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Research Article

On the Existence and Stability of Bounded Solutions for Abstract Dynamic Equations on Time Scales

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In this article we study the existence and stability of bounded solutions for semilinear abstract dynamic equations on time scales in Banach spaces. In order to do so, we use the definition of the Riemann delta-integral to prove a result about closed operator in Banach spaces and then we just use the representation of bounded solutions as an improper delta-integral from minus infinite to t. We prove the existence, uniqueness, and exponential stability of such bounded solutions. As particular cases, we study the existence of periodic and almost periodic solutions as well. Finally, we present some equations on time scales where our results can be applied.

1. Introduction

The time scales theory was introduced by Hilger (see [1, 2]) with the purpose to study difference and differential equations from a unified perspective. In recent decades, a large number of researchers have directed their efforts to study this powerful tool which has relevant applications in economics, population dynamics, quantum physics, controllability, and among others (see, for instance [3–9] and the references therein).

Particularly, this paper is devoted to study the existence and stability of bounded solutions of certain abstract dynamic equations on times scales, but before presenting the subject of this paper we recall that a time scale, denoted by $\mathbb{T}$, is any arbitrary nonempty closed subset of $\mathbb{R}$. On $\mathbb{T}$, the forward and backward jump operators σ, ρ: $\mathbb{T} \longrightarrow \mathbb{T}$ are defined, respectively, as $\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}$and $\rho(t) = \sup\{s \in \mathbb{T}: s < t\}$. A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$. The function μ: $\mathbb{T} \longrightarrow [0, \infty)$ defined by $\mu(t)\colon = \sigma(t) - t$ is known as graininess function. $\mathbb{T}$ has the topology inherited from standard topology on the real numbers, and the time scale interval $[a, b]_{\mathbb{T}}$ is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}: a \le t \le b\}$, with $a, b \in \mathbb{T}$, similarly is defined open intervals and open neighborhoods. The set $\mathbb{T}^{\kappa} \subseteq \mathbb{T}$ is given by $\mathbb{T}^{\kappa} = \mathbb{T} \setminus (\rho(m), m]$ if $\sup \mathbb{T} = m < \infty$ or $\mathbb{T}^{\kappa} = \mathbb{T}$ if $\sup \mathbb{T} = \infty$.

A function z: $\mathbb{T} \longrightarrow \mathbb{X}$ is called rd-continuous (right-dense continuous) if z is continuous at every right-dense point of $\mathbb{T}$ and its left-sided limit exists and is finite at left-dense points of $\mathbb{T}$. The set of all rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{X})$. The function z is said to be Δ-differentiable (delta differentiable) at $t \in \mathbb{T}^{k}$ if there exists a vector $z^{\Delta}(t)$ with the following property: given $\varepsilon > 0$, there exists a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$, for some $\delta > 0$, such that $\|z(\sigma(t)) - z(s) - z^{\Delta}(t)(\sigma(t) - s)\| \le \varepsilon|\sigma(t) - s)|$ for all $s \in U$. In this case the vector $z^{\Delta}(t)$ is called the delta derivative of z in t. When z is Δ-differentiable at $t \in \mathbb{T}^{\kappa}$ it is easy to show that $z^{\Delta}(t) = (f(\sigma(t)) - f(t))/(\sigma(t) - t)$ if t is right-scattered, or $z^{\Delta}(t) = \lim_{s \longrightarrow t}(f(t) - f(s))/(t - s)$ if t is right-dense. For more details about calculus on time scale we refer the reader to Bohner and Peterson's book [4].

For our purposes, in the sequel, we will assume that $0 \in \mathbb{T}$, $\inf \mathbb{T} = -\infty$, $\sup \mathbb{T} = +\infty$, $b - a \in \mathbb{T}$ if $a, b \in \mathbb{T}$ and $a < b$, and $\mu^{*} = \sup\{\mu(t)\colon t \in \mathbb{T}\} < \infty$. We will also denote by $\mathbb{T}_0^{+} = \mathbb{T} \cap [0, \infty)$.

The main concern of this paper is to study the existence and stability of bounded solutions for the following abstract dynamic equation on time scales:

$$\begin{cases} z^{\Delta}(t) = Az(t) + f(t, z(t)), t \in [t_0, \infty_{\mathbb{T}}), \\ z(t_0) = z_0. \end{cases} \tag{1}$$

Here, we are considering $f: \mathbb{T} \times \mathbb{X} \longrightarrow \mathbb{X}$ as a rd-continuous function which is locally Lipschitz on a Banach space $\mathbb{X}$, uniformly on t and $A: \mathcal{D}(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ is a linear operator which generates a C_0-semigroup of bounded linear operator $\{T(t): t \in \mathbb{T}_0^+\}$.

The motivation to study equation (1) was given by the works presented in [10, 11] where the authors studied the existence and stability of bounded solutions of the discrete-dependent system of difference equation

$$\begin{cases} z(n+1) = Az(n) + f(n, z(n)), n \in \mathbb{N} \cup \{0\}, \\ z(n_0) = z_0, \end{cases} \tag{2}$$

and the evolution equation,

$$\begin{cases} z'(t) = Az(t) + f(t, z(t)), t > t_0, \\ z(t_0) = z_0, \end{cases} \tag{3}$$

respectively. Therefore, this paper allows us to unify the results presented in [10, 11] and extend them.

Now, to accomplish this task, we will assume the following hypotheses for problem (1):

(H1) $\|f(t,0)\| \le L_f$ for some $L_f > 0$ constant

(H2) Given $\varrho > 0$ there exits $L_\varrho > 0$ such that $|f(t, z_1) - f(t, z_1)| \le L_\varrho |z_1 - z_2|$ for all $t \in \mathbb{T}$ and $z_1, z_2 \in B_\varrho = \{z \in \mathbb{X}: \|z\| < \varrho\}$

The organization of this paper is as follows: Next section is devoted to present some preliminary results about integration, generalized exponential function, and semigroup theory on time scales. In the third section, we analyze the existence and stability of bounded solutions. The fourth section is devoted to study, as particular case, the existence of periodic and almost periodic solutions. In the fifth section, we show that the obtained bounded mild solution is also, under certain conditions, a classical solution. Finally, we presented some examples as applications of our results.

2. Preliminaries

We start to present some facts about integration, the exponential function, and semigroup theory on time scales, that will be of utility in the development of this work.

A partition on $[a,b]_{\mathbb{T}}$ (see [12]) is a finite sequence of points $\{t_0, t_1, \ldots, t_n\} \subset [a,b]_{\mathbb{T}}$ such that $a = t_0 < t_1 < \cdots < t_n = b$, this partition will be denoted by $\mathcal{P}$. It is clear that n depends on the partition $\mathcal{P}$, so we get $n = n(\mathcal{P})$.

