# Existence and Uniqueness of Renormalized Solution to Nonlinear Anisotropic Elliptic Problems with Variable Exponent and $L^{1}$-Data 

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Received 26 September 2022; Revised 18 February 2023; Accepted 25 March 2023; Published 10 April 2023
Academic Editor: Davood D. Ganji
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#### Abstract

Nonlinear partial differential equations are considered as an essential tool for describing the behavior of many natural phenomena. The modeling of some phenomena requires to work in Sobolev spaces with constant exponent. But for others, such as electrorheological fluids, the properties of classical spaces are not sufficient to have precision. To overcome this difficulty, we work in the appropriate spaces called Lebesgue and Sobolev spaces with variable exponent. In recent works, researchers are attracted by the study of mathematical problems in the context of variable exponent. This great interest is motivated by their applications in many fields such as elastic mechanics, fluid dynamics, and image restoration. In this paper, we combine the technic of monotone operators in Banach spaces and approximation methods to prove the existence of renormalized solutions of a class of nonlinear anisotropic problem involving $\vec{p}$ (.)-Leray-Lions operator, a graph, and $L^{1}$ data. In particular, we establish the uniqueness of the solution when the graph data are considered a strictly increasing function.


## 1. Introduction

Partial differential equations (PDEs for short) are considered a fundamental tool for modeling and thus understanding many real-world phenomena. These partial differential equations make it possible to take into account many parameters related to the course of phenomena and the role of these parameters. They also make it possible to predict, sometimes extremely accurately, how the phenomenon evolves over time. This prediction may exist in the very special case of linear PDEs, but when the phenomenon is modeled by a nonlinear PDE, prediction becomes almost impossible.

Nonlinear PDEs appear in many fields including chemistry, physics, and engineering science (see [1-5]). For example, in [6], Gandji et al. used nonlinear equations to study the three-dimensional Bödewadt hybrid nanofluid flow where fluids are composed of water and hexanol. Note that some nonhomogeneous materials such as aluminum
oxide or alumina $\left(\mathrm{Al}_{2} \mathrm{O}_{3}\right)$ have the ability to change state very quickly (in a few milliseconds) physically when an electric field of very small intensity is applied to them. To model the behavior of these materials, classical Lebesgue and Sobolev spaces with constant exponent are not efficient enough to have accuracy. To this end, we commonly work in the Lebesgue and Sobolev spaces with variable exponent. The properties of these nonhomogeneous materials are widely exploited in many technological applications such as shock absorbers and equipment rehabilitation.

The study of PDEs with variable exponent has increased intensively in recent years. The importance of studying such problems is due to the discovery of their applications in the modeling of behavior of certain nonhomogeneous devices in physics, mechanical process, electrorheological fluids, and stationary thermo-rheological viscous flows of nonNewtonian fluids (see [7-11] for more details). They are also used in modeling the propagation of epidemic diseases (see [12]) and image processing ([13]).

In this paper, we are interested in the existence and uniqueness of renormalized solution of the following anisotropic problem:

$$
(E, f)\left\{\begin{array}{l}
\beta(u)-\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}} a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right)+\operatorname{div} F(u) \ni f \text { in } \Omega  \tag{1}\\
u=0 \text { on } \partial \Omega
\end{array}\right.
$$

where $\Omega$ is an open bounded domain of $\mathbb{R}^{N}(N \geq 3)$ with Lipschitz boundary $\partial \Omega, f \in L^{1}(\Omega), F: \mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}}$ is locally Lipschitz continuous, and $\beta: \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is a set-valued maximal monotone mapping such that $0 \in \beta(0)$. We allow the term $\beta(u)$ to be multivalued, not necessarily defined in the whole of $\mathbb{R}$.

Under our assumptions, problem $(E, f)$ is generally not well posed in the framework of weak solution because $F(u)$ may not belong to $\left(L_{\mathrm{loc}}^{1}(\Omega)\right)^{N}(F$ is just continuous on $\mathbb{R})$. To overcome this difficulty, we use the framework of renormalized solutions which requires low regularity than the weak one. This concept of solution first appeared in the work of Lions and Diperna [14] and used later by Lions and Murat to tackle elliptic equation with low summability data (i.e. when the data are $L^{1}$ or a measure).

Analysis of problems involving graph data, Sobolev space $W^{1, p}(\Omega)$ with constant exponent, and generalized Orlicz spaces is already a classical topic investigated since [15-17]. Then, differential inclusion problems have been extended to variable exponent setting in [18-21] and the references therein. In [22], Akdim and Allalou ensured the existence of renormalized solution of a problem close to $(E, f)$ but in the framework of weighted space.

In the literature, special cases of problem $(E, f)$ have been explored in the framework of anisotropic Sobolev space (see [23-25]) and have concerned the problem below:

$$
(P)\left\{\begin{array}{l}
\beta(u)-\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}} a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \ni f \text { in } \Omega  \tag{2}\\
u=0 \text { on } \partial \Omega
\end{array}\right.
$$

where $f$ is a bounded Radon measure or $L^{1}$-function.

Let us recall that the first elliptic problems studied in anisotropic Sobolev space with variable exponent were the works of Mihailescu et al. [26, 27].

For the case where $\beta(.) \equiv 0$, Koné et al. $[28,29]$ used the minimization technics to prove the existence of weak solutions of problem $(P)$ (see also [30, 31]). The case in which $f \in L^{1}(\Omega)$ and $\beta$ are continuous nondecreasing functions from $\mathbb{R}$ to $\mathbb{R}$ is studied in [23]. In [24], Konaté and Ouaro have proved the existence and uniqueness of an entropy solution of problem $(P)$ when $f$ is a Radon measure and $\beta$ is a maximal monotone graph.

When the components of the vector $\vec{p}=\left(p_{1}, \ldots, p_{N}\right)$ are constants, the authors in [32] studied the problem ( $E, f$ ) and established the existence and uniqueness of renormalized solution in the anisotropic Sobolev space $W^{1, \vec{p}}(\Omega)$ with constant components of the vector $\vec{p}$. Since the components of the vector $\vec{p}($.$) are able to vary, the$ $\vec{p}($.$) -Leray-Lions operator A u=-\sum_{i=1}^{N} \partial / \partial x_{i} a_{i}\left(x, \partial u / \partial x_{i}\right)$ which appears in the left-hand side of problem $(E, f)$ is more general than the one which appears in [32].

To our knowledge, all the previous studies dealing with similar problem $(E, f)$ in the framework of variable exponent spaces are focused on particular cases.

In this paper, we extend the recent works [21, 24, 25, 32] by using the ideas developed in [21, 32]. More precisely, we used the technic of monotone operators in Banach spaces and approximation methods to prove the existence and uniqueness of a renormalized solution of problem $(E, f)$ in the context of anisotropic space involving variable exponents $W^{1, \vec{p}}().(\Omega)$. As the novelty of this study, the components of the vector $\vec{p}()=.\left(p_{1}(),. \ldots, p_{N}().\right)$ are able to vary and the diffusion convection term div $F(u)$ is not null. The main difficulty we encounter is how to establish the a priori estimates and convergence results.

