theta_2}\right)\\
&= -\frac{1}{2}B\left(e^{\theta_1}.e^{\theta_2}\right) - \frac{1}{2}B\left(e^{\theta_2}.e^{\theta_1}\right)\\
&= -(a_1 - a_2)\left((b_1 - b_2) + (a_1 - a_2)^3\right)e^{\theta_1+\theta_2}\\
&= -(a_1 - a_2)\left(-\left(a_1^3 - a_2^3\right) + (a_1 - a_2)^3\right)e^{\theta_1+\theta_2}\\
&= (a_1 - a_2)^2\left(a_1^2 + a_1 a_2 + a_2^2 - a_1^2 + 2a_1 a_2 - a_2^2\right)e^{\theta_1+\theta_2}\\
&= 3a_1 a_2 (a_1 - a_2)^2 e^{\theta_1+\theta_2}\\
\frac{\partial}{\partial_x}\left(\frac{\partial}{\partial_t}+\frac{\partial^3}{\partial_x^3}\right)f_2 &= 3a_1 a_2 (a_1 - a_2)^2 e^{\theta_1+\theta_2}.
\end{aligned}
\tag{72}
$$

Since the equation is linear in f_2 it has an obvious solution of the form $f_2 = Ae^{\theta_1+\theta_2}$. So, we just have to determine A. Substituting into the f_2 equation; that is, let

$$
\begin{aligned}
&\partial_x = (a_1 + a_2),\\
&\partial_t = \left(-a_1^3 + a_2^3\right),\\
&\partial_x^3 = (a_1 + a_2)^3,\\
&\quad \cdot \frac{\partial}{\partial_x}\left(\frac{\partial}{\partial_t} + \frac{\partial^3}{\partial_x^3}\right) f_2 = 3a_1 a_2 (a_1 - a_2)^2 e^{\theta_1+\theta_2}\\
&\quad \cdot (a_1 + a_2)\left[\left(-a_1^3 + a_2^3\right) + (a_1 + a_2)^3\right] A e^{\theta_1+\theta_2} = 3a_1 a_2 (a_1 - a_2)^2 e^{\theta_1+\theta_2}\\
&\quad \cdot (a_1 + a_2)\left[\left(-a_1^3 + a_2^3\right) + (a_1 + a_2)^3\right] A = 3a_1 a_2 (a_1 - a_2)^2\\
&\quad \cdot 3a_1 a_2 (a_1 + a_2)^2 A = 3a_1 a_2 (a_1 - a_2)^2\\
&A = \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2}.
\end{aligned}
\tag{73}
$$

The solution is then, $f = 1 + \varepsilon e^{\theta_1} + \varepsilon e^{\theta_2} + \varepsilon^2 (a_1 - a_2)^2/(a_1 + a_2)^2 e^{\theta_1+\theta_2}$, which is a two-soliton solution of the KdV equation. Setting $\varepsilon = 1$, we have

$$
f = 1 + e^{\theta_1} + e^{\theta_2} + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_1+\theta_2}. \tag{74}
$$

Figure 5 shows follow of the presence of a solitary wave to the front, having two kinks shaped with long amplitude, dark of high speed.

Figure 6 shows the solitary wave to the front, having two kinks shaped, a valley of the wave, and dark high speed.

Figure 7 shows the solitary wave from left to right, having two kink-shaped solitons, a valley-shaped wave, long amplitudes, and dark high speed.

Figure 8 shows follow of the solitary wave to the front, having two kink--shaped soliton profiles, periodic, and dark and bell slow speed.

Finally, we just need to differentiate to get the original w or (u) using $w = 2f_x/f = a_1 e^{\theta_1} + a_2 e^{\theta_2} + (a_1 - a_2)^2/(a_1 + a_2)^2 e^{\theta_1+\theta_2}/1 + e^{\theta_1} + e^{\theta_2} + (a_1 - a_2)^2/(a_1 + a_2)^2 e^{\theta_1+\theta_2}$. It is easier in this format to study the asymptotic of the two soliton solutions. As in the BT discussion, first, we choose a coordinate system with the origin at the center of the around $i = 1, 2$ soliton; taking $t \longrightarrow t_i + \delta_t$ with $\delta_x = x - x_0 - v_i t_i$, where $v_i = a_i^2$ and let $t_i \longrightarrow \pm\infty$, we have $\theta_i = \sqrt{vi}/2(\delta_x - v_i \delta_t)$

$$
\theta_{j\neq i} = \frac{\sqrt{v_j}}{2}\left(\delta_x - v_j \delta_t\right) + \frac{\sqrt{v_j}}{2}\left(v_i - v_j\right) t_i. \tag{75}
$$

Near the i soliton, we have (by definition) $\delta_x, \delta_t \approx 0$, so that θ_i is small but $\theta_{j\neq i} \longrightarrow \pm\infty$ as follows: ($t \longrightarrow -\infty$, if $i = 1$, θ_1 is fixed and $\theta_{j=2} \longrightarrow -\infty$, and if $i = 2$, $\theta_{j=1} \longrightarrow -\infty$), ($t \longrightarrow +\infty$, if $i = 1$, $\theta_{j=1} \longrightarrow -\infty$, and if $i = 2$, $\theta_{j=2} \longrightarrow +\infty$). When $\theta_{j\neq i} \longrightarrow -\infty$ the f clearly reduces to the single soliton case, $f = 1 + e^{\theta_i}$. When $\theta_{j\neq i} \longrightarrow +\infty$, we have to do a little more work. To lead order we have

$$
f = e^{\theta_j}\left(1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_i} + \ldots.\right). \tag{76}
$$

where the ellipsis indicates terms of order $e^{-\theta_2}$. But then $w = 2\partial x \log f$

$$
w = 2a_j + 2\partial x \log\left(1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_i} + \ldots.\right). \tag{77}
$$

Since $w \longrightarrow w + const$ gives equivalent solutions we might ignore the constant piece. The remaining solution is equivalent to an f of the for

$$
f = \left(1 + \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2} e^{\theta_i}\right). \tag{78}
$$

This is the original single i-soliton solution but with a phase shift of

$$
\delta_{x_0} - \frac{2}{\sqrt{v_i}} \log\left(\frac{\left|\sqrt{v_1} - \sqrt{v_2}\right|}{\sqrt{v_1} - \sqrt{v_2}}\right). \tag{79}
$$

