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Research Article

Group Analysis Explicit Power Series Solutions and Conservation Laws of the Time-Fractional Generalized Foam Drainage Equation

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In this study, the classical Lie symmetry method is successfully applied to investigate the symmetries of the time-fractional generalized foam drainage equation with the Riemann–Liouville derivative. With the help of the obtained Lie point symmetries, the equation is reduced to nonlinear fractional ordinary differential equations (NLFODEs) which contain the Erdélyi–Kober fractional differential operator. The equation is also studied by applying the power series method, which enables us to obtain extra solutions. The obtained power series solution is further examined for convergence. Conservation laws for this equation are obtained with the aid of the new conservation theorem and the fractional generalization of the Noether operators.

1. Introduction

Fractional calculus, which deals with fractional integrals and derivatives of arbitrary order, emerged towards the end of the 17th century. Since then, numerous researchers have dedicated their efforts to understanding, studying the properties, and applying fractional order differential equations. The interpretation, properties, and applications of such equations have garnered significant attention within the scientific community [1–3]. In recent years, there has been a surge of interest in studying fractional differential equations (FDEs) as they prove to be effective in describing various physical phenomena and processes across diverse fields. These equations have found applications in hydrology, viscoelasticity, mechanics, physics, fluid dynamics, biology, chemistry, control theory, electrochemistry, and finance [4–6]. Numerous efficient methods have been developed to obtain both analytical and numerical solutions for fractional order differential equations. Some prominent methods include homotopy perturbation method, subequation method, the first integral method, and Lie group method; for more details see [7–10].

The Lie symmetry method was initially introduced by Sophus Lie (1842–1899) in order to study the (DE) of integer order. This method is an algorithmic procedure to obtain the point symmetry which leaves the considered differential equation invariant. Later, Gazizov proposed the generalization of the Lie symmetry method for fractional differential equations (FDEs) by developing prolongation formulas for fractional derivatives. Since then, numerous studies have been conducted to investigate FDEs using the Lie symmetry method, see [11–13].

Conservation laws play a significant role in investigating various properties of nonlinear partial differential equations (PDEs). The relationship between the Lie symmetry group and conservation laws of PDEs was established by Noether's theorem [14] which provides a powerful framework for constructing conservation laws of differential equations. In recent developments, Ibragimov [15] has introduced a new conservation theorem based on the concept of nonlinear self-adjoint equations to study the conservation laws for arbitrary differential equations.

The analysis on the equations of foam drainage is of considerable significance as foams omnipresent in daily activities, either naturel or industrial. The generalized foam drainage equation describes the evolution of the vertical density of foam under the gravity, for more details see [16–19].

In this study, we are interesting in the following fractional generalized foam drainage equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (u^m u_x)_x + 2uu_x = 0,\ 0<\alpha\leq 1, \tag{1}$$

where $\partial_t^\alpha u$ is the Riemann–Liouville (R-L) fractional derivative of order α with respect to t. In [20], the generalized fractional foam drainage equation with $\alpha = 1$ is reduced to the following classical generalized foam drainage equation:

$$u_t - (u^m u_x)_x + 2uu_x = 0, \tag{2}$$

for $m = (1/2)$, equation (1) becomes the well-known foam drainage equation which has been studied in both cases, fractional and integer order by using Lie symmetry analysis see ([21, 22]), also for the case and $m = 1, 2$ the equation is studied in [21].

The paper is structured as follows. In Section 2, we provide some of the most important Lie symmetry analysis results in the context of fractional partial differential equations FPDEs in general. In Section 3, we present Lie point symmetries and similarity reduction of generalized fractional foam drainage equation. In Section 4, we propose another type of solutions in the form of power series solution by using the power series method. By using the nonlinear self-adjointness method, the conservation laws of equation (1) are calculated in Section 5. Finally, some conclusions are given in Section 6.

2. A Review on Lie Symmetry Analysis

The main idea of this section is to describe the Lie symmetry method for FPDEs; so, let us consider a general form of the FPDE expressed as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(x,t,u,u_x,u_{xx}), \tag{3}$$

where the subscripts indicate the partial derivatives and $(\partial^\alpha/\partial t^\alpha)$ is R-L fractional derivative operator presented by

$$\frac{\partial^\alpha}{\partial t^\alpha} g(x,t) = \begin{cases} \dfrac{\partial^n}{\partial t^n} g(x,t), & \text{if } \alpha = n, \\[2ex] \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} g(x,s)\mathrm{d}s, & 0\leq n-1<\alpha<n, \end{cases} \tag{4}$$

where $g(x,t)$ is a real valued function, and $n\in\mathbb{N}^*$ (the set of natural numbers).