Our main results rely on the following assumptions.
Throughout this paper, $\vec{p}()=.\left(p_{1}(),. \ldots, p_{N}().\right)$ is a vector such that the components $p_{i}():. \bar{\Omega} \longrightarrow \mathbb{R}$ are continuous functions (for any $i=1, \ldots, N$ ) satisfying

$$
\begin{equation*}
1<p_{i}^{-}:=\inf _{x \in \Omega} p_{i}(x) \leq p_{i}^{+}:=\sup _{x \in \Omega} p_{i}(x)<\infty, \tag{3}
\end{equation*}
$$

and we set

$$
\begin{equation*}
p_{M}(x):=\max \left(p_{1}(x), \ldots, p_{N}(x)\right) \text { and } p_{m}(x):=\min \left(p_{1}(x), \ldots, p_{N}(x)\right) \tag{4}
\end{equation*}
$$

For any $i=1, \ldots, N$, let $a_{i}: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function verifying the following assumptions.

There exists positive constants $C_{1}, C_{2}, C_{3}$ such that
(i) For a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}$,

$$
\begin{equation*}
\left|a_{i}(x, \xi)\right| \leq C_{1}\left(j_{i}(x)+|\xi|^{p_{i}(x)-1}\right) \tag{5}
\end{equation*}
$$

where $j_{i}$ is a nonnegative function in $L^{p_{i}^{\prime}(.)}(\Omega)$, with $1 / p_{i}(x)+1 / p_{i}^{\prime}(x)=1$.
(ii) For $\xi, \eta \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \Omega$,

$$
\left(a_{i}(x, \xi)-a_{i}(x, \eta)\right)(\xi-\eta) \geq \begin{cases}C_{2}|\xi-\eta|^{p_{i}(x)}, & \text { if }|\xi-\eta| \geq 1  \tag{6}\\ C_{2}|\xi-\eta|^{p_{i}^{-}}, & \text {if }|\xi-\eta|<1\end{cases}
$$

(iii) For $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$
\begin{equation*}
a_{i}(x, \xi) . \xi \geq C_{3}|\xi|^{p_{i}(x)} \tag{7}
\end{equation*}
$$

We assume that

$$
\begin{align*}
& \frac{\bar{p}(N-1)}{N(\bar{p}-1)}<p_{i}^{-}<\frac{\bar{p}(N-1)}{N-\bar{p}}, \frac{p_{i}^{+}-p_{i}^{-}-1}{p_{i}^{-}}<\frac{\bar{p}-N}{\bar{p}(N-1)}  \tag{8}\\
& \quad \sum_{i=1}^{N} \frac{1}{p_{i}^{-}}>1 \tag{9}
\end{align*}
$$

where $N / \bar{p}=\sum_{i=1}^{N} 1 / p_{i}^{-}$.
This paper is structured as follows. In Section 2, we recall some fundamental preliminaries which are useful in this work and we give our main results. In Section 3 we study the
case where $f \in L^{\infty}(\Omega)$. In Section 4, we study the existence and uniqueness of a renormalized solution when $f \in L^{1}(\Omega)$. Finally, in Section 5, we give an example for illustrating our abstract result.

## 2. Preliminary and Main Results

This section is devoted to some definitions and basic properties of anisotropic Lebesgue with Sobolev spaces and variable exponents. Set

$$
\begin{equation*}
C_{+}(\bar{\Omega})=\left\{p \in C(\bar{\Omega}): \min _{x \in \bar{\Omega}} p(x)>1\right\} . \tag{10}
\end{equation*}
$$

For any $p \in C_{+}(\bar{\Omega})$, the variable exponent Lebesgue space is defined by

$$
\begin{equation*}
L^{p(.)}(\Omega):=\left\{u: \Omega \longrightarrow \mathbb{R} \text { a measurable function such that } \int_{\Omega}|u|^{p(x)} \mathrm{d} x<\infty\right\} \tag{11}
\end{equation*}
$$

endowed with the so-called Luxemburg norm

$$
\begin{equation*}
|u|_{p(.)}:=\inf \left\{\lambda>0: \int_{\Omega}\left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d} x \leq 1\right\} \tag{12}
\end{equation*}
$$

The $p($.$) -modular of the space L^{p(.)}(\Omega)$ is the mapping $\rho_{p(.)}: L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
\rho_{p(.)}(u):=\int_{\Omega}|u|^{p(x)} \mathrm{d} x . \tag{13}
\end{equation*}
$$

For any $u \in L^{p(.)}(\Omega)$, we have (see $[33,34]$ )

$$
\begin{equation*}
\min \left\{|u|_{p(.)}^{p^{-}} ;|u|_{p(.)}^{p^{+}}\right\} \leq \rho_{p(.)}(u) \leq \max \left\{|u|_{p(.)}^{p^{-}} ;|u|_{p(.)}^{p^{+}}\right\} . \tag{14}
\end{equation*}
$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, with $1 / p(x)+$ $1 / q(x)=1$ for any $x \in \Omega$, we have the Hölder type inequality

$$
\begin{equation*}
\left|\int_{\Omega} u v \mathrm{~d} x\right| \leq\left(\frac{1}{p^{-}}+\frac{1}{q^{-}}\right)|u|_{p(.)}|v|_{q(.)} . \tag{15}
\end{equation*}
$$

If $\Omega$ is bounded and $p, q \in C_{+}(\bar{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(.)}(\Omega) \longrightarrow L^{q(.)}(\Omega)$ is continuous (see [35], Theorem 2.8).

We defined the anisotropic Sobolev space with variable exponent as follows:

$$
\begin{equation*}
W^{1, \vec{p}(.)}(\Omega):=\left\{u \in L^{p_{M}(.)}(\Omega): \frac{\partial u}{\partial x_{i}} \in L^{p_{i}(.)}(\Omega), i=1, \ldots, N\right\} \tag{16}
\end{equation*}
$$

which is a separable and reflexive Banach space (see [26]) under the norm

$$
\begin{equation*}
\|u\|_{\vec{p}_{(.)}}=|u|_{p_{M}(.)}+\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_{i}}\right|_{p_{i}(.)} \tag{17}
\end{equation*}
$$

We have the following embedding results.