Suppose that φ is a bounded function on interval $[a,b]_{\mathbb{T}}$ and let $\mathcal{P}$ be a partition of $[a,b]_{\mathbb{T}}$, the sum

$$\sum_{i=1}^{n} \varphi(\xi_i)(t_i - t_{i-1}), \tag{4}$$

is called the Riemann Δ-sum of φ on $[a,b]_{\mathbb{T}}$, where ξ_i is an arbitrary point on $[\xi_{i-1}, \xi_i)_T$ for each $1 \le i \le n$. If $\delta > 0$, the set $\mathcal{P}_\delta[a,b]_{\mathbb{T}}$ denotes the set of all partitions of $[a,b]_{\mathbb{T}}$ such that for every $j \in \{1, \ldots, n\}$ either $t_j - t_{j-1} \le \delta$ or $t_j - t_{j-1} > \delta$ and $\rho(t_j) = t_{j-1}$. The set $\mathcal{P}_\delta[a,b]_{\mathbb{T}} \ne \varnothing$ by Lemma 2.7 in [12].

Definition 1 (see [12]). The function $\varphi: [a,b]_{\mathbb{T}} \longrightarrow \mathbb{X}$ is said to be Riemann Δ-integrable on $[a,b]_{\mathbb{T}}$ if there exists an element $I \in \mathbb{X}$, denoted by $\int_a^b \varphi(t)\Delta t$, with the property for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| \sum_{i=1}^{n} \varphi(\xi_i)(t_i - t_{i-1}) - \int_a^b \varphi(t)\Delta t \right\| < \varepsilon, \tag{5}$$

for all $\mathcal{P} \in \mathcal{P}_\delta[a,b]_{\mathbb{T}}$ independently *of* $\xi_i \in [t_i, t_{i-1})_{\mathbb{T}}$ *for* $1 \le i \le n$.

The element $\int_a^b \varphi(t)\Delta t$ is unique and it is called the Riemann Δ- integral of φ from a to b.

Let $\varphi \in C_{\mathrm{rd}}([a,\infty]_{\mathbb{T}}, \mathbb{X})$. The improper Δ-integral of φ is defined by (see [13])

$$\int_a^{\infty} \varphi(s)\Delta s = \lim_{b \longrightarrow \infty} \int_a^b \varphi(s)\Delta s, \tag{6}$$

provided that this limit exists. Analogously,

$$\int_{-\infty}^{b} \varphi(s)\Delta s = \lim_{a \longrightarrow -\infty} \int_a^b \varphi(s)\Delta s. \tag{7}$$

The function $p: \mathbb{T} \longrightarrow \mathbb{R}$ is called regressive (resp. positively regressive) if $1 + \mu(t)p(t) \ne 0$ (resp. $1 + \mu(t)\, p(t) > 0$) for $t \in \mathbb{T}$. The set of all regressive (resp. positively regressive) and rd-continuous functions is denoted by $\mathcal{R}$ (resp. $\mathcal{R}^+$).

Definition 2 (see [4]). Let $p \in \mathcal{R}$, the generalized exponential function is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right), \tag{8}$$

where

$$\xi_h(z) = \begin{cases} \dfrac{1}{h} \operatorname{Log}(1 + \mathrm{hz}) & \text{if } h > 0, \\ z, & \text{if } h = 0, \end{cases} \tag{9}$$

is the cylinder transformation defined on the Hilger complex numbers $z \in \mathbb{C}_h: = \{z \in \mathbb{C}: z \ne -1/h\}$ and $\operatorname{Log} z = \log|z| + i \arg z$, $-\pi < \arg z \le \pi$.

For $p, q \in \mathcal{R}$ we define the operations

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t),$$
$$(\ominus p)(t) := \frac{-p(t)}{1 + \mu(t)p(t)}. \tag{10}$$

It easy to see that $(\mathcal{R}, \oplus)$ forms an Abelian group, for details see [4].

Theorem 1 (see [4, 14]). *For $p, q \in \mathscr{R}$, the generalized exponential function $e_p(t,s)$ satisfy the following properties: For $t, s \in \mathbb{T}$,*

(1) $e_0(t,s) \equiv 1$, $e_p(t,t) \equiv 1$, $e_p(t,r)e_p(r,s) = e_p(t,s)$

(2) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$, $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$

(3) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$

(4) $e_p^{\Delta}(t,s) = p(t)e_p(t,s)$

(5) if $p \in \mathscr{R}^+$ and $p(t) \le q(t)$ for all $t \ge s$, $t \in \mathbb{T}$, then $e_p(t,s) \le e_q(t,s)$ for all $t \ge s$

(6) if $p > 0$, then for $t > s$, $0 < e_{\ominus p}(t,s) \le 1$, $\lim_{t\longrightarrow\infty} e_{\ominus p}(t,s) = 0$ and $\lim_{s\longrightarrow-\infty} e_{\ominus p}(t,s) = 0$.

Definition 3 (see [15]). A one parameter family $\{T(t)\colon t \in \mathbb{T}_0^+\} \subset L(\mathbb{X})$ is a C_0-semigroup if it satisfies the following properties:

(i) $T(0) = I_{\mathbb{X}}$

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \in \mathbb{T}_0^+$

(iii) $\lim_{s\longrightarrow 0^+} T(t)z = z$ for each $z \in \mathbb{X}$

If $\lim_{t\longrightarrow 0^+}\|T(t) - I_{\mathbb{X}}\| = 0$ then the semigroup $T(t)$ is called uniformly continuous.

The linear operator A defined by

$$\begin{aligned}\mathscr{D}(A) &= \left\{z \in \mathbb{X}\colon \lim_{s\longrightarrow 0^+} \frac{T(\mu(t))z - T(s)z}{\mu(t) - s} \text{ exists}\right\},\\ Az &= \lim_{s\longrightarrow 0^+} \frac{T(\mu(t))z - T(s)z}{\mu(t) - s}, z \in \mathscr{D}(A),\end{aligned} \tag{11}$$

is the infinitesimal generator of the semigroup $T(t)$ and $\mathscr{D}(A)$ is the domain of A.

One class of important functions are the generalized polynomial functions (see [4]) $h_k\colon \mathbb{T}\times\mathbb{T}\longrightarrow\mathbb{R}$, $k \in \mathbb{N}_0$, which are defined recursively of the following form: $h_0(t,s) \equiv 1$ and $h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau$.

Lemma 1 (see [4]). *Let $k \in \mathbb{N}_0$. Then, $0 \le h_k(t,s) \le ((t-s)^k/k!)$ for all $t \ge s$.*

Theorem 2. *If A is a bounded linear operator on $\mathbb{X}$, then A is the generator of a uniformly continuous semigroup $\{T(t)\colon t \in \mathbb{T}_0^+\}$ which is given by*

$$T(t) = e_A(t,0) = \sum_{n=0}^{\infty} A^n h_n(t,0). \tag{12}$$

Proof. Note that for $m, n \in \mathbb{N}$, with $m > n$,

$$\left\|\sum_{i=n}^{m} A^i h_i(t,0)z\right\| \le \sum_{i=n}^{m} \|A\|^i h_i(t,0)\|Z\| \le \sum_{i=n}^{m} \|A\|^i \frac{t^i}{i!}\|z\|, \tag{13}$$

which implies the convergence $e_A(t,0)$ in the uniform operator topology for any $z \in \mathbb{X}$ and defines a bounded linear operator for each t.