Precisely what we found for the 2 KdV soliton solution using the much more complicated BT method. Finally, it was just presented the general solution which might be proved by induction. Define an NxN matrix S_N with elements. Considering the 2-soliton solution for equation (68), $f = 1 + \varepsilon e^{\theta_1} + \varepsilon e^{\theta_2} + \varepsilon^2 (a_1 - a_2)^2/(a_1 + a_2)^2 e^{\theta_1+\theta_2}$

$$
\begin{aligned}
&= \left(1 + \varepsilon e^{\theta_1}\right)\left(1 + \varepsilon e^{\theta_2}\right) - \varepsilon^2 e^{\theta_1+\theta_2} + \varepsilon^2 \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 e^{\theta_1+\theta_2}\\
&= \left(1 + \varepsilon e^{\theta_1}\right)\left(1 + \varepsilon e^{\theta_2}\right) - \varepsilon^2 \frac{4a_1 a_2}{\left(a_1 + a_2\right)^2} e^{\theta_1+\theta_2}\\
&= \begin{bmatrix} 1 + \varepsilon e^{\theta_1} & \varepsilon \dfrac{2a_1}{\left(a_1 + a_2\right)} e^{\theta_2} \\ \varepsilon \dfrac{2a_2}{\left(a_1 + a_2\right)} e^{\theta_1} & 1 + \varepsilon e^{\theta_2} \end{bmatrix}.\\
&f = \det(S), \text{where} \left(S_N\right)_{ij} = \delta_{ij} + \varepsilon \frac{2a_i}{a_i + a_j} e^{\theta_j}.
\end{aligned}
\tag{80}
$$

and $i, j = 1, 2$. Then, the N-soliton solution $U = 2\partial_x^2 \log f$, where $f = \det(S_N)$. □

2.3. Illustrative Examples

Example 1. Consider the fractional-order nonlinear KdV system is given as follows:

$$
\begin{aligned}
\frac{\partial^\rho \mu}{\partial_{\tau\rho}} &= -a \frac{\partial^3 \mu}{\partial_\zeta^3} - 6a\mu \frac{\partial \mu}{\partial_\zeta} + 6\nu \frac{\partial \nu}{\partial_\zeta},\\
\frac{\partial^\rho \nu}{\partial_{\tau\rho}} &= -a \frac{\partial^3 \nu}{\partial_\zeta^3} - 3a\mu \frac{\partial \nu}{\partial_{\zeta'}}. \quad 0 < \rho < 1.
\end{aligned}
\tag{81}
$$

with the initial condition,

$$
\begin{aligned}
\mu(\zeta, 0) &= \eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\mu\zeta}{2}\right),\\
\nu(\zeta, 0) &= \sqrt{\frac{\alpha}{2}} \eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\mu\zeta}{2}\right).
\end{aligned}
\tag{82}
$$

For $\rho = 1$, the exact results of the KdV scheme (82) are given by

$$
\begin{aligned}
\mu(\zeta, \tau) &= \eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2} - \frac{a\eta^3\tau}{2}\right),\\
\nu(\zeta, \tau) &= \sqrt{\frac{\alpha}{2}} \eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2} - \frac{a\eta^3\tau}{2}\right).
\end{aligned}
\tag{83}
$$

Using the ELzaki transform to (83), we obtain,

$$
\begin{aligned}
\frac{1}{s^\rho} E[\mu(\zeta, \tau)] - \sum_{k=0}^{m-1} \mu_{(k)}(\zeta, 0) s^{2-\rho+k} &= E\left[-a \frac{\partial^3 \mu}{\partial_\zeta^3} - 6a\mu \frac{\partial \mu}{\partial_\zeta} + 6\nu \frac{\partial \nu}{\partial_\zeta}\right],\\
\frac{1}{s^\rho} E[\nu(\zeta, \tau)] - \sum_{k=0}^{m-1} \nu_{(k)}(\zeta, 0) s^{2-\rho+k} &= E\left[-a \frac{\partial^3 \nu}{\partial_\zeta^3} - 3a\mu \frac{\partial \nu}{\partial_\zeta}\right],
\end{aligned}
\tag{84}
$$

$$
\begin{aligned}
\frac{1}{s^\rho} E[\mu(\zeta, \tau)] &= \mu_{(k)}(\zeta, 0) s^{2-\rho} + E\left[-a \frac{\partial^3 \mu}{\partial_\zeta^3} - 6a\mu \frac{\partial \mu}{\partial_\zeta} + 6\nu \frac{\partial \nu}{\partial_\zeta}\right],\\
\frac{1}{s^\rho} E[\nu(\zeta, \tau)] &= \nu_{(k)}(\zeta, 0) s^{2-\rho} + E\left[-a \frac{\partial^3 \nu}{\partial_\zeta^3} - 3a\mu \frac{\partial \nu}{\partial_\zeta}\right],\\
E[\mu(\zeta, \tau)] &= s^2 \mu(\zeta, 0) + E s^\rho \left[-a \frac{\partial^3 \mu}{\partial_\zeta^3} - 6a\mu \frac{\partial \mu}{\partial_\zeta} + 6\nu \frac{\partial \nu}{\partial_\zeta}\right],\\
E[\nu(\zeta, \tau)] &= s^2 \nu(\zeta, 0) + E s^\rho \left[-a \frac{\partial^3 \nu}{\partial_\zeta^3} - 3a\mu \frac{\partial \nu}{\partial_\zeta}\right].
\end{aligned}
\tag{85}
$$

Applying the inverse ELzaki transform of (85), we have

$$
\begin{aligned}
\mu(\zeta, \tau) &= E^{-1}\left[s^2 \mu(\zeta, 0)\right] + E^{-1}\left[s^\rho E\left(-a \frac{\partial^3 \mu}{\partial_\zeta^3} - 6a\mu \frac{\partial \mu}{\partial_\zeta} + 6\nu \frac{\partial \nu}{\partial_\zeta}\right)\right],\\
\nu(\zeta, \tau) &= E^{-1}\left[s^2 \nu(\zeta, 0)\right] + E^{-1}\left[s^\rho E\left(-a \frac{\partial^3 \nu}{\partial_\zeta^3} - 3a\mu \frac{\partial \nu}{\partial_\zeta}\right)\right].
\end{aligned}
\tag{86}
$$

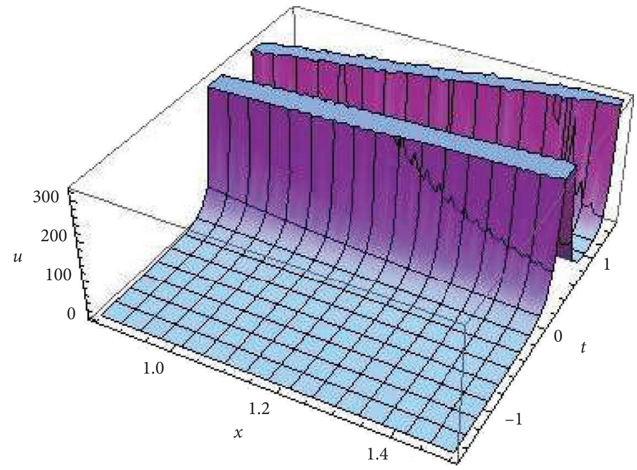

Figure 5: The result of the two-soliton solution for $a_1 = 1, a_2 = 1.5, c = 0.5$.