Theorem 1 (see [33], Corollary 2.1). Let $\Omega \subset \mathbb{R}^{N}(N \geq 3)$ be a bounded open set and for all $i=1, \ldots, N, p_{i} \in L^{\infty}(\Omega), p_{i}(x) \geq 1$ a.e. $x \in \Omega$. Then, for any $q \in L^{\infty}(\Omega)$ with $q(x) \geq 1$ a.e. $x \in \Omega$ such that

$$
\begin{equation*}
\text { ess } \inf _{x \in \Omega}\left(p_{M}(x)-q(x)\right)>0 \tag{18}
\end{equation*}
$$

we have the compact embedding

$$
\begin{equation*}
W^{1, \vec{p}(.)}(\Omega) \longrightarrow L^{q(.)}(\Omega) \tag{19}
\end{equation*}
$$

We defined the numbers

$$
\begin{align*}
q & =\frac{N(\bar{p}-1)}{N-1} \\
q^{*} & =\frac{N(\bar{p}-1)}{N-\bar{p}}=\frac{N q}{N-q} . \tag{20}
\end{align*}
$$

Theorem 2 (see [36]). Let $p_{1}, \ldots, p_{N} \in[1, \infty)$; $g \in W^{1,\left(p_{1}, \ldots, p_{N}\right)}(\Omega)$ and

$$
\begin{cases}q=(\bar{p})^{*}, & \text { if }(\bar{p})^{*}<N  \tag{21}\\ q \in[1, \infty), & \text { if }(\bar{p})^{*} \geq N\end{cases}
$$

Then, there exists a constant $C_{4}>0$ depending on $N, p_{1}, \ldots, p_{N}$ if $\bar{p}<N$ and also on $q$ and meas $(\Omega)$ if $\bar{p} \geq N$ such that

$$
\begin{equation*}
\|g\|_{L^{q}(\Omega)} \leq C_{4} \prod_{i=1}^{N}\left[\|g\|_{L^{p_{M}}(\Omega)}+\left\|\frac{\partial g}{\partial x_{i}}\right\|_{L^{p_{i}}(\Omega)}\right]^{1 / N} \tag{22}
\end{equation*}
$$

where $1 / \bar{p}=\sum_{i=1}^{N} p_{i}$ and $(\bar{p})^{*}=N \bar{p} / N-\bar{p}$.
The Marcinkiewicz space $\mathscr{M}^{q}(\Omega)(1<q<+\infty)$ is introduced as the set of measurable function $g: \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$
\begin{equation*}
\lambda_{g}(k):=\operatorname{meas}\{x \in \Omega:|g(x)|>k\}, k \geq 0 \tag{23}
\end{equation*}
$$

satisfies the following:

$$
\begin{equation*}
\lambda_{g}(k) \leq C k^{-q}, \text { for some finite constant } C>0 \tag{24}
\end{equation*}
$$

We will use the following pseudonorm in $\mathscr{M}^{q}(\Omega)$.

$$
\begin{equation*}
\|g\|_{M^{q}(\Omega)}:=\inf \left\{C>0: \lambda_{g}(k) \leq C k^{-q}, \forall k>0\right\} . \tag{25}
\end{equation*}
$$

We defined the truncation function $T_{k},(k>0)$ by

$$
\begin{equation*}
T_{k}(s)=\max \{-k, \min \{k ; s\}\} \tag{26}
\end{equation*}
$$

We observe that $\lim _{k \rightarrow+\infty} T_{k}(s)=s$ and $\left|T_{k}(s)\right|=\min \{|s| ; k\}$.

For any $v \in W^{1, \vec{p}(.)}(\Omega)$, we use $v$ instead of $\left.v\right|_{\partial \Omega}$ for the trace of $v$ on $\partial \Omega$.

Set $\mathscr{T}^{1, \vec{p}(.)}(\Omega)$ as the set of the measurable functions $u: \Omega \longrightarrow \mathbb{R}$ such that for any $k>0, T_{k}(u) \in W^{1, \vec{p}(.)}(\Omega)$.

Lemma 1 (see [30]). Let $g$ be a nonnegative function in $W^{1, \vec{p}(.)}(\Omega)$. Assume $\bar{p}<N$ and there exists a constant $C>0$ such that

$$
\begin{equation*}
\int_{\Omega}\left|T_{k}(g)\right|^{p_{M}^{-}} \mathrm{d} x+\sum_{i=1}^{N} \int_{\{|g| \leq k\}}\left|\frac{\partial g}{\partial x_{i}}\right|^{p_{i}^{-}} \mathrm{d} x \leq C(k+1) \tag{27}
\end{equation*}
$$

for every $k>0$.
Then, there exists a constant $D$, depending on $C$ such that

$$
\begin{equation*}
\|g\|_{M^{q^{*}}(\Omega)} \leq D \tag{28}
\end{equation*}
$$

where $q^{*}=N(\bar{p}-1) / N-\bar{p}$.
We introduce some useful functions as follows.
For $r \in \mathbb{R}$, let $r^{+}:=\max (r, 0)$ and $\operatorname{sign}_{0}^{+}$be the function defined by

$$
\operatorname{sign}_{0}^{+}(r)= \begin{cases}1, & \text { if } r>0  \tag{29}\\ 0, & \text { if } r \leq 0\end{cases}
$$

Let $\quad h_{l}: \mathbb{R} \longrightarrow \mathbb{R} \quad$ be $\quad$ defined $h_{l}(r):=\min \left((l+1-|r|)^{+}, 1\right)$ for each $r, l \in \mathbb{R}$.

For $\sigma>0$, we define $H_{\sigma}^{+}: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$
H_{\sigma}^{+}(r)= \begin{cases}0, & \text { if } r<0 \\ \frac{1}{\sigma} r, & \text { if } 0 \leq r \leq \sigma \\ 1, & \text { if } r>\sigma\end{cases}
$$

and $H_{\sigma}: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$
H_{\sigma}(r)= \begin{cases}l-1, & \text { if } r<-\sigma  \tag{31}\\ \frac{1}{\sigma} r, & \text { if }-\sigma \leq r \leq \sigma \\ 1, & \text { if } r>\sigma\end{cases}
$$

Now, we give our main results.
Theorem 3. Under assumptions (3)-(9) and $f \in L^{1}(\Omega)$, there exists at least one renormalized solution $(u, b)$ to problem $(E, f)$ in the sense that
(i) $u \in \mathscr{T}^{1, \vec{p}(.)}(\Omega), b \in L^{1}(\Omega), \quad u(x) \in \operatorname{dom}(\beta(x))$, $b(x) \in \beta(u(x))$ for a.e. in $\Omega$.
(ii) For all $h \in C_{c}^{1}(\mathbb{R})$ and $\varphi \in W_{0}^{1,} \vec{p}^{(.)}(\Omega) \cap L^{\infty}(\Omega)$,

$$
\begin{align*}
& \sum_{i=1}^{N} \int_{\Omega} a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}[h(u) \varphi] d x+\int_{\Omega} b h(u) \varphi \mathrm{d} x \\
& -\int_{\Omega} F(u) \cdot \nabla[h(u) \varphi] \mathrm{d} x=\int_{\Omega} f h(u) \varphi \mathrm{d} x . \tag{32}
\end{align*}
$$



Theorem 4. Let $(u, b)$ and $(\widetilde{u}, \widetilde{b})$ be two renormalized solutions of problem $(E, f)$. Then,

$$
\left\{\begin{array}{l}
u=\tilde{u} \text { a.e.in } \Omega  \tag{33}\\
b=\widetilde{b} \text { a.e.in } \Omega
\end{array}\right.
$$

## 3. Existence Result for $L^{\infty}$-Data

Theorem 5. Assuming that (3)-(9) hold, $f \in L^{\infty}(\Omega)$, then the problem $(E, f)$ admits at least one renormalized solution.