Now,

(i) $T(0) = e_A(0,0) = A^0 h_0(0,0) + \sum_{n=1}^{\infty} A^n h_n(0,0) = I_{\mathbb{X}}$

(ii) $T(t)T(s) = e_A(t,0)e_A(s,0) = (\sum_{n=1}^{\infty} A^n h_n(t,0)) (\sum_{m=0}^{\infty} A^m h_m(s,0)) = \sum_{n=0}^{\infty} (\sum_{k=0}^{n} (A^k h_k(t,0)) (A^{n-k} h_{n-k}(s,0))) = \sum_{n=0}^{\infty} A^n (\sum_{k=1}^{n} h_k(t,0)h_{n-k}(s,0))$

We shall see that

$$\sum_{k=0}^{n} h_k(t,0)h_{n-k}(s,0) = h_n(t+s,0). \tag{14}$$

Indeed, we proceed by using induction over n.

If $n = 0$, then $h_0(t+s,0) = 1 = h_0(t,0)h_0(s,0)$; if $n = 1$, then $h_1(t+s,0) = \int_0^{t+s} h_0(\tau,0)\Delta\tau = t + s = h_0(t,0)h_1(s,0) + h_1(t,0)h_0(s,0)$. Suppose that (14) holds for n, then

$$\begin{aligned}h_{n+1}(t+s,0) &= \int_0^{t+s} h_n(\tau,0)\Delta\tau = \int_0^{t} h_n(\tau,0)\Delta\tau + \int_t^{t+s} h_n(\tau,0)\Delta\tau\\ &= h_{n+1}(t,0) + \int_0^{s} h_n(\tau+t,0)\Delta\tau = h_{n+1}(t,0) + \int_0^{s} \sum_{k=0}^{m} h_k(t,0)h_{n-k}(\tau,0)\Delta\tau\\ &= h_{n+1}(t,0) + \sum_{k=1}^{n} h_k(t,0) \int_0^{s} h_{n-k}(\tau,0)\Delta\tau\\ &= h_{n+1}(t,0) + \sum_{k=0}^{n} h_k(t,0)h_{n+1-k}(s,0) = \sum_{k=1}^{n+1} h_k(t,0)h_{n+1-k}(s,0).\end{aligned} \tag{15}$$

Therefore, $T(t)T(s) = \sum_{n=0}^{\infty} A^n h_n(t+s,0) = e_A(t+s,0) = T(t+s)$.

(iii) Estimating the power series yields

$$\begin{aligned}\left\|T(t) - I_{\mathbb{X}}\right\| &\leq \sum_{n=1}^{\infty} \|A\|^n h_n(t,0) \leq \sum_{n=1}^{\infty} \|A\|^n \frac{t^n}{n!} \\ &= e^{t\|A\|} - 1 \longrightarrow 0,\end{aligned} \tag{16}$$

as $t \longrightarrow 0$.

Finally, we see that A is the infinitesimal generator of $T(t)$.

$$\begin{aligned}\lim_{s\longrightarrow 0^+} \frac{T(\mu(t))z - T(s)z}{\mu(t) - s} &= \lim_{s\longrightarrow 0^+} \frac{1}{\mu(t) - s} \sum_{n=1}^{\infty} A^n \left(h_n(\mu(t),0) - h_n(s,0)\right)z \\ &= \lim_{s\longrightarrow 0^+} \frac{1}{\mu(t) - s} \sum_{n=1}^{\infty} A^n \int_s^{\mu(t)} h_{n-1}(\tau,0)\Delta\tau z \\ &= \sum_{n=1}^{\infty} A^n \lim_{s\longrightarrow 0^+} \frac{1}{s + t - \sigma(t)} \int_{\sigma(t)+t}^{s} h_{n-1}(\tau,0)\Delta\tau z \\ &= \sum_{n=1}^{\infty} A^n \lim_{r\longrightarrow t^+} \frac{1}{r - \sigma(t)} \int_{\sigma(t)}^{r} h_{n-1}(\tau - t,0)\Delta\tau z \\ &= Az + \sum_{n=2}^{\infty} h_{n-1}(0,0)z = Az.\end{aligned} \tag{17}$$

□

Remark 1. The converse of Theorem 2 is also true and was proved in [15] Theorem 2.7.

Definition 4 (see [16]). Let T be a C_0-semigroup. The semigroup is called exponentially stable if there exists $K \geq 1$ and $\beta > 0$ such that

$$\left\|T(t - t_0)\right\| \leq K e_{\ominus\beta}(t,t_0), \tag{18}$$

for all $t, t_0 \in \mathbb{T}$ with $t > t_0$.

It is known (see [16, 17]) that solution of system (1) satisfies the integral equation

$$z(t) = T(t - t_0)z_0 + \int_{t_0}^{t} T(t - \sigma(s)) f(s, z(s))\Delta s, \tag{19}$$

but not conversely since a solution of (19) is not necessarily Δ-differentiable. We shall refer to a rd-continuous solution of (19) as a mild solution of equation (1). $z\colon \mathbb{T}_0^+ \longrightarrow \mathbb{X}$ is a classical solution of (1) on T_0^+ if z is rd-continuously Δ-differentiable, $z(t) \in \mathcal{D}(A)$ for $t \in \mathbb{T}_0^+$ and (1) is satisfied on $\mathbb{T}_0^+$.

3. Existence of Bounded Solutions

In this section, we study the existence and stability of bounded solutions for system (1). The natural space to study this problem is

$$\begin{aligned}\mathbb{X}_b &= \mathrm{BC}_{\mathrm{rd}}(\mathbb{T}, \mathbb{X}) \\ &= \{z\colon \mathbb{T} \longrightarrow \mathbb{X}\colon z \text{ is rd} - \text{continuous and bounded}\},\end{aligned} \tag{20}$$

which we will endow with the norm

$$\|z\|_b = \sup\{\|z(t)\|\colon t \in \mathbb{T}\}. \tag{21}$$

It is easy to show that the space $(\mathbb{X}_b, \|\cdot\|_b)$ is a Banach space, and for $\varrho > 0$ we define

$$B_\varrho^b = \{x \in \mathbb{X}_b\colon \|z\|_b < \varrho\}. \tag{22}$$

Lemma 2. *Let T a exponentially stable C_0-semigroup with infinitesimal generator A, and let $z \in \mathbb{X}_b$. Then, z is a mild solution of (1) if and only if z is solution of the integral equation*

$$z(t) = \int_{-\infty}^{t} T(t - \sigma(s)) f(s, z(s))\Delta s. \tag{23}$$

Proof. If z is a mild solution of (1), then

$$z(t) = T(t - t_0)z_0 + \int_{t_0}^{t} T(t - \sigma(s)) f(s, z(s))\Delta s. \tag{24}$$

Since the semigroup T is exponentially stable, we have that there exist $M \geq 1$ and $\beta > 0$ such that

$$\left\|T(t-t_0)z_0\right\| \leq M e_{\ominus\beta}(t,t_0)\left\|z_0\right\|, \quad t \geq t_0. \tag{25}$$