The Lie symmetry method is based on assuming that equation (3) is invariant under the following transformations introduced by

$$\begin{aligned} \hat{t} &= t + \epsilon\tau(x,t,u) + O(\epsilon), \\ \hat{x} &= x + \epsilon\xi(x,t,u) + O(\epsilon), \\ \hat{u} &= u + \epsilon\eta(x,t,u) + O(\epsilon), \\ \widehat{u_t^\alpha} &= u_t^\alpha + \epsilon\eta_t^\alpha(x,t,u) + O(\epsilon), \\ \widehat{u_x} &= u_x + \epsilon\eta^x(x,t,u) + O(\epsilon), \\ \widehat{u_{xx}} &= u_{xx} + \epsilon\eta^{xx}(x,t,u) + O(\epsilon), \end{aligned} \tag{5}$$

where (5) is a one-parameter Lie group and ϵ is the group parameter and $\xi, \tau, \text{and}\, \eta$ are the infinitesimals and $\eta^x \text{and}\, \eta^{xx}$ are the extended infinitesimals of order 1 and 2 and are given by the following explicit form:

$$\begin{aligned} \eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau), \\ &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \eta^{xx} &= D_x(\eta^x) - u_{xx}D_x(\xi) - u_{xt}D_x(\tau) \\ &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 \\ &\quad - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\eta_u - 2\xi_x)u_{xx} \\ &\quad - 2\xi_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_{xx}u_t - 2\tau_u u_x u_{xx}, \end{aligned} \tag{6}$$

where D_x, is the total derivative operator with respect to x, defined by

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \ldots. \tag{7}$$

Now, the explicit form of the extended infinitesimal η_t^α of order α is written as follows:

$$\begin{aligned} \eta_t^\alpha &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha-1}(\tau u) + \tau D_t^{\alpha+1}(u), \\ &= \partial_t^\alpha(\eta) + (\eta_u - \alpha D_t(\tau))\partial_t^\alpha u - u\partial_t^\alpha(\eta_u) + \mu + \sum_{n=1}^{\infty}\binom{\alpha}{n}\partial_t^n\eta_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)D_t^{\alpha-n}u - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)\partial_t^{\alpha-n}(u_x), \end{aligned} \tag{8}$$

where D_t^α is the total time fractional derivative, and μ is given by

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{r}{k}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \times [-u]^r \frac{d^m}{dt^m}\left[u^{k-r}\right]\frac{\partial^{n-m+k}\eta_u}{\partial t^{n-m}\partial u^k}. \quad (9)$$

We should mention here that μ vanishes when the infinitesimal $\eta(x,t,u)$ is linear in the variable u, that is,

$$\eta(x,t,u) = u(x,t)f(x,t) + h(x,t). \quad (10)$$

One can now present the Lie algebra of the one-parameter Lie group (5), which is generated by the vector fields in the following form:

$$X = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \quad (11)$$

The prolonged $X^{(\alpha,2)}$ operator of the infinitesimal generator X of order $(\alpha, 2)$ is written in the following form:

$$X^{(\alpha,2)} = X + \eta_t^{\alpha}\frac{\partial}{\partial_t^{\alpha}u} + \eta^{x}\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}}. \quad (12)$$

Theorem 1 (Infinitesimal criterion of invariance). *Equation (1) is invariant under (5), if and only if equation (1) satisfies the following invariant condition described by*

$$X^{\alpha,2}(\Delta)\big|(\Delta=0) = 0, \Delta = \partial_t^{\alpha}u - F. \quad (13)$$

Remark 2 (Invariance condition). In equation (3), the lower limit of the integral must be invariant under (5), which means

$$X(t)\mid_{(t=0)} = 0, \Longrightarrow \quad \tau(x,t,u)\mid_{t=0} = 0. \quad (14)$$

Definition 3. A solution $u = f(x,t)$ is an invariant solution of (5) if it satisfies the following conditions:

(i) $u = f(x,t)$ is an invariant surface of (11), which is equivalent to

$$Xf = 0, \Longrightarrow \left[\tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}\right]f = 0. \quad (15)$$

(ii) $u = f(x,t)$ satisfies equation (3).