Proof. We demonstrate Theorem 5 in five steps.
Step 1. Approximate problem.
Let $\beta_{\epsilon}: \mathbb{R} \longrightarrow \mathbb{R}$ be the Yosida regularization of $\beta$ (see [37]), defined by $\beta_{\epsilon}=1 / \epsilon\left(I-(I+\epsilon \beta)^{-1}\right)$ such that $0<\epsilon \leq 1$. We consider the approximate problem

$$
\left(E_{\epsilon}, f\right)\left\{\begin{array}{l}
\beta_{\epsilon}\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right)+\epsilon \arctan \left(u_{\epsilon}\right)-\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}} a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right)-\operatorname{div} F\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right)=f \text { in } \Omega  \tag{34}\\
u_{\epsilon}=0 \text { on } \partial \Omega
\end{array}\right.
$$

Lemma 2. The problem $\left(E_{\epsilon}, f\right)$ has at least one weak solution $u_{\epsilon} W_{0}^{1, \vec{p}(.)}(\Omega)$ in the sense that

$$
\begin{align*}
& \int_{\Omega} \sum_{i=1}^{N} a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \frac{\partial \varphi}{\partial x_{i}} \mathrm{~d} x+\int_{\Omega}\left(\beta_{\epsilon}\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)+\epsilon \arctan \left(u_{\epsilon}\right)\right) \varphi \mathrm{d} x\right.  \tag{35}\\
& +\int_{\Omega} F\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right) \cdot \nabla \varphi \mathrm{d} x=\langle f, \varphi\rangle
\end{align*}
$$

where $\varphi \in W_{0}^{1, \vec{p}(.)}(\Omega) \cap L^{\infty}(\Omega)$ and $\langle.,$.$\rangle denotes the d u$ ality pairing between $W_{0}^{1, \vec{p}(.)}(\Omega)$ and $\left(W_{0}^{1, \vec{p}(.)}(\Omega)\right)^{*}$.

Proof. We define the operators $A_{1, \epsilon}, A_{2, \epsilon}$, and $A_{\epsilon}:=A_{1, \epsilon}+A_{2, \epsilon}$, acting from $W_{0}^{1, \vec{p}(.)}(\Omega)$ into its dual $\left(W_{0}^{1, \vec{p}(.)}(\Omega)\right)^{*}$ as follows:

$$
\begin{equation*}
\left\langle A_{1, \epsilon} u, \varphi\right\rangle=\langle A u, \varphi\rangle+\int_{\Omega}\left(\beta_{\epsilon}\left(T_{1 / \epsilon}(u)+\epsilon \arctan \left(u_{\epsilon}\right)\right) \varphi \mathrm{d} x, \forall u, \varphi \in W_{0}^{1, \vec{p}}(\Omega),\right. \tag{36}
\end{equation*}
$$

where

$$
\begin{align*}
& \langle A u, \varphi\rangle=\int_{\Omega} \sum_{i=1}^{N} a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \frac{\partial \varphi}{\partial x_{i}} \mathrm{~d} x,  \tag{37}\\
& \left\langle A_{2, \epsilon} u, \varphi\right\rangle=-\int_{\Omega} F\left(T_{1 / \epsilon}(u)\right) \cdot \nabla \varphi \mathrm{d} x, \forall u, \varphi \in W_{0}^{1, \vec{p}}(\Omega) .
\end{align*}
$$

Reasoning as in [25] (see also [21, 32]), we can prove that the operator $A_{\epsilon}$ is pseudomonotone, coercive, and bounded. Then, we deduce from [38] (Theorem 2.7) that $A_{\epsilon}$ is surjective. Since $f \in\left(W_{0}^{1, \vec{p}(.)}(\Omega)\right)^{*}$, it follows that the problem $\left(E_{\epsilon}, f\right)$ admits at least one solution $u_{\epsilon} \in W_{0}^{1, \vec{p}(.)}(\Omega)$.

Taking into account the monotonicity of $a_{i}$ and $\beta_{\epsilon}$ and following the same lines as in [21, 32], we establish the following comparison principle which will be essential in the proof of uniqueness of the solution.

Proposition 1. Let $f, \tilde{f} \in L^{\infty}(\Omega)$ and $u_{\epsilon}, \tilde{u}_{\epsilon} \in W_{0}^{1,} \vec{p}().(\Omega)$ such that $u_{\epsilon}$ is a solution of $\left(E_{\epsilon}, f\right)$ and $\tilde{u}_{\epsilon}$ is a solution of $\left(E_{\epsilon}, \tilde{f}\right)$. Then, the following comparison principle holds:

$$
\epsilon \int_{\Omega}\left(\arctan \left(u_{\epsilon}\right)-\arctan \left(\tilde{u}_{\epsilon}\right)^{+} \leq \int_{\Omega}(f-\tilde{f}) \operatorname{sign}_{0}^{+}\left(u_{\epsilon}-\tilde{u_{\epsilon}}\right) .\right.
$$

Remark 1 (see $[21,32]$ ). By assuming that $f \leq \tilde{f}$ a.e. in $\Omega$, an immediate consequence of the proposition above is the inequality $u_{\epsilon} \leq \tilde{u}_{\epsilon}$. In addition, we have $\beta_{\epsilon}\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right) \leq \beta_{\epsilon}\left(T_{1 / \epsilon}\left(\tilde{u}_{\epsilon}\right)\right)$ a.e. in $\Omega$.

Step 2. A apriori estimates.
Lemma 3 (see [25]). If $u_{\epsilon}$ is a solution of problem $\left(E_{\epsilon}, f\right)$, then

$$
\begin{aligned}
& \sum_{i=1}^{N} \int_{\left\{\left|u_{\epsilon}\right| \leq k\right\}}\left|\frac{\partial u_{\epsilon}}{\partial x_{i}}\right|^{p_{i}(x)} \mathrm{d} x \leq \frac{k\|f\|_{\infty}}{C_{5}}, \\
& \left\|\beta_{\epsilon}\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right)\right\|_{\infty} \leq\|f\|_{\infty} \\
& \sum_{i=1}^{N} \int_{\left\{l\left|u_{\epsilon}\right|<l+k\right\}} a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \frac{\partial u_{\epsilon}}{\partial x_{i}} \mathrm{~d} x \leq k \int_{\left\{\left|u_{\epsilon}\right|>l\right\}}|f| \mathrm{d} x \\
& \int_{\left\{\left|u_{\epsilon}\right| \leq k\right\}}\left|\nabla T_{k}\left(u_{\epsilon}\right)\right|^{p_{m}^{-}} \mathrm{d} x \leq C_{6}
\end{aligned}
$$

where $k, C_{5}, C_{6}>0$.

Lemma 4 (see [23, 30]). There exist some constants $C_{7}, C_{8}>0$ such that
(i) $\left\|u_{\epsilon}\right\|_{M^{q^{*}}(\Omega)} \leq C_{7}$.
(ii) $\left\|\left(\partial u_{\epsilon} / \partial x_{i}\right)\right\|_{M^{p_{i}^{q} q \bar{p}}(\Omega)} \leq C_{8}, \forall i=1, \ldots, N$.