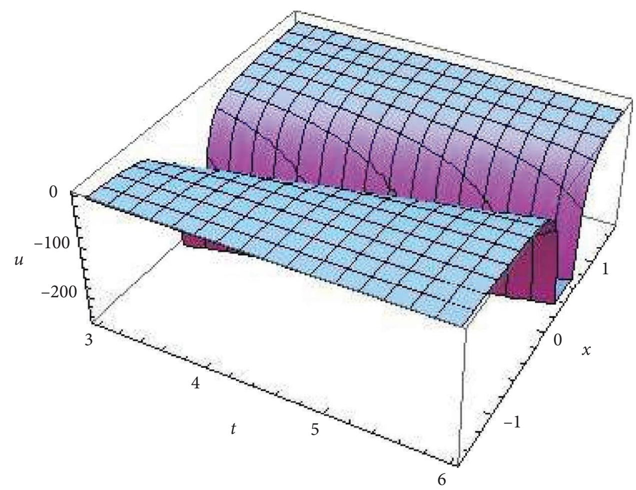

Figure 6: The result of the two-soliton solution for $a_1 = -1, a_2 = -0.5, c = -1.5$.

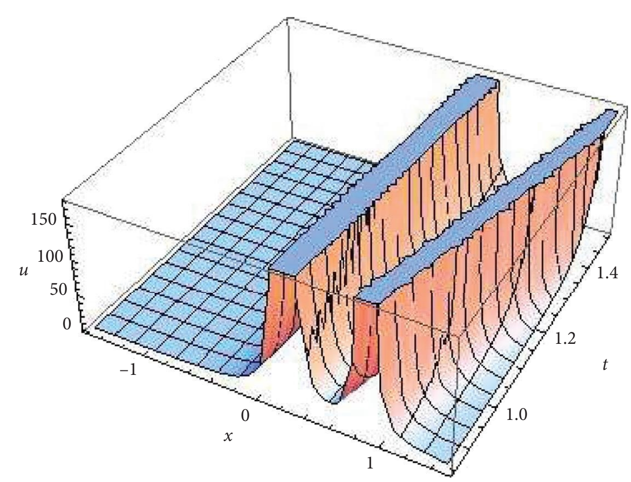

Figure 7: The result of the two-soliton solution for $a_1 = -3, a_2 = -2.5, c = -3.5$.

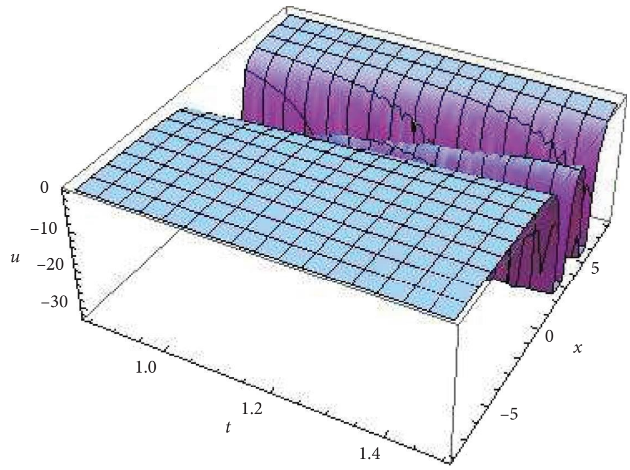

Figure 8: The result of the two-soliton solution for $a_1 = -5, a_2 = -3.5, c = -4.5$.

Now, by using the suggested analytical method, we get

$$\mu_0(\zeta,\tau) = \eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\mu\zeta}{2}\right),$$

$$\nu_0(\zeta,\tau) = \sqrt{\frac{\alpha}{2}}\eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\mu\zeta}{2}\right),$$

$$\mu_1(\zeta,\tau) = E^{-1}\left[s^\rho E\left(-a\frac{\partial^3\mu_0}{\partial_\zeta^3} - 6a\mu_0\frac{\partial\mu_0}{\partial_\zeta} + 6\nu_0\frac{\partial\nu_0}{\partial_\zeta}\right)\right],$$

$$\nu_1(\zeta,\tau) = E^{-1}\left[s^\rho E\left(-a\frac{\partial^3\nu_0}{\partial_\zeta^3} - 3a\mu_0\frac{\partial\nu_0}{\partial_\zeta}\right)\right],$$

$$\mu_1(\zeta,\tau) = \eta^5 a \tanh\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2}\right)\sec h^2\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2}\right)\frac{\tau^\rho}{\Gamma(\rho+1)},$$

$$\nu_1(\zeta,\tau) = \frac{\eta^5 a^{3/2}}{\sqrt{2}}\tanh\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2}\right)\sec h^2\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2}\right)\frac{\tau^\rho}{\Gamma(\rho+1)},$$

$$\mu_2(\zeta,\tau) = E^{-1}\left[s^\rho E\left(-a\frac{\partial^3\mu_1}{\partial_\zeta^3} - 6a\mu_1\frac{\partial\mu_1}{\partial_\zeta} + 6\nu_1\frac{\partial\nu_1}{\partial_\zeta}\right)\right],$$

$$\nu_2(\zeta,\tau) = E^{-1}\left[s^\rho E\left(-a\frac{\partial^3\nu_1}{\partial_\zeta^3} - 3a\mu_1\frac{\partial\nu_1}{\partial_\zeta}\right)\right],$$

$$\mu_2(\zeta,\tau) = \frac{\eta^8 a^2}{2}\left(2\cosh^2\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2}\right) - 3\right)\sec h^4\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)},$$

$$
\begin{aligned}
\nu_2(\zeta,\tau) &= \frac{\eta^5 a^{5/2}\sqrt{2}}{4}\left[2\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)-3\right]\sec h^4\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)},\\
\mu_3(\zeta,\tau) &= E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\mu_2}{\partial_\zeta^3}-6a\mu_2\frac{\partial\mu_2}{\partial_\zeta}+6\nu_2\frac{\partial\nu_2}{\partial_\zeta}\right)\right],\\
\nu_3(\zeta,\tau) &= E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\nu_2}{\partial_\zeta^3}-3a\mu_2\frac{\partial\nu_2}{\partial_\zeta}\right)\right],\\
\mu_3(\zeta,\tau) &= \frac{\sinh\left(\alpha/2+\eta\zeta/2\right)\tau^{3\rho}a^3\eta^4}{2\Gamma(3\rho+1)\Gamma(\rho+1)^2\cosh^7\left(\alpha/2+\eta\zeta/2\right)}\left[2\Gamma(\rho+1)^2\cosh^4\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)\right.\\
&\quad\left.-18\Gamma(\rho+1)^2\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)+6\Gamma(2\rho+1)\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)-18\Gamma(\rho+1)^2-9\Gamma(2\rho+1)\right],\\
\nu_3(\zeta,\tau) &= \frac{\sqrt{2}\sinh\left(\alpha/2+\eta\zeta/2\right)\tau^{3\rho}a^{7/2}\eta^{11}}{4\Gamma(3\rho+1)\Gamma(\rho+1)^2\cosh^7\left(\alpha/2+\eta\zeta/2\right)}\left[2\Gamma(\rho+1)^2\cosh^4\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)\right.\\
&\quad\left.-18\Gamma(\rho+1)^2\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)+6\Gamma(2\rho+1)\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)-18\Gamma(\rho+1)^2-9\Gamma(2\rho+1)\right].
\end{aligned}
\tag{87}
$$