3. Application of the Proposed Method for Generalized Foam Drainage Equation

Suppose that equation (1) is invariant under equation (5), so we have that

$$\frac{\partial^{\alpha}\hat{u}}{\partial \hat{t}^{\alpha}} - \left(\hat{u}^m \hat{u}_{\hat{x}}\right)_{\hat{x}} + 2\hat{u}\hat{u}_{\hat{x}} = 0, \quad (16)$$

with $u = u(t,x)$ satisfies equation (1). Now, we start by applying the second prolongation $X^{(\alpha,2)}$ to (1), then the infinitesimal criterion (13) becomes

$$\eta_t^{\alpha} - \left[m(m-1)u^{m-1}u_x^2 + mu^{m-1}u_{xx} - 2u_x\right]\eta + 2\left(u - mu^{m-1}u_x\right)\eta^x - u^m\eta^{xx} = 0. \quad (17)$$

Substituting the explicit expressions η^x, η^{xx}, and η_t^{α} into (17) and equating powers of derivatives up to zero, we obtain the determining equations, by analyzing the determining equations with the initial condition (14), and the infinitesimals are determined as follows:

$$\begin{aligned} \tau(t,x,u) &= C_1 mt - 2C_1 t, \\ \xi(t,x,u) &= C_1\alpha mx - C_1\alpha x + C_2, \\ \eta(t,x,u) &= u\alpha C_1, \end{aligned} \quad (18)$$

where C_1 and C_2 are arbitrary constants. The corresponding Lie algebra is written as follows:

$$X = (C_1\alpha mx - C_1\alpha x + C_2)\frac{\partial}{\partial x} + (C_1 mt - 2C_1 t)\frac{\partial}{\partial t} + (u\alpha C_1)\frac{\partial}{\partial u}. \quad (19)$$

If we set

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= (m-2)t\frac{\partial}{\partial t} + \alpha(m-1)x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u}. \end{aligned} \quad (20)$$

It is clear to show that the vector fields X_1, X_2 are closed under the Lie bracket defined by $[X_i, X_j] = X_iX_j - X_jX_i$; thus, the Lie algebra X is generated by the vectors fields X_i (1, 2) and is rewritten as follows:

$$X = C_1X_1 + C_2X_2. \quad (21)$$

In order to find the reduced form and exact solution, we should solve the characteristic equation corresponding of each infinitesimal generator, which is described by

$$\frac{dt}{\tau(x,t,u)} = \frac{dx}{\xi(x,t,u)} = \frac{du}{\eta(x,t,u)}, \tag{22}$$

Case 4. Reduction with $X_1 = (\partial/\partial x)$.

By integrating the characteristic equation

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0}, \tag{23}$$

the symmetry, X_1, leads to the group invariant solution

$$u = f(t), \tag{24}$$

and $f(t)$ satisfies

$$D_t^{\alpha} f(t) = 0. \tag{25}$$

Therefore, the group invariant solution corresponding to X_1, is given by

$$u_1 = k_1 t^{\alpha-1}, \tag{26}$$

with k_1 as an arbitrary constant.

Figure 1 presents the graph of solution $u_1(x,t)$ for some different values of α,

$$u_1 = k_1 t^{\alpha-1}, \tag{27}$$

with k_1 as an arbitrary constant. Figure 1 presents the graph of solution $u_1(x,t)$ for some different values of α.

Case 5. Reduction with $X_2 = (m-2)t(\partial/\partial t) + \alpha(m-1)x(\partial/\partial x) + \alpha u(\partial/\partial u)$.

The similarity variable z and similarity transformation $f(z)$ corresponding to the infinitesimal generator X_2 is obtained by solving the associated characteristic equation given by

$$\frac{dt}{(m-2)t} = \frac{dx}{\alpha(m-1)x} = \frac{du}{\alpha u}, \tag{28}$$

then, for $m \neq 2, 1$, the similarity variables are

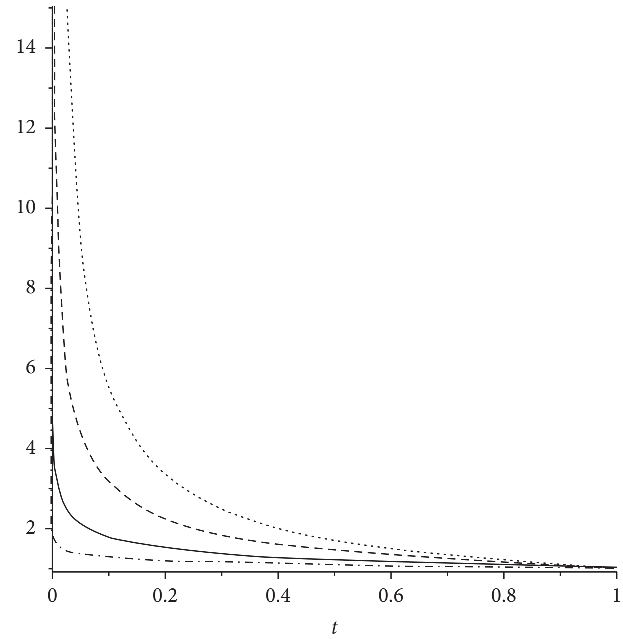

Figure 1: The solution $u_1(x,t)$ for equation (1) for $K_1 = 1$ and (-·-·- $\alpha = 0.9$, —— $\alpha = 0.75$, ■ ■ ■ $\alpha = 0.5$, and ······· $\alpha = 0.25$).