Remark 2 (see [25]). There exist $C_{9}, C_{10}>0$ such that

$$
\begin{align*}
& \sum_{i=1}^{N} \int_{\Omega}\left|\frac{\partial u}{\partial x_{i}}\right|^{p_{i}(x)} \mathrm{d} x \geq C_{9} \nabla u\| \| \|_{L^{P_{m}}(\Omega)}^{p_{m}^{-}}-N \operatorname{meas}(\Omega), \\
& \sum_{i=1}^{N} \int_{\left\{l\left\langle u_{\epsilon}\right|<l+k\right\}} a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \frac{\partial u_{\epsilon}}{\partial x_{i}} \mathrm{~d} x \leq k\|f\|_{\infty}\left|\left\{\left|u_{\epsilon}\right| \geq l\right\}\right| \leq C_{10}(k) l^{-p_{m}^{-}} . \tag{38}
\end{align*}
$$

## Step 3. Convergence results.

Lemma 5 (see [23, 25]). Assume that $0<\epsilon \leq \rightarrow$ and $u_{\epsilon}$ is a solution of $\left(E_{\epsilon}, f\right)$. Then, there exist $u \in W_{0}^{1,}{ }^{(.)}(\Omega)$ and $b \in L^{\infty}(\Omega)$ such that for a non-relabelled subsequence of $\left(u_{\epsilon}\right)_{0<\epsilon \leq 1}$ as $\epsilon \downarrow 0$,
$u_{\epsilon} \longrightarrow u$ in $L^{\vec{p}(.)}(\Omega)$ and a.e. in $\Omega ;$ $\frac{\partial u_{\epsilon}}{\partial x_{i}}$ converges in measure to the weak partial gradi ent of $u$;
$a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \longrightarrow a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right)$ in $L^{1}(\Omega)$ a.e. $x \in \Omega ;$
$a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \frac{\partial u_{\epsilon}}{\partial x_{i}} \longrightarrow a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \frac{\partial u}{\partial x_{i}}$ in $L^{1}(\Omega)$ and $a . e . x \in \Omega$
and $\beta_{\epsilon}\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right)-b$ weakly $-*$ in $L^{\infty}(\Omega)$.

Moreover, for any $k>0$,

$$
\begin{gather*}
\frac{\partial T_{k}\left(u_{\epsilon}\right)}{\partial x_{i}}-\frac{\partial T_{k}(u)}{\partial x_{i}} \text { in } L^{p_{i}(.)}(\Omega), \\
a_{i}\left(x, \frac{\partial T_{k}\left(u_{\epsilon}\right)}{\partial x_{i}}\right) \longrightarrow a_{i}\left(x, \frac{\partial T_{k}(u)}{\partial x_{i}}\right) \text { in } L^{1}(\Omega) . \tag{42}
\end{gather*}
$$

Step 4. Passing to limit.
Let $h \in C_{c}^{1}(\mathbb{R})$ and $\in W_{0}^{\vec{p}^{(.)}}(\Omega) \cap L^{\infty}(\Omega)$. We apply the test function $h_{l}\left(u_{\epsilon}\right) h(u) \varphi$ in (35) to get

$$
\begin{equation*}
J_{\epsilon, l}^{1}+J_{\epsilon, l}^{2}+J_{\epsilon, l}^{3}+J_{\epsilon, l}^{4}=J_{\epsilon, l}^{5}, \tag{43}
\end{equation*}
$$

where

$$
\begin{align*}
& J_{\epsilon, l}^{1}=\int_{\Omega} \beta_{\epsilon}\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right) h_{l}\left(u_{\epsilon}\right) h(u) \varphi \mathrm{d} x, \\
& J_{\epsilon, l}^{2}=\sum_{i=1}^{N} \int_{\Omega} a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}\left[h_{l}\left(u_{\epsilon}\right) h(u) \varphi\right] \mathrm{d} x, \\
& J_{\epsilon, l}^{3}=\int_{\Omega} F\left(T_{1 / \epsilon}\left(u_{\epsilon}\right)\right) \cdot \nabla\left[h_{l}\left(u_{\epsilon}\right) h(u) \varphi\right] \mathrm{d} x,  \tag{44}\\
& J_{\epsilon, l}^{4}=\epsilon \int_{\Omega} \arctan \left(u_{\epsilon}\right) h_{l}\left(u_{\epsilon}\right) h(u) \varphi \mathrm{d} x, \\
& J_{\epsilon, l}^{5}=\int_{\Omega} f h_{l}\left(u_{\epsilon}\right) h(u) \varphi \mathrm{d} x .
\end{align*}
$$

We first observe that $\lim _{\epsilon\rfloor 0} J_{\epsilon, l}^{4}=0$. Then, letting $\epsilon \downarrow 0$ and $l \uparrow \infty$ in (43), we obtain

$$
\begin{align*}
& \int_{\Omega} \sum_{i=1}^{N} a_{i}\left(x, \frac{\partial u_{\epsilon}}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}[h(u) \varphi] \mathrm{d} x+\int_{\Omega} b h(u) \varphi \mathrm{d} x  \tag{45}\\
& \quad+\int_{\Omega} F(u) . \nabla[h(u) \varphi] \mathrm{d} x=\int_{\Omega} f h(u) \varphi \mathrm{d} x
\end{align*}
$$

where $h \in C_{c}^{1}(\mathbb{R})$ and $\varphi \in W_{0}^{1,} \vec{p}^{(.)}(\Omega) \cap L^{\infty}(\Omega)$ (for the convergence result, see [25]).

Step 5. Subdifferential argument.
To end the proof, we have (see $[25,39]$ )
(i) $u(x) \in \operatorname{dom}(\beta(x)), b(x) \in \beta(u(x))$ a.e. in $\Omega$.
(ii) $\int_{\substack{i k<|u|<k+1\} \\+\infty}} a_{i}\left(x, \partial u_{\epsilon} / \partial x_{i}\right) \partial / \partial x_{i} \mathrm{~d} x \longrightarrow 0 \quad$ as $\quad k \longrightarrow$

Remark 3 (see [21]). If ( $u, b$ ) is a renormalized solution of $(E, f)_{1,} \vec{p}_{(.)}^{\text {for }}$ second member $f \in L^{\infty}(\Omega)$, then $u \in W_{0}^{1, p(.)}(\Omega) \cap L^{\infty}(\Omega)$. Moreover, $u$ is a weak solution of ( $E, f$ ).