On the other hand $\|z\|_b \leq m$ for $t \in \mathbb{T}$, therefore we obtain the estimate

$$\left\|T(t-t_0)z_0\right\| \leq m M e_{\ominus\beta}(t,t_0), \quad t \geq t_0, \tag{26}$$

and hence $\lim_{t_0 \longrightarrow -\infty}\|T(t-t_0)z_0\| = 0$. Now, passing to limit $t_0 \longrightarrow -\infty$ in (24), it follows that

$$z(t) = \int_{-\infty}^{t} T(t-\sigma(s)) f(s, z(s))\Delta s. \tag{27}$$

Our next step is to show the improper integral is convergent. For this end, let us consider $\varrho > 0$ such that $\|z\|_b < \varrho$ and let L_ϱ be the Lipschitz constant of f in $B_\varrho \subset \mathbb{X}$. Then,

$$\begin{aligned}
\int_{-\infty}^{t} \|T(t-\sigma(s)) f(s,z(s))\|\Delta s &\leq \int_{-\infty}^{t} M e_{\ominus\beta}(t,\sigma(s))\|f(s,z(s))\|\Delta s \\
&\leq M \int_{-\infty}^{t} (1+\mu(s)\beta) e_{\beta}(s,t)\left[\|f(s,z(s)) - f(s,0)\| + \|f(s,0)\|\right]\Delta s \\
&\leq M\left[L_\varrho\|z\|_b + L_f\right]\frac{(1+\mu^*\beta)}{\beta}\int_{-\infty}^{t} \beta e_\beta(s,t)\Delta s \\
&\leq M\left[L_\varrho\|z\|_b + L_f\right]\frac{(1+\mu^*\beta)}{\beta}\left(1 - \lim_{r\longrightarrow-\infty} e_{\ominus\beta}(t,r)\right) \\
&\leq M\left[L_\varrho\|z\|_b + L_f\right]\frac{(1+\mu^*\beta)}{\beta} < \infty.
\end{aligned} \tag{28}$$

Therefore, (23) is well defined.

Now, suppose that z is solution of (23). Then, for each $t_0 \in \mathbb{T}$ we have that

$$\begin{aligned}
z(t) &= \int_{-\infty}^{t_0} T(t-\sigma(s)) f(s,z(s))\Delta s \\
&\quad + \int_{t_0}^{t} T(t-\sigma(s)) f(s,z(s))\Delta s.
\end{aligned} \tag{29}$$

Hence, for $t \geq t_0$ we get that

$$\begin{aligned}
z(t) &= T(t-t_0)\int_{-\infty}^{t_0} T(t_0-\sigma(s)) f(s,z(s))\Delta s + \int_{t_0}^{t} T(t-\sigma(s)) f(s,z(s))\Delta s \\
&= T(t-t_0) z(t_0) + \int_{t_0}^{t} T(t-\sigma(s)) f(s,z(s))\Delta s,
\end{aligned} \tag{30}$$

where $z(t_0) = \int_{-\infty}^{t_0} T(t_0-\sigma(s)) f(s,z(s))\Delta s$. Therefore, $z(t)$ is a mild solution of equation (1). □

Theorem 3. *Let T be a exponentially stable C_0-semigroup with infinitesimal generator A. If*

$$\mathrm{ML}_f < \left[\frac{\beta}{1+\mu^*\beta} - \mathrm{ML}_\varrho\right]\varrho, \tag{31}$$

where L_ϱ is the Lipschitz constant of f in B_ϱ, then equation (1) has a unique mild solution $\overline{z}$ in B_ϱ^b. Moreover, $\overline{z}$ is exponentially stable.

Proof. Let us consider the operator $\mathcal{T}: \mathbb{X}_b \longrightarrow \mathbb{X}_b$ defined by

$$(\mathcal{T}z)(t) = \int_{-\infty}^{t} T(t-\sigma(s)) f(s,z(s))\Delta s. \tag{32}$$

By considering Lemma 2 we will show that the operator $\mathcal{T}$ has a unique fixed point in B_ϱ^b.

For $z \in B^b_\varrho$, we have that

$$\begin{aligned}\|(\mathcal{T}z)(t)\| &\leq \int_{-\infty}^{t} Me_{\ominus\beta}(t,\sigma(s))\left[L_\varrho\|z\|_b + L_f\right] \\ &\leq M\left[L_\varrho\|X\|_b + L_f\right]\frac{1+\mu^*\beta}{\beta} \\ &\leq M\left[L_\varrho\varrho + L_f\right]\frac{1+\mu^*\beta}{\beta} < \varrho.\end{aligned} \tag{33}$$

So, $\mathcal{T}z \in B^b_\varrho$ and therefore $\mathcal{T}\colon B^b_\varrho \longrightarrow B^b_\varrho$. Now, for $z_1, z_2 \in B^b_\varrho$ it follows

$$\begin{aligned}\left\|(\mathcal{T}z_1)(t) - (\mathcal{T}z_2)(t)\right\| &\leq \int_{-\infty}^{t} Me_{\ominus\beta}(t,\sigma(s))L_\varrho\big\|z_1(s) \\ &\quad - z_2(s)\big\|\Delta s \\ &\leq \mathrm{ML}_\varrho\frac{1+\mu^*\beta}{\beta}\|z_1 - z_2\|_b.\end{aligned} \tag{34}$$

Since $ML_\varrho(1+\mu^*+\beta/\beta) < 1$, we get $\mathcal{T}$ is a contraction mapping. Thus, by using the Banach fixed point theorem, $\mathcal{T}$ has a unique fixed point $\overline{z}$ in B^b_ϱ, which satisfies

$$\overline{z}(t) = \int_{-\infty}^{t} T(t-\sigma(s))f(s,\overline{z}(s))\Delta s. \tag{35}$$

Hence by the preceding lemma, $\overline{z}$ is a bounded mild solution of equation (1).

Finally, let us prove that $\overline{z}(t)$ is exponentially stable. Consider an arbitrary solution $z(t)$ of (1) such that $\|z(t_0) - \overline{z}(t_0)\| < \varrho/2M$, with $t_0 \geq 0$, then $\|z(t_0)\| < 2\varrho$. As long as $\|z(t)\|$ remains less than 2ϱ, we get the following estimate:

$$\begin{aligned}\|z(t) - \overline{z}(t)\| &\leq \left\|T(t-t_0)\right\|\left\|z(t_0) - \overline{z}(t_0)\right\| + \int_{t_0}^{t} \|T(t-\sigma(s))\|\|f(s,z(s)) - f(s,\overline{z}(s)\|\Delta s \\ &\leq Me_{\ominus\beta}(t,t_0)\left\|z(t_0) - \overline{z}(t_0)\right\| + \int_{t_0}^{t} M(1+\mu^*\beta)L_\varrho e_{\ominus\beta}(t,s)\|z(s) - \overline{z}(s)\|\Delta s.\end{aligned} \tag{36}$$

So,

$$e_\beta(t,0)\|z(t) - \overline{z}(t)\| \leq Me_\beta(t_0,0)\left\|z(t_0) - \overline{z}(t_0)\right\| + \int_{t_0}^{t} M(1+\mu^*\beta)L_\varrho e_\beta(s,0)\|z(s) - \overline{z}(s)\|\Delta s. \tag{37}$$