$$z = t, f(z) = u, \tag{29}$$

$$z = xt^{(\alpha(1-m)/m-2)}, f(z) = ut^{(-\alpha/m-2)}. \tag{30}$$

Therefore,

$$u = t^{(\alpha/m-2)} f\left(xt^{(\alpha(1-m)/m-2)}\right). \tag{31}$$

Theorem 6. *Using the abovementioned similarity transformation (30) in (1), we find that the time fractional foam drainage equation is transformed into a nonlinear ODE of fractional order in the following form:*

$$\left(P_{(m-2/\alpha(m-1))}^{(\alpha(3-m)/m-2)+1,\alpha} f\right)(z) - \left(f^m f_z\right)_z + \left(f^2\right)_z = 0, \tag{32}$$

with $(P_\lambda^{\delta,\alpha} f)(\zeta)$ is the Erdélyi–Kober differential operator given by

$$\left(P_\lambda^{\delta,\alpha} f\right)(\zeta) = \prod_{i=0}^{m-1}\left(\delta + i - \frac{1}{\lambda}\zeta\frac{d}{d\zeta}\right)\left(K^{\delta+\alpha,m-\alpha} f\right)(\zeta), m = \begin{cases} [\alpha]+1, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \tag{33}$$

where $K_\lambda^{\delta,\alpha}$ is the Erdélyi–Kober fractional integral operator introduced by

$$\begin{cases} \left(K_\lambda^{\delta,\alpha} f\right)(\zeta) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{\infty} (z-1)^{\alpha-1} z^{-(\delta+\alpha)} f\left(\zeta z^{(1/\lambda)}\right) dz, & \alpha > 0, \\ f(\zeta), & \alpha = 0. \end{cases} \tag{34}$$

Proof. By using the Riemann–Liouville fractional derivative definition for the similarity transformation, we have

$$u = t^{(\alpha/m-2)} f\left(xt^{(\alpha(1-m)/m-2)}\right), \tag{35}$$

We have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} s^{(\alpha/m-2)} f\left(xs^{(\alpha(1-m)/m-2)}\right)\mathrm{d}s\right]. \tag{36}$$

Let $= (t/s)$, we have $ds = (-t/v^2)$, so the above-mentioned expression can be expressed as follows:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^n}{\partial t^n}\left[t^{n-(\alpha(m-3)/m-2)}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1} v^{-(n+1-(\alpha(m-3)/m-2))} f\left(zv^{(\alpha(m-1)/m-2)}\right)\mathrm{d}v\right],\\ &= \frac{\partial^n}{\partial t^n}\left[t^{n-(\alpha(m-3)/m-2)}\left(K^{1+(\alpha/m-2),n-\alpha}_{((m-2)/\alpha(m-1))} f\right)(z)\right].\end{aligned} \tag{37}$$

On the other hand, we have

$$\begin{aligned} t\frac{\partial}{\partial t}\phi(z) &= tx\frac{\alpha(1-m)}{m-2}t^{(\alpha(1-m)/m-2)-1}\phi'(z),\\ &= \frac{\alpha(1-m)}{m-2}z\phi'(z),\end{aligned} \tag{38}$$

thus

$$\begin{aligned}\frac{\partial^n u}{\partial t^n} &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-(\alpha(m-3)/m-2)}\left(K^{1+(\alpha/m-2),n-\alpha}_{((m-2)/\alpha(m-1))} f\right)(z)\right)\right],\\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-(\alpha(m-3)/m-2)-1}\left(n-\frac{\alpha(m-3)}{m-2}-\frac{m-2}{\alpha(m-1)}z\frac{\partial}{\partial z}\right)\left(K^{1+(\alpha/m-2),n-\alpha}_{(m-2)/\alpha(m-1)} f\right)(z)\right)\right].\end{aligned} \tag{39}$$

We repeat this procedure $n-1$ times, we get

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-(\alpha(m-3)/m-2)-1}\left(n-\frac{\alpha(m-3)}{m-2}-\frac{m-2}{\alpha(m-1)}z\frac{\partial}{\partial z}\right)\left(K^{1+(\alpha/m-2),n-\alpha}_{((m-2)/\alpha(m-1))} f\right)(z)\right)\right],\\ &\vdots\\ &= t^{-\frac{\alpha(m-3)}{m-2}}\prod_{j=0}^{n-1}\left(1-\frac{\alpha(m-3)}{m-2}+j-\frac{m-2}{\alpha(m-1)}z\frac{\partial}{\partial z}\right)\left(K^{1+(\alpha/m-2),n-\alpha}_{(m-2)/\alpha(m-1)} f\right)(z), \quad = t^{(\alpha(3-m)/m-2)}\left(P^{(\alpha(3-m)/m-2)+1,\alpha}_{(m-2)/\alpha(m-1)} f\right)(z).\end{aligned} \tag{40}$$