## 4. The Case of $L^{1}$-Data

4.1. Proof of Theorem 1. This proof is made in several steps.
4.1.1. Step 1: Approximate Problem. The first step consists in approximating the second member by bounded function. For $f \in L^{1}(\Omega)$ and $m, n \in \mathbb{N}$, we define $f_{m, n}: \Omega \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
f_{m, n}(x)=\max \{\min (f(x), m),-n\} \text { a.e. } x \Omega . \tag{46}
\end{equation*}
$$

Note that $f_{m, n} \in L^{\infty}(\Omega),\left|f_{m, n}(x)\right| \leq|f(x)|$ a.e. in $\Omega$ and $f_{m, n} \rightarrow f$ in $L^{1}(\Omega)$ as $n, m \rightarrow \infty$. We have also

$$
\begin{equation*}
\left\|f_{m, n}\right\|_{1} \leq f\|f\|_{1} \tag{47}
\end{equation*}
$$

According to Theorem 5, the problem ( $E, f_{m, n}$ ) admits a renormalized solution $\left(u_{m, n}, b_{m, n}\right) \in W_{0}^{1,} \vec{p}^{(.)}(\Omega) \times L^{\infty}(\Omega)$. That is,

$$
\begin{align*}
& \int_{\Omega} \sum_{i=1}^{N} a_{i}\left(x, \frac{\partial u_{m, n}}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}\left[h\left(u_{m, n}\right) \varphi\right] \mathrm{d} x+\int_{\Omega} b_{m, n} h\left(u_{m, n}\right) \varphi \mathrm{d} x \\
& \quad+\int_{\Omega} F\left(u_{m, n}\right) \cdot \nabla\left[h\left(u_{m, n}\right) \varphi\right] \mathrm{d} x=\int_{\Omega} f_{m, n} h\left(u_{m, n}\right) \varphi \mathrm{d} x, \tag{48}
\end{align*}
$$

where $h \in C_{c}^{1}(\mathbb{R})$ and $\varphi \in W_{0}^{1, \vec{p}(.)}(\Omega) \cap L^{\infty}(\Omega)$.
Our goal is to show that these approximated solutions $\left(u_{m, n}, b_{m, n}\right)$ tend, as $n, m$ go to $\infty$, to a couple of functions $(u, b)$ which are renormalized solutions of problem ( $E, f$ ). We begin by giving some useful a priori estimates.

Lemma 6. If $\left(u_{m, n}, b_{m, n}\right)$ is a renormalized solution of for $k>0$ and $m, n \in \mathbb{N}$. problem $\left(E, f_{m, n}\right)$, then

$$
\begin{gather*}
\sum_{i=1}^{N} \int_{\Omega}\left|\frac{\partial u_{m, n}}{\partial x_{i}}\right|^{p_{i}(x)} \mathrm{d} x \leq \frac{k\|f\|_{1}}{C_{5}}  \tag{49}\\
\left\|b_{m, n 1} \leq\right\| f_{1} \tag{50}
\end{gather*}
$$

Proof. For any $k, l>0$, by applying $h_{l}\left(u_{m, n}\right) T_{k}\left(u_{m, n}\right)$ as test function in (48), we obtain

$$
\begin{align*}
& \int_{\Omega} \sum_{i=1}^{N} a_{i}\left(x, \frac{\partial u_{m, n}}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}\left[h\left(u_{m, n}\right) T_{k}\left(u_{m, n}\right)\right] \mathrm{d} x+\int_{\Omega} b_{m, n} h\left(u_{m, n}\right) T_{k}\left(u_{m, n}\right) \mathrm{d} x  \tag{51}\\
& \quad+\int_{\Omega} F\left(u_{m, n}\right) \cdot \nabla\left[h\left(u_{m, n}\right) T_{k}\left(u_{m, n}\right)\right] \mathrm{d} x=\int_{\Omega} f_{m, n} h\left(u_{m, n}\right) T_{k}\left(u_{m, n}\right) \mathrm{d} x .
\end{align*}
$$

Using the same arguments as used in the proof of Lemma 2, we get (49).

For the proof of (50), see [21, 32, 39].
Remark 4 (see $[21,32,39]$ ). Letting $m, n$ go to $\infty$, we get the following convergences.

$$
\begin{align*}
& b_{m, n} \longrightarrow b \text { in } L^{1}(\Omega) \text { and a.e.in } \Omega \\
& u_{m, n} \longrightarrow u \text { a.e.in } \Omega \tag{52}
\end{align*}
$$

where $u: \Omega \longrightarrow \overline{\mathbb{R}}$ is a measurable function.
The next lemma will be used to show that $u$ is finite a.e. in $\Omega$.

Lemma 7. If $\left(u_{m, n}, b_{m, n}\right)$ is a renormalized solution of $\left(E, f_{m, n}\right)$, then there exists a constant $C_{13}>0$, not depending on $m, n \in \mathbb{N}$, such that

$$
\begin{equation*}
\left|\left\{\left|u_{m, n}\right| \geq l\right\}\right| \leq C_{13} l^{-\left(p_{m}^{-}\right)} \tag{53}
\end{equation*}
$$

for all $l \geq 1$.

$$
\begin{equation*}
\sum_{i=1}^{N} \int_{l\left|u_{m, n}\right|<l+k} a_{i}\left(x, \frac{\partial u_{m, n}}{\partial x_{i}}\right) \frac{\partial u_{m, n}}{\partial x_{i}} \mathrm{~d} x \leq k\left(\int_{\left\{\left|u_{m, n}\right|>l\right\} \cap\{|f|<\delta\}}|f| \mathrm{d} x+\int_{\{|f|>\delta\}}|f| d x\right), \tag{55}
\end{equation*}
$$

for any $k, l, \delta>0$. Now using (53) in (55), we get

$$
\begin{equation*}
\sum_{i=1}^{N} \int_{l\left\langle u_{m, n}\right|<l+k} a_{i}\left(x, \frac{\partial u_{m, n}}{\partial x_{i}}\right) \frac{\partial u_{m, n}}{\partial x_{i}} d x \leq \delta k C_{13} l^{-\left(p_{m}^{-}-1\right)}+k \int_{\{|f|>\delta\}}|f| \mathrm{d} x \tag{56}
\end{equation*}
$$

where $k, \delta>0, l \geq 1$, and $m, n \in \mathbb{N}$.
4.1.3. Step 3: Basic Convergence

Proof. Using Remark 2, we get

$$
\begin{equation*}
\left|\left\{\left|u_{m, n}\right| \geq l\right\}\right| \leq C\left(p_{m}^{-}, N\right) l^{p_{m}^{-}}\left(\sum_{i=1}^{N} \int_{\Omega}\left|\frac{\partial u}{\partial x_{i}}\right|^{p_{i}(x)} \mathrm{d} x+N . \operatorname{meas}(\Omega)\right), \tag{54}
\end{equation*}
$$

where $m, n \in \mathbb{N}$ and $C\left(p_{m}^{-}, N\right)$ is a constant coming from Sobolev embedding in (19). From (49) and (54), we deduce (53).

Remark 5. $u$ is finite a.e. in $\Omega$ and $b \in \beta(u)$ a.e. in $\Omega$. Indeed, the proof relies on Lemma 4.6 and subdifferential argument (see [21, 32, 39] for details).