By using Gronwall's inequality (see [18]), we obtain that

$$\begin{aligned}e_\beta(t,0)\|z(t) - \overline{z}(t)\| &\leq Me_\beta(t_0,0)\left\|z(t_0) - \overline{z}(t_0)\right\| \\ &\quad e_{\mathrm{ML}_\varrho(1+\mu^*\beta)}(t,t_0),\end{aligned} \tag{38}$$

which implies that

$$\begin{aligned}\|z(t) - \overline{z}(t)\| &\leq M\left\|z(t_0) - \overline{z}(t_0)\right\| \\ &\quad e_{\ominus\left(\beta\ominus\mathrm{ML}_\varrho(1+\mu^*\beta)\right)}(t,t_0),\ for\ t \in [t_0, t_{1\mathbb{T}}).\end{aligned} \tag{39}$$

Since $\|z(t)\| < 2\varrho$ and

$$\beta\ominus\mathrm{ML}_\varrho(1+\mu^*\beta) = \frac{\beta - \mathrm{ML}_\varrho(1+\mu^*\beta)}{1+\mu(t)\mathrm{ML}_\varrho(1+\mu^*\beta)} > 0, \tag{40}$$

then we get

$$\|z(t) - \overline{z}(t)\| \leq M\left\|z(t_0) - \overline{z}(t_0)\right\| \leq \frac{\rho}{2}, \tag{41}$$

for $t \in [t_0, t_1)_\mathbb{T}$. Due that $\|z(t)\| < 2\varrho$ for $t \in [t_0, t_1)_\mathbb{T}$, then $t_1 = \infty$ or $\|z(t_1)\| = 2\varrho$, but in the second case, this contradicts the estimate $\|z(t) - \overline{z}(t)\| \leq \varrho/2$. Therefore, $t_1 = \infty$ and $z(t)$ remains for all $t \geq t_0$ in the ball $B^b_{2\varrho}$ and

$$\begin{aligned}\|z(t) - \overline{z}(t)\| &\leq M\left\|z(t_0) - \overline{z}(t_0)\right\| \\ &\quad e_{\ominus\left(\beta\ominus\mathrm{ML}_\varrho(1+\mu^*\beta)\right)}(t,t_0), \text{for all } t \geq t_0.\end{aligned} \tag{42}$$

If we suppose that f is globally Lipschitz, then the bounded solutions given by Theorem 3 is globally. □

Theorem 4. *Let T be a exponentially stable C_0-semigroup with infinitesimal generator A, suppose that f is globally Lipschitz with constant L and such that*

$$\mathrm{ML} < \frac{\beta}{1+\mu^*\beta}. \tag{43}$$

Then, there exists a unique mild solution of (1) defined on $\mathbb{T}$ and exponentially stable.

Proof. If $\varrho > (\mathrm{ML}_f(1+\mu^*\beta))/(\beta - \mathrm{ML}(1+\mu^*\beta))$, then the condition (43) implies that

$$\mathrm{ML}_f < \left[\frac{\beta}{1+\mu^*\beta} - \mathrm{ML}\right]\varrho. \tag{44}$$

Thus, from the preceding theorem, we have that equation (1) has a unique mild solution $\overline{z}$ in B_ϱ^b. Since the condition (44) is satisfied for ϱ large enough, then $\overline{z}$ is the unique solution of (1) on $\mathbb{X}_b$.

Now, if z is any other solution of (1), then

$$\begin{aligned}\|z(t)-\overline{z}(t)\| &\le Me_{\ominus\beta}(t,t_0)\left\|z(t)-z(t_0)\right\| \\ &\quad + \int_{t_0}^{t} MLe_{\ominus\beta}(t,\sigma(s))\|z(s)-\overline{z}(s)\|\Delta s,\end{aligned} \tag{45}$$

and by using Gronwall's inequality, it is obtained that

$$\begin{aligned}\|z(t)-\overline{z}(t)\| &\le M\left\|z(t_0)-\overline{z}(t_0)\right\| \\ &\quad e_{\ominus(\beta\ominus\mathrm{ML}(1+\mu^*\beta))}(t,t_0), t>t_0.\end{aligned} \tag{46}$$

Since $\beta\ominus\mathrm{ML}(1+\mu^*\beta)>0$, $\overline{z}$ is exponentially stable for all $t>t_0$. □

Corollary 1. *Suppose that*

$$f(t,z) = h(z) + \varphi(t), \tag{47}$$

where $\varphi \in BC_{rd}(\mathbb{T},\mathbb{X})$ and $h\colon \mathbb{X}\longrightarrow\mathbb{X}$ is a Lipschitz function with constant L. Then, the unique bounded solution $x_\varphi(t)$ giving by Theorems 3 and 4 depends continuously on φ.

Proof. Let $\varphi_1,\varphi_2 \in \mathrm{BC}_{\mathrm{rd}}(\mathbb{T},\mathbb{X})$ and $z_{\varphi_1}(\cdot)$, $z_{\varphi_2}(\cdot)$ be the bounded solutions of (1) given by Theorems 3 and 4. Then,

$$\begin{aligned}z_{\varphi_1}(t)-z_{\varphi_2}(t) &= \int_{-\infty}^{t} T(t-\sigma(s))\left[h\left(z_{\varphi_1}(s)\right)-h\left(z_{\varphi_2}(s)\right)\right]\Delta s \\ &\quad + \int_{-\infty}^{t} T(t-\sigma(s))\left[\varphi_1(s)-\varphi_2(s)\right]\Delta s.\end{aligned} \tag{48}$$

Hence,

$$\begin{aligned}\left\|z_{\varphi_1}(t)-z_{\varphi_2}(t)\right\| &\le \int_{-\infty}^{t} MLe_{\ominus\beta}(t,\sigma(s))\left\|z_{\varphi_1}(s)-z_{\varphi_2}(s)\right\|\Delta s \\ &\quad + \int_{-\infty}^{t} Me_{\ominus\beta}(t,\sigma(s))\left\|\varphi_1(s)-\varphi_2(s)\right\|\Delta s,\end{aligned} \tag{49}$$

so

$$\begin{aligned}\left\|z_{\varphi_1}-z_{\varphi_2}\right\|_b &\le \mathrm{ML}\left(\frac{1+\mu^*\beta}{\beta}\right)\left\|z_{\varphi_1}-z_{\varphi_2}\right\|_b \\ &\quad + M\left(\frac{1+\mu^*\beta}{\beta}\right)\left\|\varphi_1-\varphi_2\right\|_b,\end{aligned} \tag{50}$$

and therefore,

$$\left\|z_{\varphi_1}-z_{\varphi_2}\right\|_b \le \frac{M(1+\mu^*\beta)}{\beta - ML(1+\mu^*\beta)}\left\|\varphi_1-\varphi_2\right\|_{b.} \tag{51}$$

□

4. Periodic and Almost Periodic Cases

This section is dedicated to study the cases when the function f is either periodic or almost periodic. We begin with the periodic case.