Continuing further by calculating u_x and u_{xx} for (31) and replacing in equation (1) we find that time-fractional generalized foam drainage equation is reduced to a FODE written as follows:

$$\left(P^{(\alpha(3-m)/m-2)+1,\alpha}_{(m-2)/\alpha(m-1)} f\right)(z) - mf^{m-1}f_z^2 - f^m f_{zz} + 2f_z f = 0, \tag{41}$$

which is equivalent to

$$\left(P^{(\alpha(3-m)/m-2)+1,\alpha}_{(m-2)/\alpha(m-1)} f\right)(z) - \left(f^m f_z\right)_z + \left(f^2\right)_x = 0. \quad (42)$$

So, the proof becomes complete. □

4. Conservation Laws

In this present section, some of the conservation laws of the foam drainage equation are derived.

The conservation laws of equation (1) are that each vector (C_t, C_x) can be satisfied by the following conservation equation for all solutions of equation (1)

$$D_t(C_t) + D_x(C_x) = 0, \quad (43)$$

where D_t and D_x are the total derivative operators with respect to t and x, respectively. So, by observing equation (1), we can easily see that equation (1) can be rewritten as follows:

$$u_t^\alpha - (u^m u_x)_x - 2uu_x = D_t\left(D^{\alpha-1}u\right) + D_x\left(-u^m u_x + u^2\right), \quad (44)$$

thus, $C_t = D^{\alpha-1}u$ and $C_x = -u^m u_x + u^2$ is a conserved vector of equation (1).

Now, let us introduce the formulation of formal Lagrangian which is written as follows:

$$\mathfrak{F} = (\psi(u_t^\alpha - (u^m u_x)_x + 2uu_x), \quad (45)$$

with $\psi(x,t)$ as a new dependent variable. The adjoint equation of the foam drainage equation is determined by

$$\frac{\delta \mathfrak{F}}{\delta u} = 0, \quad (46)$$

where $(\delta/\delta u)$ is the Euler–Lagrange operator described by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \left(D_t^\alpha\right)^* \frac{\partial}{\partial\left(D_t^\alpha u\right)} - D_x \frac{\partial}{\partial(u_x)} + D_x^2 \frac{\partial}{\partial(u_{xx})} - \ldots. \quad (47)$$

$(D_t^\alpha)^*$ presents the adjoint operator of D_t^α

$$\begin{aligned}\left(D_t^\alpha\right)^* &= (-1)^n P_T^{n-\alpha}\left(D_T^n\right),\\ &= {}_c^t D_t^\alpha,\end{aligned} \quad (48)$$

with

$$\begin{aligned}P_t^{m-\alpha} f(x,t) &= \frac{1}{\Gamma(m-\alpha)} \int_t^s \frac{f(x,s)}{(s-t)^{1+\alpha-m}} \mathrm{d}s,\\ m &= [\alpha] + 1,\end{aligned} \quad (49)$$

and $(D_t^\alpha)^*$ is the right-side Caputo operator. The construction of CLs for FPDEs is in the same way of PDEs; therefore, the fundamental Noether identity is given by

$$X^{(\alpha,t)} + D_t(\tau) + D_x(\xi) = W_i \frac{\delta}{\delta u} + D_t(N_t) + D_x(N_x), \quad (50)$$

where N_x, N_t are Noether operators, $X^{(\alpha,2)}$ is defined by (12), and W_i is the characteristic function represented as follows:

$$W_i = \eta_i - \tau_i u_t - \xi_i u_x. \quad (51)$$

For the x-component of the Noether operator, it is clear to define N_x as follows:

$$N_x = \xi + W_i \left[\frac{\partial}{\partial u_x} - D_x\left(\frac{\partial}{\partial u_{xx}}\right)\right] + D_x(W_i) \frac{\partial}{\partial u_{xx}}. \quad (52)$$

For the R-L time fractional derivative, N_t is determined by

$$N_t = \tau \mathfrak{F} + \sum_{k=0}^{m-1} (-1)^k D^{\alpha-1-k}(W_i) D_t^k \frac{\partial}{\partial D_t^\alpha u} - (-1)^m I\left(W_i, D_t^m \frac{\partial}{\partial\left(D_t^\alpha\right)}\right), \quad (53)$$

where I is presented by

$$I(f,h) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x) h(\phi,x)}{(\phi-\tau)^{\alpha+1-m}} \mathrm{d}\phi\, \mathrm{d}\tau. \quad (54)$$