Generally, the n^{th} solution result was obtained as follows:

$$
\begin{aligned}
\mu_n(\zeta,\tau) &= E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\mu_{n-1}}{\partial_\zeta^3}-6a\mu_{n-1}\frac{\partial\mu_{n-1}}{\partial_\zeta}+6\nu_{n-1}\frac{\partial\nu_{n-1}}{\partial_\zeta}\right)\right],\\
\nu_n(\zeta,\tau) &= E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\nu_{n-1}}{\partial_\zeta^3}-3a\mu_{n-1}\frac{\partial\nu_{n-1}}{\partial_\zeta}\right)\right], n\geq 0.
\end{aligned}
\tag{88}
$$

The series form result is as follows:

$$
\begin{aligned}
\mu(\zeta,\tau) &= \mu_0(\zeta,\tau)+\mu_1(\zeta,\tau)+\mu_2(\zeta,\tau)+\mu_3(\zeta,\tau)+\ldots..\mu_n(\zeta,\tau),\\
\nu(\zeta,\tau) &= \nu_0(\zeta,\tau)+\nu_1(\zeta,\tau)+\nu_2(\zeta,\tau)+\nu_3(\zeta,\tau)+\ldots..\nu_n(\zeta,\tau),\\
\mu(\zeta,\tau) &= \eta^2\sec h^2\left(\frac{\alpha}{2}+\frac{\mu\zeta}{2}\right)+E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\mu_0}{\partial_\zeta^3}-6a\mu_0\frac{\partial\mu_0}{\partial_\zeta}+6\nu_0\frac{\partial\nu_0}{\partial_\zeta}\right)\right]+E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\mu_1}{\partial_\zeta^3}-6a\mu_1\frac{\partial\mu_1}{\partial_\zeta}+6\nu_1\frac{\partial\nu_1}{\partial_\zeta}\right)\right]\\
&\quad+\frac{\sinh\left(\alpha/2+\eta\zeta/2\right)\tau^{3\rho}a^3\eta^4}{2\Gamma(3\rho+1)\Gamma(\rho+1)^2\cosh^7\left(\alpha/2+\eta\zeta/2\right)}\left[2\Gamma(\rho+1)^2\cosh^4\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)-18\Gamma(\rho+1)^2\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)\right.\\
&\quad\left.+6\Gamma(2\rho+1)\cosh^2\left(\alpha/2+\eta\zeta/2\right)-18\Gamma(\rho+1)^2-9\Gamma(2\rho+1)\right]+\ldots.\\
\nu(\zeta,\tau) &= \sqrt{\frac{\alpha}{2}}\eta^2\sec h^2\left(\frac{\alpha}{2}+\frac{\mu\zeta}{2}\right)+E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\nu_0}{\partial_\zeta^3}-3a\mu_0\frac{\partial\nu_0}{\partial_\zeta}\right)\right]+E^{-1}\left[s^{\rho}E\left(-a\frac{\partial^3\nu_1}{\partial_\zeta^3}-3a\mu_1\frac{\partial\nu_1}{\partial_\zeta}\right)\right]\\
&\quad+\frac{\sqrt{2}\sinh\left(\alpha/2+\eta\zeta/2\right)\tau^{3\rho}a^{7/2}\eta^{11}}{4\Gamma(3\rho+1)\Gamma(\rho+1)^2\cosh^7\left(\alpha/2+\eta\zeta/2\right)}\left[2\Gamma(\rho+1)^2\cosh^4\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)-18\Gamma(\rho+1)^2\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)\right.\\
&\quad\left.+6\Gamma(2\rho+1)\cosh^2\left(\frac{\alpha}{2}+\frac{\eta\zeta}{2}\right)-18\Gamma(\rho+1)^2-9\Gamma(2\rho+1)\right]+\ldots.
\end{aligned}
\tag{89}
$$

For $\rho = 1$, the exact results of the KdV scheme (81) are given by

$$\mu(\zeta,\tau) = \eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2} - \frac{a\eta^3\tau}{2}\right),$$
$$\nu(\zeta,\tau) = \sqrt{\frac{\alpha}{2}}\eta^2 \sec h^2\left(\frac{\alpha}{2} + \frac{\eta\zeta}{2} - \frac{a\eta^3\tau}{2}\right). \tag{90}$$

Example 2. Consider the fractional-order nonlinear dispersive long wave scheme.

$$\frac{\partial^\rho \mu}{\partial_{\tau\rho}} = -\frac{\partial \nu}{\partial_\zeta} - \frac{1}{2}\frac{\partial \mu^2}{\partial_\zeta},$$
$$\frac{\partial^\rho \mu}{\partial_{\tau\rho}} = -\frac{\partial^3 \mu}{\partial_\zeta^3} - \frac{\partial \mu}{\partial_\zeta} - \frac{\partial \mu\nu}{\partial_\zeta}, 0 < \rho < 1, \tag{91}$$

and the initial condition;

$$\mu(\zeta,0) = a\left[\tanh\left(\frac{\eta}{2} + \frac{\alpha\zeta}{2}\right) + 1\right],$$
$$\nu(\zeta,0) = -1 + \frac{1}{2}\sec h^2\left(\frac{\eta}{2} + \frac{\alpha\zeta}{2}\right). \tag{92}$$

While using ELzaki transform of (91), we obtained

$$\frac{1}{s^\rho}E[\mu(\zeta,\tau)] - \sum_{k=0}^{m-1}\mu_{(k)}(\zeta,0)s^{2-\rho+k} = E\left[-\frac{\partial \nu}{\partial_\zeta} - \frac{1}{2}\frac{\partial \mu^2}{\partial_\zeta}\right],$$
$$\frac{1}{s^\rho}E[\nu(\zeta,\tau)] - \sum_{k=0}^{m-1}\nu_{(k)}(\zeta,0)s^{2-\rho+k} = E\left[-\frac{\partial^3 \mu}{\partial_\zeta^3} - \frac{\partial \mu}{\partial_\zeta} - \frac{\partial \mu\nu}{\partial_\zeta}\right],$$
$$\frac{1}{s^\rho}E[\mu(\zeta,\tau)] = \sum_{k=0}^{m-1}\mu_{(k)}(\zeta,0)s^{2-\rho+k} + E\left[-\frac{\partial \nu}{\partial_\zeta} - \frac{1}{2}\frac{\partial \mu^2}{\partial_\zeta}\right],$$
$$\frac{1}{s^\rho}E[\nu(\zeta,\tau)] = \sum_{k=0}^{m-1}\nu_{(k)}(\zeta,0)s^{2-\rho+k} + E\left[-\frac{\partial^3 \mu}{\partial_\zeta^3} - \frac{\partial \mu}{\partial_\zeta} - \frac{\partial \mu\nu}{\partial_\zeta}\right],$$
$$\frac{1}{s^\rho}E[\mu(\zeta,\tau)] = \mu_{(0)}(\zeta,0)s^{2-\rho} + E\left[-\frac{\partial \nu}{\partial_\zeta} - \frac{1}{2}\frac{\partial \mu^2}{\partial_\zeta}\right],$$
$$\frac{1}{s^\rho}E[\nu(\zeta,\tau)] = \nu_{(0)}(\zeta,0)s^{2-\rho} + E\left[-\frac{\partial^3 \mu}{\partial_\zeta^3} - \frac{\partial \mu}{\partial_\zeta} - \frac{\partial \mu\nu}{\partial_\zeta}\right],$$
$$E[\mu(\zeta,\tau)] = s^2\mu(\zeta,0) + s^\rho E\left[-\frac{\partial \nu}{\partial_\zeta} - \frac{1}{2}\frac{\partial \mu^2}{\partial_\zeta}\right],$$
$$E[\nu(\zeta,\tau)] = s^2\nu(\zeta,0) + s^\rho E\left[-\frac{\partial^3 \mu}{\partial_\zeta^3} - \frac{\partial \mu}{\partial_\zeta} - \frac{\partial \mu\nu}{\partial_\zeta}\right]. \tag{93}$$