Let $\tau \in \mathbb{T}_0^+$ fixed. The time scale $\mathbb{T}$ is called τ-periodic if for each $t \in \mathbb{T}$ we get that $t+\tau \in \mathbb{T}$, and the function $f\colon \mathbb{T}\times\mathbb{X}\longrightarrow\mathbb{X}$ is said to be τ-periodic on $\mathbb{T}$ if $f(t+\tau,z) = f(t,z)$ for all $t\in\mathbb{T}$. From Theorem 2.1 in [19] we get that $\sigma(t+\tau)=\sigma(t)+\tau$, $\rho(t+\tau)=\rho(t)+\tau$ and $\mu(t+\tau) = \mu(t)+\tau$. For more details about periodic functions on times scales see [4, 19].

Theorem 5. *Assume that $\mathbb{T}$ is τ-periodic, and $f\colon \mathbb{T}\times\mathbb{X}\longrightarrow\mathbb{X}$ is τ-periodic in t, then the solution $\overline{z}$ obtained in Theorems 3 and 4 is also τ-periodic.*

Proof. Let $\overline{z}$ be the unique solution of (19) on B_ϱ^b. We will prove that $z(t)\colon = \overline{z}(t+\tau)$ is also a solution of (19) on the ball B_ϱ^b. In fact, given $z_0 = \overline{z}(t_0)$ we have

$$\begin{aligned}
\overline{z}(t+\tau) &= T(t+\tau-t_0)z_0 + \int_{t_0}^{t+\tau} T(t+\tau-\sigma(s))f(s,\overline{z}(s))\Delta s \\
&= T(t-t_0)\left\{T(\tau)x_0 + \int_{t_0}^{t_0+\tau} T(t_0+\tau-\sigma(s))f(s,\overline{z}(s))\Delta s\right\} \\
&\quad + \int_{t_0+\tau}^{t+\tau} T(t+\tau-\sigma(s))f(s,\overline{z}(s))\Delta s \\
&= T(t-t_0)\overline{z}(t_0+\tau) + \int_{t_0}^{t} T(t+\tau-\sigma(s+\tau))f(s+\tau,\overline{z}(s+\tau))\Delta s \\
&= T(t-t_0)\overline{z}(t_0+\tau) + \int_{t_0}^{t} T(t-\sigma(s))f(s,\overline{z}(s+\tau))\Delta s.
\end{aligned} \tag{52}$$

Thus, $z(t) := \overline{z}(t+\tau)$ satisfies (19). Since the operator $\mathcal{T}$ has a unique fixed point which is defined by (32), then $\overline{z}(t) = \overline{z}(t+\tau)$ for $t \in \mathbb{T}$.

For studying the almost periodic case, we need some definitions and results about almost periodic functions on time scales, for details of it, one can read [20–22].

A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\omega \in \mathbb{R}: t \pm \omega \in \mathbb{T}, \quad \text{for all } t \in \mathbb{T}\} \neq \{0\}. \tag{53}$$

Suppose that $\mathbb{T}$ is an almost periodic time scale. A function $f \in C(\mathbb{T},\mathbb{X})$ is called an almost periodic function in $t \in \mathbb{T}$ if the ε-translation set of f

$$E\{\varepsilon, f\} = \{\omega \in \Pi: \|f(t+\omega) - f(t)\| < \varepsilon, \quad \forall t \in \mathbb{T}\}, \tag{54}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exist a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon) > 0$ contains a $\omega \in E\{\varepsilon, f\}$ such that

$$\|f(t+\omega) - f(t)\| < \varepsilon, \text{for all } t \in \mathbb{T}. \tag{55}$$

ω is called the ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

For a given $g \in C(\mathbb{T},\mathbb{X})\}$, the hull of g is defined as

$$H(g) = \left\{\psi: \mathbb{T} \longrightarrow \mathbb{X}: \text{there exist}\{\alpha_n\} \in \mathbb{T} \text{ such that } \quad T_{\{\alpha_n\}}g(t) = \psi(t) \text{ exists uniformly on } \mathbb{T}\right\}, \tag{56}$$

where $T_{\{\alpha_n\}}g(t) = \psi(t)$ means that $\psi(t) = \lim_{n\longrightarrow\infty} g(t+\alpha_n)$, when this limit exists. □

Theorem 6. *Suppose that $f(t,z) = h(z) + \varphi(t)$ where φ almost periodic and h globally Lipschitz with constant L_h, then the unique solution $\overline{z}$ given by Theorems 3 and 4 is almost periodic.*

Proof. To prove this theorem, we shall use the Theorem 3.18 in [20]. A function $g \in C(\mathbb{T},\mathbb{X})$ is almost periodic if and only if the hull $H(g)$ is compact in the topology of uniform convergence.

Now, let $\varrho > 0$ and we consider the set $\mathcal{A}_\varrho = \left\{z \in B_\varrho^b: z \text{ is almost periodic}\right\}$, by Theorem 3.27 in [21] we have that $\mathcal{A}_\varrho$ is closed. If $z \in \mathcal{A}_\varrho$, then $g(t) = h(z(t)) + \varphi(t)$ is also an almost periodic function. Let us consider the function given by (32).

$$\begin{aligned}
\phi(t) = (\mathcal{T}z)(t) &= \int_{-\infty}^{t} T(t-\sigma(s))[h(z(s)) + \varphi(s)]\Delta s \\
&= \int_{-\infty}^{t} T(t-s(s))g(s)\Delta s, t \in \mathbb{T}.
\end{aligned} \tag{57}$$

It is enough to establish that $H(\phi)$ is compact in the topology of uniform convergence. In fact, if we consider the sequence $\{\phi_n\}$ in $H(\phi)$, then by using the diagonal procedure, there is a sequence $\{\alpha_n\}$ in $\mathbb{T}$ such that

$$\left\|\phi_n(\cdot+\alpha_n) - \phi_n(\cdot)\right\| < \frac{1}{n}. \tag{58}$$

Due g is almost periodic, then $H(g)$ is relatively compact, and putting $\{g_n(t)\}_{n\geq 1} = \{g(t+\alpha_n)\}_{n\geq 1} \subset H(g)$, then there exists a convergent subsequence $\left\{g_{n_k}\right\}_{k\geq 1}$. On the other hand,

$$\phi_{n_k}\left(t+\alpha_{n_k}\right) = \int_{-\infty}^{t+\alpha_{n_k}} T\left(t+\alpha_{n_k}-\sigma(s)\right)g(s)\Delta s = \int_{-\infty}^{t} T(t-\sigma(s))g\left(s+\alpha_{n_k}\right)\Delta s. \tag{59}$$

Then,

$$\begin{aligned} \left\|\phi_{n_k}(t)-\phi_{n_p}(t)\right\| &\le \left\|\phi_{n_k}(t)-\phi_{n_k}\left(t+\alpha_{n_k}\right)\right\|+\left\|\phi_{n_k}\left(t+\alpha_{n_k}\right)-\phi_{n_p}\left(t+\alpha_{n_p}\right)\right\| \\ &+\left\|\phi_{n_p}\left(t+\alpha_{n_p}\right)-\phi_{n_p}(t)\right\| \le \frac{1}{n_k}+\int_{-\infty}^{t}\|T(t-\sigma(s))\|\left\|g\left(s+\alpha_{n_k}\right)-g\left(s+\alpha_{n_p}\right)\right\|\Delta s \\ &+\frac{1}{n_p} \le \left\|g_{n_k}-g_{n_p}\right\|_b \int_{-\infty}^{t} M e_{\ominus\beta}(s,\sigma(s))\Delta s+\frac{1}{n_k}+\frac{1}{n_p} \\ &\le \left\|g_{n_k}-g_{n_p}\right\|_b\left(\frac{1+\mu^*\beta}{\beta}\right)+\frac{1}{n_k}+\frac{1}{n_p}. \end{aligned} \tag{60}$$