Now, we apply (50) to (45) for X_i (1, 2), and for all solutions, we conclude that $X^{(\alpha,2)} + D_t(\tau)\mathfrak{F} + D_x(\xi)\mathfrak{F} = 0$; also, we have $(\delta\mathfrak{F}/\delta u) = 0$, therefore

$$D_t(N_t \mathfrak{F}) + D_x(N_x \mathfrak{F}) = 0, \quad (55)$$

which is identical to the definition of conservation laws of (1). So, the components of the conserved vector become

$$\begin{aligned} C_t &= N_t\mathfrak{F} = \tau\mathfrak{F} + \sum_{k=0}^{m-1} (-1)^k D^{\alpha-1-k}(W_i)D_t^k\frac{\partial\mathfrak{F}}{\partial D_t^{\alpha}u} - (-1)^m I\left(W_i, D_t^m\frac{\partial\mathfrak{F}}{\partial(D_t^{\alpha})}\right),\\ &= \sum_{k=0}^{m-1} (-1)^k D^{\alpha-1-k}(W_i)D_t^k\frac{\partial\mathfrak{F}}{\partial D_t^{\alpha}u} - (-1)^m I\left(W_i, D_t^m\frac{\partial\mathfrak{F}}{\partial(D_t^{\alpha})}\right),\\ C_x &= N_x\mathfrak{F} = \xi\mathfrak{F} + W_i\left[\frac{\partial\mathfrak{F}}{\partial u_x} - D_x\left(\frac{\partial\mathfrak{F}}{\partial u_{xx}}\right)\right] + D_x(W_i)\frac{\partial\mathfrak{F}}{\partial u_{xx}},\\ &= W_i\left[\frac{\partial\mathfrak{F}}{\partial u_x} - D_x\left(\frac{\partial\mathfrak{F}}{\partial u_{xx}}\right)\right] + D_x(W_i)\frac{\partial\mathfrak{F}}{\partial u_{xx}}. \end{aligned} \tag{56}$$

Now, we can compute the conservation laws of equation (1) by using (56). As we have seen in the previous section, the time-fractional foam drainage equation admits two infinitesimals generators defined in Section 3 by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x},\\ X_2 &= (m-2)t\frac{\partial}{\partial t} + \alpha(m-1)x\frac{\partial}{\partial x} + \alpha u\frac{\partial}{\partial u}. \end{aligned} \tag{57}$$

The characteristic functions corresponding to each generator are given by the following formulas:

$$\begin{aligned} W_1 &= -u_x,\\ W_2 &= \alpha u + (2-m)tu_t + \alpha(1-m)xu_x. \end{aligned} \tag{58}$$

Substituting $Wi\,(i = 1; 2; 3)$ into the vector components (56), we obtain the conserved vectors of equation (1) as follows:

$$\begin{aligned} C_t^1 &= \psi D_t^{\alpha-1}(-u_x) + I(-u_x, \psi_t),\\ C_x^1 &= -u_x\left[2\left(u - mu^{m-1}u_x\right)\psi + D_x(u^m\psi)\right] + D_x(-u_x)(-u^m\psi),\\ &= -2\psi uu_x + \psi mu^{m-1}u_x^2 - \psi_x u_x u^m + \psi u^m u_{xx}. \end{aligned} \tag{59}$$

5. Power Series Solution

In this section, by using the power series method, we can extract another type of exact solution in the form of power series solution. Moreover, the convergence of power series solutions is shown.

With the aid of the fractional complex transformation given by

$$\begin{aligned} u(x,t) &= u(z),\\ z &= cx - \frac{vt^{\alpha}}{\Gamma(1+\alpha)}, \end{aligned} \tag{60}$$

where C and v are two arbitrary constants, and equation (1) is reduced to a nonlinear ODE by substituting (60) into (1), so we get

$$-vu' - c^2u^m u'' - C^2 mu^{m-1}u'^2 + 2cuu' = 0. \tag{61}$$

Now, we integrate (61) with respect to z and obtain

$$-vu - C^2 mu^m u' + Cu^2 + k = 0, \tag{62}$$

where k is an integration constant. Then, we assume that the solution of (62) is in a power series form written as follows:

$$\begin{aligned} u(z) &= \sum_{n=0}^{\infty} a_n z^n,\\ &= \sum_{n=0}^{\infty} a_n\left(cx - \frac{vt^{\alpha}}{\Gamma(1+\alpha)}\right)^n, \end{aligned} \tag{63}$$

moreover, we have

$$u_z = \sum_{n=0}^{\infty} (n+1) a_{n+1} \zeta^n, \tag{64}$$

substituting (63) and (64) into (62), we obtain

$$\begin{aligned} &- v \sum_{n=0}^{\infty} a_n z^n - C^2 m \left(\sum_{n=0}^{\infty} a_n z^n \right)^m \sum_{n=0}^{\infty} (n+1) a_{n+1} \zeta^n \\ &+ C \left(\sum_{n=0}^{\infty} a_n z^n \right)^2 + k \\ &= 0, \end{aligned} \tag{65}$$