Applying the inverse ELzaki transform of equation (93), we get

$$\mu(\zeta,\tau) = E^{-1}\left[s^2\mu(\zeta,0)\right] + E^{-1}\left[s^\rho E\left(-\frac{\partial \nu}{\partial_\zeta} - \frac{1}{2}\frac{\partial \mu^2}{\partial_\zeta}\right)\right],$$
$$\nu(\zeta,\tau) = E^{-1}\left[s^2\nu(\zeta,0)\right] + E^{-1}\left[s^\rho E\left(-\frac{\partial^3 \mu}{\partial_\zeta^3} - \frac{\partial \mu}{\partial_\zeta} - \frac{\partial \mu\nu}{\partial_\zeta}\right)\right]. \tag{94}$$

Now, by using the current analytical method, we get

$$\mu_0(\zeta,0) = a\left[\tanh\left(\frac{\eta}{2} + \frac{\alpha\zeta}{2}\right) + 1\right],$$
$$\nu_0(\zeta,0) = -1 + \frac{1}{2}a^2 \sec h^2\left(\frac{\eta}{2} + \frac{\alpha\zeta}{2}\right)$$
$$\mu_1(\zeta,\tau) = E^{-1}\left[s^\rho E\left(-\frac{\partial \nu_0}{\partial_\zeta} - \frac{1}{2}\frac{\partial \mu_0^2}{\partial_\zeta}\right)\right],$$
$$\nu_1(\zeta,\tau) = E^{-1}\left[s^\rho E\left(-\frac{\partial^3 \mu_0}{\partial_\zeta^3} - \frac{\partial \mu_0}{\partial_\zeta} - \frac{\partial \mu_0\nu_0}{\partial_\zeta}\right)\right]$$

$$\begin{aligned}
\mu_1(\zeta,\tau) &= -\frac{a^2}{2}\sec h^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^\rho}{\Gamma(\rho+1)}\\
\nu_1(\zeta,\tau) &= \frac{a^3}{2}\sinh\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\sec h^3\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^\rho}{\Gamma(\rho+1)}\\
\mu_2(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(-\frac{\partial \nu_1}{\partial_\zeta}-\frac{1}{2}\frac{\partial \mu_1^2}{\partial_\zeta}\right)\right],\\
\nu_2(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(-\frac{\partial^3 \mu_1}{\partial_\zeta^3}-\frac{\partial \mu_1}{\partial_\zeta}-\frac{\partial \mu_1\nu_1}{\partial_\zeta}\right)\right]\\
\mu_2(\zeta,\tau) &= -\frac{a^5}{4}\sec h^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}+\frac{3a^5}{4}\sinh^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\\
&\quad+\frac{a^7}{4}\sinh\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\sec h^4\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}+\sec h^5\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\Gamma(2\rho+1)\tau^{3\rho}}{\Gamma(3\rho+1)\Gamma(\rho+1)^2}\\
\nu_2(\zeta,\tau) &= \frac{a^6}{4}\left[2\cosh^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)-3\right]\sec h^4\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}\\
\mu_n(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(-\frac{\partial \nu_{n-1}}{\partial_\zeta}-\frac{1}{2}\frac{\partial(\mu_{n-1})^2}{\partial_\zeta}\right)\right],\\
\nu_n(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(-\frac{\partial^3 \mu_{n-1}}{\partial_\zeta^3}-\frac{\partial \mu_{n-1}}{\partial_\zeta}-\frac{\partial \mu_{n-1}\nu_{n-1}}{\partial_\zeta}\right)\right]\\
& n\geq 0
\end{aligned} \tag{95}$$

Thus, the series form of the result is as follows:

$$\begin{aligned}
\mu(\zeta,\tau) &= \mu_0(\zeta,\tau)+\mu_1(\zeta,\tau)+\mu_2(\zeta,\tau)+\cdots+\mu_n(\zeta,\tau),\\
\nu(\zeta,\tau) &= \nu_0(\zeta,\tau)+\nu_1(\zeta,\tau)+\nu_2(\zeta,\tau)+\cdots+\nu_n(\zeta,\tau),\\
\mu(\zeta,\tau) &= a\left[\tanh\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)+1\right]-\frac{a^2}{2}\sec h^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^\rho}{\sigma(\rho+1)}-\frac{a^5}{4}\sec h^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)},\\
&\quad+\frac{3a^5}{4}\sinh^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\sec h^4\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}+\frac{a^7}{4}\sinh\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\sec h^5\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\Gamma(2\rho+1)\tau^{3\rho}}{\sigma(3\rho+1)\Gamma(\rho+1)^2}+\cdots,\\
\nu(\zeta,\tau) &= -1+\frac{1}{2}a^2\sec h^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)+\frac{a^3}{2}\sinh\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\sec h^3\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^\rho}{\Gamma(\rho+1)}\\
&\quad+\frac{a^6}{4}\left[2\cosh^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)-3\right]\sec h^4\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}\right)\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}+\cdots.
\end{aligned} \tag{96}$$

For $\rho=1$, the exact results of the KdV scheme (91)is as follows:

$$\mu(\zeta,\tau)=a\left[\tanh\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}-\frac{a^2\tau}{2}\right)+1\right],$$
$$v(\zeta,\tau)=-1+\frac{1}{2}\sec h^2\left(\frac{\eta}{2}+\frac{\alpha\zeta}{2}-\frac{a^2\tau}{2}\right). \tag{97}$$