Thus, $\{\phi_n\}$ is a Cauchy sequence in $C(\mathbb{T},\mathbb{X})$, which implies that $H(\phi)$ is relatively compact. So $\mathcal{T}z$ is an almost periodic function. Moreover, $\mathcal{T}(\mathscr{A}_\varrho)\subset A_\varrho$. For this reason, the only fixed point of $\mathcal{T}$ on B_ϱ^b is in $\mathscr{A}_\varrho$, i.e., $\overline{z}$ is almost periodic. □

5. Existence of Classical Solution

In this section we will show that the bounded mild solution of (1) given by equation (32), obtained in Theorems 3 and 4, is also, under certain conditions, a classical solution of (1). In order to achieve this, we need the following theorem, which is an extension of Theorem 1.3.5 of [23]:

Theorem 7. *Let A be a closed linear operator defined on $\mathscr{D}(A)\subset\mathbb{X}$. Let z be a function in $C_{rd}((a,b]_{\mathbb{T}};X)$, where $a\ge-\infty$. If $z(t)\in\mathscr{D}(A)$, $\mathrm{Az}(t)$ is* rd*-continuous on $(a,b]_{\mathbb{T}}$ and the integrals*

$$\int_a^b z(s)\Delta s,\ \int_a^b \mathrm{Az}(s)\Delta s, \tag{61}$$

exist, then

$$\int_a^b z(s)\Delta s\in\mathscr{D}(A)\quad A\int_a^b z(s)\Delta s=\int_a^b \mathrm{Az}(s)\Delta s. \tag{62}$$

Proof. Suppose that $a>-\infty$. Set $c=a+\varepsilon$ where $\varepsilon>0$ is sufficiently small. Since $\int_c^b z(s)\Delta s$ and $\int_c^b \mathrm{Az}(s)\Delta s$ exist, then for each $m\in\mathbb{N}$, there exists $\delta_m>0$ such that

$$\begin{aligned} &\left|\sum_{i=1}^{n(\mathscr{P}_m)} z(\xi_i)(t_i-t_{i-1})-\int_c^b z(s)\Delta s\right|<\frac{1}{m} \\ &\left|\sum_{i=1}^{n(\mathscr{P}_m)} \mathrm{Az}(\xi_i)(t_i-t_{i-1})-\int_c^b \mathrm{Az}(s)\Delta s\right|<\frac{1}{m}, \end{aligned} \tag{63}$$

for every Riemann Δ-sum of f corresponding to a partition $\mathscr{P}_m\in\mathscr{P}_{\delta_m}[c,b]_{\mathbb{T}}$ independently of $\xi_i\in[t_{i-1},t_i)_{\mathbb{T}}$ for $1\le i\le n(\mathscr{P}_m)$.

If we put $S_m=\sum_{i=1}^{n(\mathscr{P}_m)} z(\xi_i)(t_i-t_{i-1})$, then $S_m\in\mathscr{D}(A)$ and $\mathrm{AS}_m=\sum_{i=1}^{n(\mathscr{P}_m)}\mathrm{Az}(\xi_i)(t_i-t_{i-1})$.

Moreover,

$$\lim_{m\longrightarrow\infty} S_m=\int_c^b z(s)\Delta s\quad \lim_{m\longrightarrow\infty}\mathrm{AS}_m=\int_c^b \mathrm{Az}(s)\Delta s. \tag{64}$$

Since A is a closed operator on $\mathscr{D}(A)$, then it follows that

$$\int_c^b z(s)\Delta s\in\mathscr{D}(A)\quad A\int_c^b z(s)\Delta s=\int_c^b \mathrm{Az}(s)\Delta s. \tag{65}$$

Now, making $c\longrightarrow a^+$, we have the result.

If $a=-\infty$, the result follows from the fact that

$$\begin{aligned} \int_{-\infty}^b z(s)\Delta s&=\lim_{a\longrightarrow-\infty}\int_a^b z(s)\Delta s\quad \int_{-\infty}^b \mathrm{Az}(s)\Delta s \\ &=\lim_{a\longrightarrow-\infty}\int_a^b \mathrm{Az}(s)\Delta s, \end{aligned} \tag{66}$$

and the previous result. □

Remark 2. Next theorem pretends to extend a result presented in [11] to time scales; particularly the conditions (1)–(3) of Theorem 8 can be satisfied if $T(t)$ is an analytic semigroup (see [24, 25]).

Theorem 8. *Let $\overline{z}$ be the bounded mild solution of equation (1), obtained in Theorems 3 and 4. If for each $t\in[0,\infty)_T$ and $s\in[-\infty,\tau_n)_{\mathbb{T}}$, where $\{\tau_n\}\subset\mathbb{T}$, $\tau_n<t$ for all $n\in\mathbb{N}$, such that $\tau_n\longrightarrow t$, $n\longrightarrow\infty$, and the following statements hold:*

(1) $T(t-\sigma(s))f(s,\overline{z}(s))\in\mathscr{D}(A)$

(2) $AT(t-\sigma(s))f(s,\overline{z}(s))$ is rd-continuous

(3) $\int_{-\infty}^t AT(t-\sigma(s))f(s,\overline{z}(s))\Delta s$ exists

Then, $\overline{z}$ is a classical solution of (1) on $[0,\infty)_{\mathbb{T}}$, i.e.,

$$\overline{z}^{\Delta}(t)=\mathrm{A}\overline{z}(t)+f(t,\overline{z}(t)),t\in[0,\infty)_{\mathbb{T}}. \tag{67}$$

Proof. We know that $\overline{z}$ can be expressed through

$$\overline{z}(t)=\int_{-\infty}^t T(t-\sigma(s))f(s,\overline{z}(s))\Delta s,t\in\mathbb{T}. \tag{68}$$

Conditions (1)–(3) and the previous theorem imply that

$$\begin{aligned}
&\int_{-\infty}^{\tau_n} T(t-\sigma(s)) f(s,\bar{z}(s))\Delta s \in \mathscr{D}(A),\\
&A\int_{-\infty}^{\tau_n} T(t-\sigma(s)) f(s,\bar{z}(s))\Delta s = \int_{-\infty}^{\tau_n} AT(t-\sigma(s)) f(s,\bar{z}(s))\Delta s.
\end{aligned} \tag{69}$$

Since

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\int_{-\infty}^{\tau_n} T(t-\sigma(s)) f(s,\bar{z}(s))\Delta s\\
&\lim_{n\longrightarrow\infty} A\int_{-\infty}^{\tau_n} T(t-\sigma(s)) f(s,\bar{z}(s))\Delta s = \lim_{n\longrightarrow\infty}\int_{-\infty}^{\tau_n} AT(t-\sigma(s),\bar{z}(s))\Delta s\\
&= \int_{-\infty}^{t} AT(t-\sigma(s)) f(s,\bar{z}(s))\Delta s.
\end{aligned} \tag{70}$$