therefore

$$\begin{aligned} &-v \sum_{n=0}^{\infty} a_n z^n - C^2 m \sum_{n=0}^{\infty} \sum_{j_1=0}^{n} \sum_{j_2=0}^{j_1} \ldots \sum_{j_m=0}^{j_{m-1}} (n - j_1 + 1) a_{n-j_1+1} a_{j_1-j_2} a_{j_2-j_3} \ldots a_{j_m} z^n \\ &+ C \sum_{n=0}^{\infty} \sum_{j=0}^{n} a_{n-j} a_j z^n + k = 0, \end{aligned} \tag{66}$$

Comparing the coefficient of z, when $n = 0$, we get

$$a_1 = \frac{-v a_0 + C a_0^2 + k}{C^2 m a_0^m}, \tag{67}$$

the second case is when $n \geq 0$

$$a_{n+1} = \frac{1}{C^2 m (n+1) a_0^m} \left[-v a_n + C \sum_{j=0}^{n} a_{n-j} a_j \right], n = 1, 2 \ldots, \tag{68}$$

thus, the power series solution of equation (1) can be expressed as

$$\begin{aligned} u(\zeta) &= a_0 + a_1 z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}, \\ &= a_0 + \left(\frac{-v a_0 - C a_0^2 + k}{C^2 m a_0^m} \right) \left(Cx - \frac{v t^{\alpha}}{\Gamma(1+\alpha)} \right) + \sum_{n=1}^{\infty} \frac{1}{C^2 m (n+1) a_0^m} \sum_{j=0}^{n} \left[-v a_n + C \sum_{j=0}^{n} a_{n-j} a_j \right] \\ &\times \left(Cx - \frac{v t^{\alpha}}{\Gamma(1+\alpha)} \right)^{n+1}, \end{aligned} \tag{69}$$

where a_0 is an arbitrary constant, and by using (67) and (68), all the coefficients of sequence $\{a_n\}_2^{\infty}$ can be calculated. It remains to prove the convergence of the power series solution.

We can see that

$$|a_{n+1}| \leq M \left[|a_n| + \sum_{i=0}^{n} |a_{n-j}| |a_j| \right], \tag{70}$$

where $M = \max\{|v/mC^2 a_0^m|, |1/mCa_0^m|\}$. Now, we take another power series of the form

$$\begin{aligned} Q &= Q(z) \\ &= \sum_{n=0}^{\infty} q_n z^n, \end{aligned} \tag{71}$$

with

$$\begin{aligned} q_0 &= |a_0|, \\ q_1 &= |a_1| q_{n+1} \\ &= M \left[q_n + \sum_{i=0}^{n} q_{n-j} q_j \right], \quad n = 1, 2, \ldots. \end{aligned} \tag{72}$$

We can observe that $|a_n| \leq q_n$, for $n = 0, 1, 2, \ldots,$ then $Q(z) = \sum_{n=0}^{\infty} q_n z^n$ is a majorant series of (63). We continue by showing that $Q(z)$ has a positive radius of convergence

$$\begin{aligned} Q(z) &= q_0 + q_1 z + M \left[\sum_{n=1}^{\infty} q_n z^{n+1} + \sum_{n=1}^{\infty} \sum_{i=0}^{n} q_{n-j} q_j z^{n+1} \right], \\ &= q_0 + q_1 z + M[(Q - q_0) z + (Q - q_0)(Q + q_0) z], \end{aligned} \tag{73}$$