Example 3. Considering the nonlinear fractional-order new coupled modified KdV system,we get

$$\frac{\partial^\rho\mu}{\partial_{\tau\rho}}=\frac{1}{2}\frac{\partial^3\mu}{\partial_\tau^3}-3\mu^2\frac{\partial\mu}{\partial_\zeta}+\frac{3}{2}w\frac{\partial^2 v}{\partial_{\zeta^2}}+3\frac{\partial v}{\partial\zeta}\frac{\partial w}{\partial_\zeta}+\frac{3}{2}v\frac{\partial^2 w}{\partial_{\zeta^2}}+$$
$$3vw\frac{\partial\mu}{\partial_\zeta}+3\mu w\frac{\partial v}{\partial_\zeta}+3\mu v\frac{\partial w}{\partial_\zeta},$$
$$\frac{\partial^\rho v}{\partial_{\tau\rho}}=-\frac{\partial^3 v}{\partial_\tau^3}-3\frac{\partial v}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3v\frac{\partial^2\mu}{\partial_{\zeta^2}}-3v^2\frac{\partial w}{\partial\zeta}+6\mu v\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial v}{\partial\zeta},$$
$$\frac{\partial^\rho w}{\partial_{\tau\rho}}=-\frac{\partial^3 w}{\partial_\tau^3}-3\frac{\partial w}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3w\frac{\partial^2\mu}{\partial_{\zeta^2}}-3w^2\frac{\partial v}{\partial\zeta}+6\mu w\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial w}{\partial\zeta}, \tag{98}$$

and the initial conditions are

$$\mu(\zeta,0)=1+\frac{1}{2}\tanh(\zeta),$$
$$v(\zeta,0)=\frac{1}{2}-\frac{1}{4}\tanh(\zeta), \tag{99}$$
$$w(\zeta,0)=2-\tanh(\zeta).$$

Using the ELzaki transform to (98), we get

$$\frac{1}{s^\rho}E[\mu(\zeta,\tau)]-\sum_{k=0}^{m-1}\mu_{(k)}(\zeta,0)s^{2-\rho+k}=E\left[\frac{1}{2}\frac{\partial^3\mu}{\partial_\tau^3}-3\mu^2\frac{\partial\mu}{\partial_\zeta}+\frac{3}{2}w\frac{\partial^2 v}{\partial_{\zeta^2}}+3\frac{\partial v}{\partial\zeta}\frac{\partial w}{\partial_\zeta}+\frac{3}{2}v\frac{\partial^2 w}{\partial_{\zeta^2}}+3vw\frac{\partial\mu}{\partial_\zeta}+3\mu w\frac{\partial v}{\partial_\zeta}+3\mu v\frac{\partial w}{\partial_\zeta}\right],$$
$$\frac{1}{s^\rho}E[v(\zeta,\tau)]-\sum_{k=0}^{m-1}v_{(k)}(\zeta,0)s^{2-\rho+k}=E\left[-\frac{\partial^3 v}{\partial_\tau^3}-3\frac{\partial v}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3v\frac{\partial^2\mu}{\partial_{\zeta^2}}-3v^2\frac{\partial w}{\partial\zeta}+6\mu v\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial v}{\partial\zeta}\right],$$
$$\frac{1}{s^\rho}E[w(\zeta,\tau)]-\sum_{k=0}^{m-1}w_{(k)}(\zeta,0)s^{2-\rho+k}=E\left[\frac{\partial^3 w}{\partial_\tau^3}-3\frac{\partial w}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3w\frac{\partial^2\mu}{\partial_{\zeta^2}}-3w^2\frac{\partial v}{\partial\zeta}+6\mu w\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial w}{\partial\zeta}\right], \tag{100}$$
$$\frac{1}{s^\rho}E[\mu(\zeta,\tau)]=\mu_{(0)}(\zeta,0)s^{2-\rho}+E\left[\frac{1}{2}\frac{\partial^3\mu}{\partial_\tau^3}-3\mu^2\frac{\partial\mu}{\partial_\zeta}+\frac{3}{2}w\frac{\partial^2 v}{\partial_{\zeta^2}}+3\frac{\partial v}{\partial\zeta}\frac{\partial w}{\partial_\zeta}+\frac{3}{2}v\frac{\partial^2 w}{\partial_{\zeta^2}}+3vw\frac{\partial\mu}{\partial_\zeta}+3\mu w\frac{\partial v}{\partial_\zeta}+3\mu v\frac{\partial w}{\partial_\zeta}\right],$$
$$\frac{1}{s^\rho}E[v(\zeta,\tau)]=v_{(0)}(\zeta,0)s^{2-\rho}+E\left[-\frac{\partial^3 v}{\partial_\tau^3}-3\frac{\partial v}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3v\frac{\partial^2\mu}{\partial_{\zeta^2}}-3v^2\frac{\partial w}{\partial\zeta}+6\mu v\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial v}{\partial\zeta}\right],$$
$$\frac{1}{s^\rho}E[w(\zeta,\tau)]=w_{(0)}(\zeta,0)s^{2-\rho}+E\left[\frac{\partial^3 w}{\partial_\tau^3}-3\frac{\partial w}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3w\frac{\partial^2\mu}{\partial_{\zeta^2}}-3w^2\frac{\partial v}{\partial\zeta}+6\mu w\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial w}{\partial\zeta}\right],$$

$$E[\mu(\zeta,\tau)]=s^2\mu(\zeta,0)+s^\rho E\left[\frac{1}{2}\frac{\partial^3\mu}{\partial_\tau^3}-3\mu^2\frac{\partial\mu}{\partial_\zeta}+\frac{3}{2}w\frac{\partial^2 v}{\partial_{\zeta^2}}+3\frac{\partial v}{\partial\zeta}\frac{\partial w}{\partial_\zeta}+\frac{3}{2}v\frac{\partial^2 w}{\partial_{\zeta^2}}+3vw\frac{\partial\mu}{\partial_\zeta}+3\mu w\frac{\partial v}{\partial_\zeta}+3\mu v\frac{\partial w}{\partial_\zeta}\right],$$
$$E[v(\zeta,\tau)]=s^2v(\zeta,0)+s^\rho E\left[-\frac{\partial^3 v}{\partial_\tau^3}-3\frac{\partial v}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3v\frac{\partial^2\mu}{\partial_{\zeta^2}}-3v^2\frac{\partial w}{\partial\zeta}+6\mu v\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial v}{\partial\zeta}\right], \tag{101}$$
$$E[w(\zeta,\tau)]=s^2w(\zeta,0)+s^\rho E\left[\frac{\partial^3 w}{\partial_\tau^3}-3\frac{\partial w}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta}-3w\frac{\partial^2\mu}{\partial_{\zeta^2}}-3w^2\frac{\partial v}{\partial\zeta}+6\mu w\frac{\partial\mu}{\partial_\zeta}+3\mu^2\frac{\partial w}{\partial\zeta}\right].$$