Baring in mind that A is a closed operator, we have that

$$\int_{-\infty}^{t} T(t-\sigma(s)) f(s,\bar{z}(s))\Delta s \in \mathscr{D}(A). \tag{71}$$

Now, we shall use the fact that, for a Riemann Δ-integrable function g the following equality holds:

$$\int_{t}^{\sigma(t)} g(s)\Delta s = \mu(t) g(t). \tag{72}$$

Next, for $s>0$, we consider the following equalities:

$$\begin{aligned}
\frac{\bar{z}(\sigma(t))-\bar{z}(t+s)}{\mu(t)-s} &= \frac{\bar{z}(\mu(t)+t)-\bar{z}(t+s)}{\mu(t)-s}\\
&= \frac{1}{\mu(t)-s}\left[\int_{-\infty}^{\mu(t)+t} T(t+\mu(t)-\sigma(\tau)) f(\tau,\bar{z}(\tau))\Delta\tau\right.\\
&\quad \left.-\int_{-\infty}^{t+s} T(t+s-\sigma(\tau)) f(\tau,\bar{z}(\tau))\Delta\tau\right]\\
&= \left(\frac{T(\mu(t))-T(s)}{\mu(t)-s}\right)\int_{-\infty}^{t} T(t-\sigma(\tau)) f(\tau,\bar{z}(\tau))\Delta\tau\\
&\quad + \frac{1}{\mu(t)-s}\int_{t}^{t+\mu(t)} T(t+\mu(t)-\sigma(\tau)) f(\tau,\bar{z}(\tau))\Delta\tau\\
&\quad - \frac{1}{\mu(t)-s}\int_{t}^{t+s} T(t+s-\sigma(\tau)) f(\tau,\bar{z}(\tau))\Delta\tau\\
&= \left(\frac{T(\mu(t))-T(s)}{\mu(t)-s}\right)\int_{-\infty}^{t} T(t-\sigma(\tau)) f(\tau,\bar{z}(\tau))\Delta\tau + \frac{\mu(t) f(t,\bar{z}(t))}{\mu(t)-s}\\
&\quad - \frac{1}{\mu(t)-s}\int_{t}^{t+s} T(t+s-\sigma(\tau)) f(\tau,\bar{z}(\tau))\Delta\tau.
\end{aligned} \tag{73}$$

Passing to the limit when $s\longrightarrow 0^+$, it is obtained that

$$\bar{z}^{\Delta}(t) = A\bar{z}(t) + f(t,\bar{z}(t)). \tag{74}$$

A similar discussion for $s < 0$, produces the same result. This concludes the proof of the theorem. □

6. Examples

This section is devoted to present some applications of our results.

Example 1. Consider the equation

$$\begin{aligned} z^{\Delta}(t) &= -\alpha z(t) + \xi(t) e^{-\gamma z(t)} \\ z(0) &= z_0, \end{aligned} \tag{75}$$

where α and γ are the positive constant; $-\alpha \in \mathscr{R}^+$ and ξ: $[0,\infty)_{\mathbb{T}} \longrightarrow \mathbb{R}^+$ is a rd-continuous and bounded function. Notice that $T(t) = e_{-\alpha}(t,0)$ and $\|T(t)\| \le e_{\ominus\alpha}(t,0)$. If we define $f(t,z) = \xi(t)e^{-\gamma z}$, then it follows that

$$\begin{aligned} \left|f(t,z_1) - f(t,z_2)\right| &= |\xi(t)|\left|e^{-\gamma z_1} - e^{-\gamma z_2}\right| \le \xi^* \gamma \left|z_1 - z_2\right|, \\ |f(t,0)| &= \xi(t) \le \xi^*, \end{aligned} \tag{76}$$

where $\xi^* = \sup_{t\in\mathbb{T}}\{\xi(t)\}$. If $(\xi^*)^2\gamma < (\alpha/1 + \mu^*\alpha)$ then all conditions of Theorem 3 are satisfied, therefore the equation (75) has a unique bounded solution which is exponentially stable.

Example 2. Let us consider the time scale $\mathbb{T} = (1/4)\mathbb{Z}$ and the dynamic equation

$$\begin{aligned} u^{\Delta\Delta} + 3u^{\Delta} + 2u &= \alpha \sin(\gamma t)\cos(u) \\ u(0) &= u_0, u^{\Delta}(0) = u_0^{\Delta}. \end{aligned} \tag{77}$$

By using the change of variable $v = u^{\Delta}$, we get that the equation (77) can be written as first order system of dynamic equations.

$$\begin{aligned} z^{\Delta} &= Az + f(t,z), \\ z(0) &= z_0, \end{aligned} \tag{78}$$

where

$$\begin{aligned} z &= \begin{pmatrix} u \\ v \end{pmatrix}, \\ A &= \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \\ f(t,z) &= \begin{pmatrix} 0 \\ \alpha \sin(\gamma t)\cos(u) \end{pmatrix}. \end{aligned} \tag{79}$$

In this case $\mu(t) = (1/4)$ and the eigenvalues of the matrix A are $-2, -1 \in \mathscr{R}^+$. Then, it is easy to show that

$$\begin{aligned} T(t) &= e_A(t,0) \\ &= \begin{pmatrix} e_{-1}(t,0) & e_{-2}(t,0) \\ -e_{-1}(t,0) & -2e_{-2}(t,0) \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{3}{4}\right)^{4t} & \left(\frac{1}{2}\right)^{4t} \\ -\left(\frac{3}{4}\right)^{4t} & -\left(\frac{1}{2}\right)^{4t} \end{pmatrix}, \end{aligned} \tag{80}$$

and $\|T(t)\| \le 4e_{\ominus 1}(t,0) = 4(4/5)^{4t}$. If $|\alpha| < 1/(1+\mu^*) = (4/5)$, then all hypotheses of Theorem 3 are satisfied which implies that equation (77) has a unique bounded solution which is exponentially stable.

Example 3. We consider the following dynamic equation:

$$\begin{aligned} z^{\Delta} &= -\alpha z(t) + \beta \tanh(z(t)) + \varphi(t), \\ z(0) &= z_0, \end{aligned} \tag{81}$$

where $\alpha > 0$. This dynamic equation is the representation on time scales of a single self-excited neuron without delayed excitation. Here z represents the voltage of the neuron, α is the ratio of the capacitance to the resistance, and β the feedback strength. The transfer function is given by $\tanh(z)$ and the function φ represents other input to the neuron. (see [26–28])

Notice that $f(t,z) = \beta\tanh(z) + \varphi(t)$ satisfies $|f(t,z_1) - f(t,z_2)| \le |\beta||z_1 - z_2|$; therefore, if φ is periodic or almost periodic, and $|\beta| < (\alpha/1 + \mu^*)$. Then, the unique solution of (81) is periodic or almost periodic.

Data Availability

Data sharing are not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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