Remark 6. If ( $u_{m, n}, b_{m, n}$ ) is a renormalized solution of $\left(E, f_{m, n}\right)$, choosing $h_{v}\left(u_{m, n}\right) T_{k}\left(u_{m, n}-T_{l}\left(u_{m, n}\right)\right)$ as a test function in (48), discarding positive terms, and letting $\nu$ go to $\infty$, we obtain

Lemma 8 (see [25], Lemma 3). For $i=1, \ldots, N$ and $m, n \in \mathbb{N}$, if $\left(u_{m, n}, b_{m, n}\right)$ is a renormalized solution of $\left(E, f_{m, n}\right)$, then there exists a subsequence $(m(n))_{n}$ such that
posing $f_{n}:=f_{m(n), n}, b_{n}:=b_{m(n), n}, u_{n}:=u_{m(n), n}$, there exists $u \in W_{0}^{1, \vec{p}}{ }_{(.)}^{(\Omega)}$ such that $u \in \operatorname{dom}(\beta)$ a.e. in $\Omega$ and the convergences below hold:

$$
\begin{gather*}
u_{n} \longrightarrow u \text { in measure and a.e.in } \Omega, \\
\frac{\partial u_{n}}{\partial x_{i}} \text { converges in measure to the weak partial gradi ent of } u,  \tag{57}\\
a_{i}\left(x, \frac{\partial u_{n}}{\partial x_{i}}\right) \longrightarrow a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \text { in } L^{1}(\Omega) \text { a.e. } x \in \Omega .
\end{gather*}
$$

More over, for any $k>0$,

$$
\begin{equation*}
a_{i}\left(x, \frac{\partial}{\partial x_{i}} T_{k}\left(u_{n}\right)\right) \longrightarrow a_{i}\left(x, \frac{\partial}{\partial x_{i}} T_{k}(u)\right) \text { in } L^{1}(\Omega) \text { strongly and in } L^{p_{i}^{\prime}(.)}(\Omega) \text { weakly. } \tag{58}
\end{equation*}
$$

### 4.1.4. Step 4: Strong Convergence

Remark 7. Arguing as in [25] (Lemma 1), we obtain equality (iii), namely,

$$
\begin{equation*}
l \xrightarrow{\lim _{+\infty}} \int_{\{l\langle u|<l+1\}} a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \frac{\partial u}{\partial x_{i}} \mathrm{~d} x=0 \tag{59}
\end{equation*}
$$

To complete the proof of Theorem 3, it remains to verify (ii). To this end, we choose $h_{l}\left(u_{n}\right) h(u) \varphi$ as test function in (48) to obtain

$$
\begin{equation*}
I_{n, l}^{1}+I_{n, l}^{2}+I_{n, l}^{3}=I_{n, l}^{4} \tag{60}
\end{equation*}
$$

where

$$
\begin{align*}
& I_{n, l}^{1}=\int_{\Omega} b_{n} h_{l}\left(u_{n}\right) h(u) \varphi \mathrm{d} x \\
& I_{n, l}^{2}=\int_{\Omega} \sum_{i=1}^{N} a_{i}\left(x, \frac{\partial u_{n}}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}\left[h_{l}\left(u_{n}\right) h(u) \varphi\right] \mathrm{d} x  \tag{61}\\
& I_{n, l}^{3}=\int_{\Omega} F\left(u_{n}\right) \cdot \nabla\left[h_{l}\left(u_{n}\right) h(u) \varphi\right] \mathrm{d} x \\
& I_{n, l}^{4}=\int_{\Omega} f_{n} h_{l}\left(u_{n}\right) h(u) \varphi \mathrm{d} x . \tag{67}
\end{align*}
$$

Letting $n$ go to $\infty$ and using Lemma 7 , we obtain

$$
\begin{align*}
\lim _{n \rightarrow+\infty} I_{n, l}^{1} & =\int_{\Omega} b h_{l}(u) h(u) \varphi \mathrm{d} x \\
\lim _{n \rightarrow+\infty} I_{n, l}^{4} & =\int_{\Omega} f h_{l}(u) h(u) \varphi \mathrm{d} x . \tag{62}
\end{align*}
$$

By rewriting as follows:

$$
\begin{equation*}
I_{n, l}^{2}=I_{n, l}^{2,1}+I_{n, l}^{2,2} \tag{63}
\end{equation*}
$$

where

$$
\begin{align*}
& I_{n, l}^{2,1}=\sum_{i=1}^{N} \int_{\Omega} h_{l}\left(u_{n}\right) a_{i}\left(x, \frac{\partial u_{n}}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}[h(u) \varphi] \mathrm{d} x, \\
& I_{n, l}^{2,2}=\sum_{i=1}^{N} \int_{\Omega} h_{l}^{\prime}\left(u_{n}\right) h(u) \varphi a_{i}\left(x, \frac{\partial u_{n}}{\partial x_{i}}\right) \frac{\partial u_{n}}{\partial x_{i}} \mathrm{~d} x, \tag{64}
\end{align*}
$$

and reasoning as in [25], we obtain

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} I_{n, l}^{2,1}=\sum_{i=1}^{N} \int_{\Omega} h_{l}(u) a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}[h(u) \varphi] \mathrm{d} x . \tag{65}
\end{equation*}
$$

From (56), we deduce that

$$
\left|\lim _{n \rightarrow \infty} I_{n, l}^{2,2}\right| \leq\left\|h_{\infty}\right\| \varphi_{\infty}\left(\delta C_{13} l^{-\left(p_{m}^{-}-1\right)}+\int_{\{|f|>\delta\}}|f| d x\right),
$$

where $n \in \mathbb{N}$ and $l \geq 1, \delta>0$.
Split $I_{n, l}^{3}=I_{n, l}^{3,1}+I_{n, l}^{3,2}$, where

$$
\begin{aligned}
& \left.I_{n, l}^{3,1}=\int_{\Omega} h_{l}^{\prime}\left(u_{n}\right) h\left(u_{n}\right) \varphi F\left(u_{n}\right)\right) \cdot \nabla u_{n} \mathrm{~d} x, \\
& I_{n, l}^{3,2}=\int_{\Omega} h_{l}\left(u_{n}\right) F\left(u_{n}\right) \cdot \nabla\left[h\left(u_{n}\right) \varphi\right] \mathrm{d} x .
\end{aligned}
$$

Passing to limit as $n$ goes to $\infty$, we get

$$
\begin{align*}
\lim _{n \longrightarrow \infty} I_{n, l}^{3,1} & =\int_{\Omega} h_{l}(u) F(u) \cdot \nabla[h(u) \varphi] \mathrm{d} x, \\
\lim _{n \longrightarrow \infty} I_{n, l}^{3,2} & =\int_{\Omega} h_{l}^{\prime}(u) h(u) \varphi F(u) . \nabla u \mathrm{~d} x . \tag{68}
\end{align*}
$$

For all $\delta>0$ and $l \geq 1$, letting $n \longrightarrow \infty$ in (60), we obtain

$$
\begin{equation*}
I_{l}^{1}+I_{l}^{2}+I_{l}^{3}+I_{l}^{4}+I_{l}^{5}=I_{l}^{6} \tag{69}
\end{equation*}
$$

where

$$
\begin{aligned}
I_{l}^{1} & =\int_{\Omega} b h_{l}(u) h(u) \varphi \mathrm{d} x \\
I_{l}^{2} & =\sum_{i=1}^{N} \int_{\Omega} h_{l}(u) a_{i}\left(x, \frac{\partial T_{k+1}(u)}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}[h(u) \varphi] \mathrm{d} x \\
\left|I_{l}^{3}\right| & \leq\left\|h_{\infty}\right\| \varphi_{\infty}\left(\delta C_{4} l^{-\left(p_{m}^{-}-1\right)}+\int_{\{|f|>\delta\}}|f| \mathrm{d} x\right), \forall \delta>0 \\
I_{l}^{4} & =\int_{\Omega} h_{l}^{\prime}(u) h(u) \varphi F(u) . \nabla u \mathrm{~d} x \\
I_{l}^{5} & =\int_{\Omega} h_{l}(u) F(u) . \nabla[h(u) \varphi] \mathrm{d} x \\
I_{l}^{6} & =\int_{\Omega} f h_{l}(u) h(u) \varphi \mathrm{d} x
\end{aligned}
$$