Applying the inverse ELzaki transform of (101)

$$\begin{aligned}
\mu(\zeta,\tau) &= E^{-1}\left[s^2\mu(\zeta,0)\right] + E^{-1}\left[s^\rho E\left(\frac{1}{2}\frac{\partial^3\mu}{\partial_\tau^3} - 3\mu^2\frac{\partial\mu}{\partial_\zeta} + \frac{3}{2}w\frac{\partial^2 v}{\partial_{\zeta^2}} + 3\frac{\partial v}{\partial\zeta}\frac{\partial w}{\partial_\zeta} + \frac{3}{2}v\frac{\partial^2 w}{\partial_{\zeta^2}} + 3vw\frac{\partial\mu}{\partial_\zeta} + 3\mu w\frac{\partial v}{\partial_\zeta} + 3\mu v\frac{\partial w}{\partial_\zeta}\right)\right],\\
v(\zeta,\tau) &= E^{-1}\left[s^2 v(\zeta,0)\right] + E^{-1}\left[s^\rho E\left(-\frac{\partial^3 v}{\partial_\tau^3} - 3\frac{\partial v}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta} - 3v\frac{\partial^2\mu}{\partial_{\zeta^2}} - 3v^2\frac{\partial w}{\partial\zeta} + 6\mu v\frac{\partial\mu}{\partial_\zeta} + 3\mu^2\frac{\partial v}{\partial\zeta}\right)\right],\\
w(\zeta,\tau) &= E^{-1}\left[s^2 w(\zeta,0)\right] + E^{-1}\left[s^\rho E\left(\frac{\partial^3 w}{\partial_\tau^3} - 3\frac{\partial w}{\partial\zeta}\frac{\partial\mu}{\partial_\zeta} - 3w\frac{\partial^2\mu}{\partial_{\zeta^2}} - 3w^2\frac{\partial v}{\partial\zeta} + 6\mu w\frac{\partial\mu}{\partial_\zeta} + 3\mu^2\frac{\partial w}{\partial\zeta}\right)\right].
\end{aligned} \tag{102}$$

Now, by using the suggested analytical method, we get

$$\begin{aligned}
\mu(\zeta,0) &= 1 + \frac{1}{2}\tanh(\zeta),\\
v(\zeta,0) &= \frac{1}{2} - \frac{1}{4}\tanh(\zeta),\\
w(\zeta,0) &= 2 - \tanh(\zeta),\\
\mu_1(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(\frac{1}{2}\frac{\partial^3\mu_0}{\partial_\tau^3} - 3\mu_0^2\frac{\partial\mu_0}{\partial_\zeta} + \frac{3}{2}w_0\frac{\partial^2 v_0}{\partial_{\zeta^2}} + 3\frac{\partial v_0}{\partial\zeta}\frac{\partial w_0}{\partial_\zeta} + \frac{3}{2}v_0\frac{\partial^2 w_0}{\partial_{\zeta^2}} + 3v_0w_0\frac{\partial\mu_0}{\partial_\zeta} + 3\mu_0w_0\frac{\partial v_0}{\partial_\zeta} + 3\mu_0v_0\frac{\partial w_0}{\partial_\zeta}\right)\right],\\
v_1(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(-\frac{\partial^3 v_0}{\partial_\tau^3} - 3\frac{\partial v_0}{\partial\zeta}\frac{\partial\mu_0}{\partial_\zeta} - 3v_0\frac{\partial^2\mu_0}{\partial_{\zeta^2}} - 3v_0^2\frac{\partial w_0}{\partial\zeta} + 6\mu_0v_0\frac{\partial\mu_0}{\partial_\zeta} + 3\mu_0^2\frac{\partial v_0}{\partial\zeta}\right)\right],\\
w_1(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(\frac{\partial^3 w_0}{\partial_\tau^3} - 3\frac{\partial w_0}{\partial\zeta}\frac{\partial\mu_0}{\partial_\zeta} - 3w_0\frac{\partial^2\mu_0}{\partial_{\zeta^2}} - 3w_0^2\frac{\partial v_0}{\partial\zeta} + 6\mu_0w_0\frac{\partial\mu_0}{\partial_\zeta} + 3\mu_0^2\frac{\partial w_0}{\partial\zeta}\right)\right],\\
\mu_1(\zeta,\tau) &= \frac{11}{2}\sec h^2(\zeta)\frac{\eta^\rho}{\Gamma(\rho+1)},\\
v_1(\zeta,\tau) &= -\frac{11}{8}\sec h^2(\zeta)\frac{\eta^\rho}{\Gamma(\rho+1)},\\
w_1(\zeta,\tau) &= -\frac{11}{2}\sec h^2(\zeta)\frac{\eta^\rho}{\Gamma(\rho+1)},\\
\mu_2(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(\frac{1}{2}\frac{\partial^3\mu_1}{\partial_\tau^3} - 3\mu_1^2\frac{\partial\mu_1}{\partial_\zeta} + \frac{3}{2}w_1\frac{\partial^2 v_1}{\partial_{\zeta^2}} + 3\frac{\partial v_1}{\partial\zeta}\frac{\partial w_1}{\partial_\zeta} + \frac{3}{2}v_1\frac{\partial^2 w_1}{\partial_{\zeta^2}} + 3v_1w_1\frac{\partial\mu_1}{\partial_\zeta} + 3\mu_1w_1\frac{\partial v_1}{\partial_\zeta} + 3\mu_1v_1\frac{\partial w_1}{\partial_\zeta}\right)\right],\\
v_2(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(-\frac{\partial^3 v_1}{\partial_\tau^3} - 3\frac{\partial v_1}{\partial\zeta}\frac{\partial\mu_1}{\partial_\zeta} - 3v_1\frac{\partial^2\mu_1}{\partial_{\zeta^2}} - 3v_1^2\frac{\partial w_1}{\partial\zeta} + 6\mu_1v_1\frac{\partial\mu_1}{\partial_\zeta} + 3\mu_1^2\frac{\partial v_1}{\partial\zeta}\right)\right],\\
w_2(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(\frac{\partial^3 w_1}{\partial_\tau^3} - 3\frac{\partial w_1}{\partial\zeta}\frac{\partial\mu_1}{\partial_\zeta} - 3w_1\frac{\partial^2\mu_1}{\partial_{\zeta^2}} - 3w_1^2\frac{\partial v_1}{\partial\zeta} + 6\mu_1w_1\frac{\partial\mu_1}{\partial_\zeta} + 3\mu_1^2\frac{\partial w_1}{\partial\zeta}\right)\right],\\
\mu_2(\zeta,\tau) &= -\frac{121}{8}\tanh(\zeta)\sec h^2(\zeta)\frac{\eta^{2\rho}}{\Gamma(2\rho+1)},
\end{aligned}$$