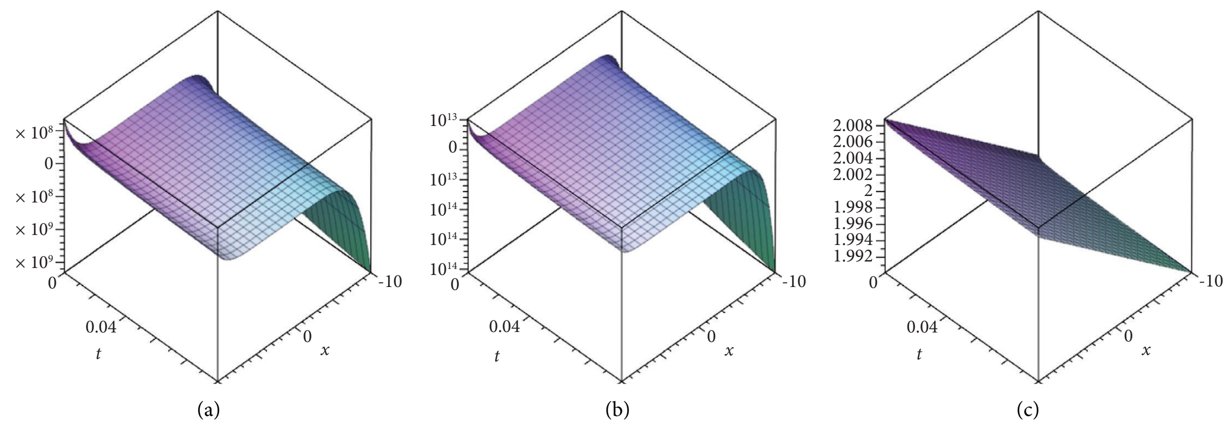

Figure 2: The solution $u_3(x,t)$ for equation (1), when $\alpha = 0.25$, $a_0 = 2, C = 1, v = 2, k = 9$. (a) $m = 1$, $n = 6$, (b) $m = 4$, $n = 10$, and (c) $m = 10$, $n = 15$.

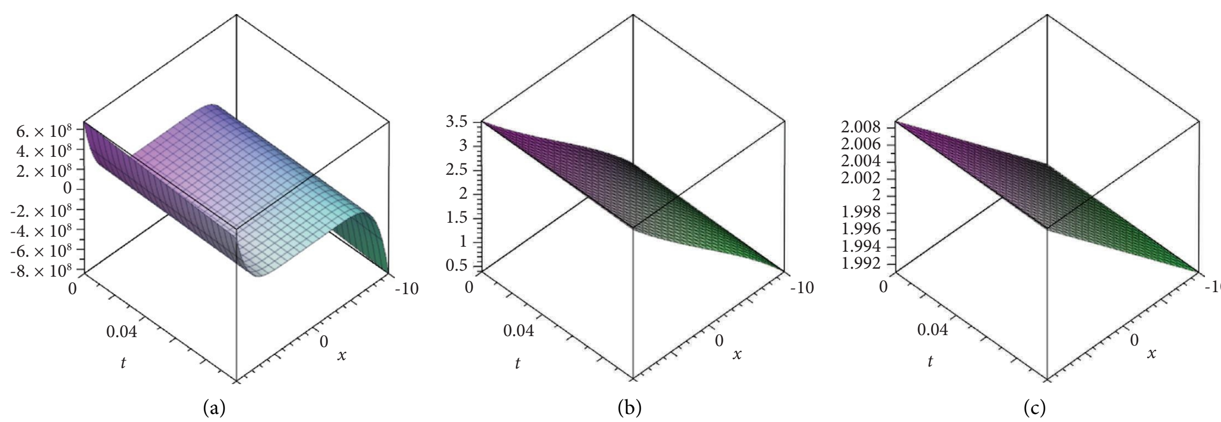

Figure 3: The solution $u_3(x,t)$ for equation (1), when $\alpha = 1$, $a_0 = 2, C = 1, v = 2, k = 9$. (a) $m = 1$, $n = 6$, (b) $m = 4$, $n = 10$, and (c) $m = 10$, $n = 15$.

then, we constitute the implicit functional equation with respect to the independent variable z

$$B(z,\delta) = \delta - q_0 - q_1 z - M\left[(Q - q_0)z + (Q - q_0)(Q + q_0)z\right]. \tag{74}$$

We can conclude from (74) that $B(z,\delta)$ is analytic in the neighborhood of $(0, q_0)$, with

$$\begin{aligned} B(0, q_0) &= 0, \\ B'_q(0, q_0) &= 1 \neq 0. \end{aligned} \tag{75}$$

By using the implicit function theorem given in [23, 24], $Q(z)$ is analytic in a neighborhood of $(0, q_0)$ with a positive radius, which shows that the power series $Q = Q(z) = \sum_{n=0}^{\infty} q_n z^n$ converges in a neighborhood of $(0, q_0)$; therefore (63) is convergent in a neighborhood of $(0, q_0)$.

Finally, we can present the graph of the power series solutions given in the following Figures 2 and 3 by choosing the suitable parameters and different values of α.

Case 1: $\alpha = 0.25$

Case 2: $\alpha = 1$.

6. Conclusion

In this paper, the Lie symmetry method is used to study the fractional generalized foam drainage equation based in Riemann–Liouville derivative. Lie symmetries are calculated and used to reduce the foam drainage equation to an ordinary differential equation of fractional order connected to the Erdélyi–Kober fractional operator, and an additional solution of equation is given by mean of the power series method. Some of CLs are obtained by using Ibragimov's method. The power series method and the fractional Lie symmetry analysis technique offer valuable and effective mathematical methods for researching other FDEs in mathematical physics and engineering.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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