$$\begin{aligned}
v_2(\zeta,\tau) &= \frac{121}{8}\tanh(\zeta)\sec h^2(\zeta)\frac{\eta^{2\rho}}{\Gamma(2\rho+1)},\\
w_2(\zeta,\tau) &= \frac{121}{4}\tanh(\zeta)\sec h^2(\zeta)\frac{\eta^{2\rho}}{\Gamma(2\rho+1)},\\
\mu_3(\zeta,\tau) &= \frac{1331}{48}(\cosh(2\zeta)-2)\sec h^4(\zeta)\frac{\eta^{3\rho}}{\Gamma(3\rho+1)},\\
v_3(\zeta,\tau) &= -\frac{1331}{96}(\cosh(2\zeta)-2)\sec h^4(\zeta)\frac{\eta^{3\rho}}{\Gamma(3\rho+1)},\\
w_3(\zeta,\tau) &= -\frac{1331}{24}(\cosh(2\zeta)-2)\sec h^4(\zeta)\frac{\eta^{3\rho}}{\Gamma(3\rho+1)},\\
\mu_n(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(\frac{1}{2}\frac{\partial^3\mu_{n-1}}{\partial_\tau^3} - 3\mu_{n-1}^2\frac{\partial\mu_{n-1}}{\partial_\zeta} + \frac{3}{2}w_{n-1}\frac{\partial^2 v_{n-1}}{\partial_{\zeta^2}} + 3\frac{\partial v_{n-1}}{\partial\zeta}\frac{\partial w_{n-1}}{\partial_\zeta} + \frac{3}{2}v_{n-1}\frac{\partial^2 w_{n-1}}{\partial_{\zeta^2}} + 3v_{n-1}w_{n-1}\frac{\partial\mu_{n-1}}{\partial_\zeta}\right.\right.\\
&\qquad \left.\left. +3\mu_{n-1}w_{n-1}\frac{\partial v_{n-1}}{\partial_\zeta} + 3\mu_{n-1}v_{n-1}\frac{\partial w_{n-1}}{\partial_\zeta}\right)\right],\\
v_n(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(-\frac{\partial^3 v_{n-1}}{\partial_\tau^3} - 3\frac{\partial v_{n-1}}{\partial\zeta}\frac{\partial\mu_{n-1}}{\partial_\zeta} - 3v_{n-1}\frac{\partial^2\mu_{n-1}}{\partial_{\zeta^2}} - 3v_{n-1}^2\frac{\partial w_{n-1}}{\partial\zeta} + 6\mu_{n-1}v_{n-1}\frac{\partial\mu_{n-1}}{\partial_\zeta} + 3\mu_{n-1}^2\frac{\partial v_{n-1}}{\partial\zeta}\right)\right],\\
w_n(\zeta,\tau) &= E^{-1}\left[s^\rho E\left(\frac{\partial^3 w_{n-1}}{\partial_\tau^3} - 3\frac{\partial w_{n-1}}{\partial\zeta}\frac{\partial\mu_{n-1}}{\partial_\zeta} - 3w_{n-1}\frac{\partial^2\mu_{n-1}}{\partial_{\zeta^2}} - 3w_{n-1}^2\frac{\partial v_{n-1}}{\partial\zeta} + 6\mu_{n-1}w_{n-1}\frac{\partial\mu_{n-1}}{\partial_\zeta} + 3\mu_{n-1}^2\frac{\partial w_{n-1}}{\partial\zeta}\right)\right].
\end{aligned} \tag{103}$$

The series form result is given as follows:

$$\begin{aligned}
\mu(\zeta,\tau) &= \mu_0(\zeta,\tau)+\mu_1(\zeta,\tau)+\mu_2(\zeta,\tau)+\mu_3(\zeta,\tau)+\cdots+\mu_n(\zeta,\tau),\\
v(\zeta,\tau) &= v_0(\zeta,\tau)+v_1(\zeta,\tau)+v_2(\zeta,\tau)+v_3(\zeta,\tau)+\cdots+v_n(\zeta,\tau),\\
w(\zeta,\tau) &= w_0(\zeta,\tau)+w_1(\zeta,\tau)+w_2(\zeta,\tau)+w_2(\zeta,\tau)+\cdots+w_n(\zeta,\tau),\\
\mu(\zeta,\tau) &= 1+\frac{1}{2}\tanh(\zeta)+\frac{11}{2}\sec h^2(\zeta)\frac{\eta^\rho}{\Gamma(\rho+1)} - \frac{121}{8}\tanh(\zeta)\sec h^2(\zeta)\frac{\eta^{2\rho}}{\Gamma(2\rho+1)}\\
&\quad + \frac{1331}{48}(\cosh(2\zeta)-2)\sec h^4(\zeta)\frac{\eta^{3\rho}}{\Gamma(3\rho+1)}+\cdots,\\
v(\zeta,\tau) &= \frac{1}{2}-\frac{1}{4}\tanh(\zeta)-\frac{11}{8}\sec h^2(\zeta)\frac{\eta^\rho}{\Gamma(\rho+1)} + \frac{121}{8}\tanh(\zeta)\sec h^2(\zeta)\frac{\eta^{2\rho}}{\Gamma(2\rho+1)}\\
&\quad - \frac{1331}{96}(\cosh(2\zeta)-2)\sec h^4(\zeta)\frac{\eta^{3\rho}}{\Gamma(3\rho+1)}+\cdots\\
w(\zeta,\tau) &= 2-\tanh(\zeta)-\frac{11}{2}\sec h^2(\zeta)\frac{\eta^\rho}{\Gamma(\rho+1)} + \frac{121}{4}\tanh(\zeta)\sec h^2(\zeta)\frac{\eta^{2\rho}}{\Gamma(2\rho+1)}\\
&\quad - \frac{1331}{24}(\cosh(2\zeta)-2)\sec h^4(\zeta)\frac{\eta^{3\rho}}{\Gamma(3\rho+1)}+\cdots.
\end{aligned} \tag{104}$$

The exact result of (98) is as follows:

$$\mu(\zeta,\tau) = 1 + \frac{1}{2}\tanh\left(\zeta - \frac{11}{2}\eta\right),$$
$$\nu(\zeta,\tau) = \frac{1}{2} - \frac{1}{4}\tanh\left(\zeta - \frac{11}{2}\eta\right), \tag{105}$$
$$w(\zeta,\tau) = 2 - \tanh\left(\zeta - \frac{11}{2}\eta\right).$$

3. Conclusion

Nonlinear partial differential equations can be solved using different methods based on their complexities to determine the exact solutions. In this work, the balance method is applied to solve the modified KdV of the NLPDEs of third order kind analytically. The modified Korteweg–de Vries equation is solved using the balanced method considered with the bilinear and Hirota methods. The obtained soliton solutions of the first and second kinds with certain parameters illustrated their physical models. The obtained results were illustrated graphically to show the geometrical interpretation of models and a few illustrative examples were presented to check the applicability of the method. The obtained solution will serve as a very useful event in the study of nonlinear partial differential equations. This work reveals that the balance method is sufficient, effective, and convenient to simplify the complexities in solving NLPDEs and can be applied appropriately for solving other modified KdV equations of solitary wave type.

Data Availability

The required data are included in the manuscript and cited as references when required.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors are thankful to the referee for their invaluable suggestions and comments which put the article in its present shape.

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