Let $k>0$ such that sup $p h \subset[-k, k]$. Replacing $u$ by $T_{k}(u)$ and passing to limit, as $l$ goes to $\infty$, in each term of (69), we get

$$
\begin{align*}
& \lim _{l \longrightarrow \infty} I_{l}^{1}=\int_{\Omega} b h(u) \varphi \mathrm{d} x  \tag{71}\\
& \lim _{l \longrightarrow \infty} I_{l}^{2}=\sum_{i=1}^{N} \int_{\Omega} a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \frac{\partial}{\partial x_{i}}[h(u) \varphi] \mathrm{d} x  \tag{72}\\
& \left|\lim _{l \longrightarrow \infty} I_{l}^{3}\right| \leq\|h\|_{\infty}\|\varphi\|_{\infty}\left(\int_{\{|f|>\delta\}}|f| \mathrm{d} x\right), \forall \delta>0  \tag{73}\\
& l \xrightarrow{\lim _{\longrightarrow}} I_{l}^{4}=0  \tag{74}\\
& \lim _{l \longrightarrow \infty} I_{l}^{5}=\int_{\Omega} F(u) . \nabla[h(u) \varphi] \mathrm{d} x \tag{75}
\end{align*}
$$

$$
\begin{equation*}
\lim _{l \longrightarrow \infty} I_{l}^{6}=\int_{\Omega} f h(u) \varphi \mathrm{d} x \tag{76}
\end{equation*}
$$

Thanks to (72)-(76), we pass to the limit in (69) as $\delta \longrightarrow \infty$, to get (32).
4.2. Proof of Theorem 2. We highlight that the uniqueness is more delicate and it is necessary to have additional hypothesis.

Here we assume that $\beta$ is strictly increasing, and we prove a uniqueness result for renormalized solution of the problem ( $E, f$ ) where $f \in L^{1}(\Omega)$.

Proposition 2. Let $f, \tilde{f} \in L^{1}(\Omega)$ and $(u, b)$ and $(\widetilde{u}, \widetilde{b})$ be renormalized solutions of $(E, f)$ and $(E, \tilde{f})$, respectively. Then, the following comparison principle holds:

$$
\begin{equation*}
\int_{\Omega}(b-\widetilde{b}) \operatorname{sign}_{0}^{+}(u-\widetilde{u}) \mathrm{d} x \leq \int_{\Omega}(f-\tilde{f}) \operatorname{sign}_{0}^{+}(u-\widetilde{u}) \mathrm{d} x . \tag{77}
\end{equation*}
$$

Proof. Let $\delta, l>0, H_{\delta}^{+}$be the Lipschitz approximation of the $\operatorname{sign}_{0}^{+}$function.

The fact that $(u, b),(\widetilde{u}, \widetilde{b})$ are renormalized solutions implies that $T_{l+1}(u), T_{l+1}(\widetilde{u}) \in W_{0}^{1, \vec{p}^{(.)}}(\Omega) \cap L^{\infty}(\Omega)$ for $l>0$. Hence, $H_{\delta}^{+}\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right)$ is an admissible test function.

Taking $h=h_{l}$ and writing the renormalized equalities corresponding to solutions $(u, b)$ and $(\widetilde{u}, \widetilde{b})$, respectively, with test function $H_{\delta}^{+}\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right)$ and adding up both results, we get

$$
\begin{equation*}
I_{l, \delta}^{1}+I_{l, \delta}^{2}+I_{l, \delta}^{3}+I_{l, \delta}^{4}+I_{l, \delta}^{5}=I_{l, \delta}^{6} \tag{78}
\end{equation*}
$$

where

$$
\begin{align*}
& I_{l, \delta}^{1}=\int_{\Omega}\left(b h_{l}(u)-\widetilde{b} h_{l}(\widetilde{u}) H_{\delta}^{+}\right)\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right) \mathrm{d} x \\
& I_{l, \delta}^{2}=\sum_{i=1}^{N} \int_{\Omega}\left(h_{l}^{\prime}(u) a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right) \frac{\partial u}{\partial x_{i}}-h_{l}^{\prime}(\widetilde{u}) a_{i}\left(x, \frac{\partial \widetilde{u}}{\partial x_{i}}\right)\right) \cdot H_{\delta}^{+}\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right) \mathrm{d} x \\
& I_{l, \delta}^{3}=\frac{1}{\delta} \sum_{i=1}^{N} \int_{K}\left(h_{l}(u) a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right)-h_{l}(\widetilde{u}) a_{i}\left(x, \frac{\partial \widetilde{u}}{\partial x_{i}}\right)\right) \frac{\partial}{\partial x_{i}}\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right) \mathrm{d} x  \tag{79}\\
& I_{l, \delta}^{4}=\int_{\Omega}\left(h_{l}^{\prime}(u) F(u) . \nabla u-h_{l}^{\prime}(\widetilde{u}) F(\widetilde{u}) . \nabla \widetilde{u}\right) H_{\delta}^{+}\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right) \mathrm{d} x \\
& I_{l, \delta}^{5}=\frac{1}{\delta} \int_{K}\left(h_{l}(u) F(u)-h_{l}(\widetilde{u}) F(\widetilde{u})\right) \cdot \nabla\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right) \mathrm{d} x \\
& I_{l, \delta}^{6}=\int_{\Omega}\left(f h_{l}(u)-\widetilde{f} h_{l}(\widetilde{u})\right) H_{\delta}^{+}\left(T_{l+1}(u)-T_{l+1}(\widetilde{u})\right) \mathrm{d} x
\end{align*}
$$

with $K=\left\{0<T_{l+1}(u)-T_{l+1}(\widetilde{u})<\delta\right\}$. Reasoning as in [21], i.e., neglecting the positive part of $I_{l, \delta}^{3}$ and using the fact that
$F$ is locally Lipschitz continuous, we can pass to the limit as $\delta \longrightarrow 0$.

Using the condition (iii) of Theorem 3, we pass to limit as $l \longrightarrow \infty$ to obtain (77).

To end the proof, we assume $f=\tilde{f}$. Then, following the same lines as in [40], we obtain

$$
\begin{aligned}
& \int_{\Omega}(b-\tilde{b}) \operatorname{sign}_{0}^{+}(u-\tilde{u}) d x \leq 0, \\
& u=\widetilde{u}, b=\widetilde{b} \text { a.e. in } \Omega .
\end{aligned}
$$

## 5. Example

An example that is covered by our assumption is the following anisotropic $\vec{p}$ (.)-harmonic problem: set

$$
\begin{gather*}
a_{i}\left(x, \frac{\partial u}{\partial x_{i}}\right)=\left|\frac{\partial u}{\partial x_{i}}\right|^{p_{i}(x)-2} \frac{\partial u}{\partial x_{i}} \text {, where } p_{i}(.)=p \text { for any } i=1, \ldots, N, \\
\beta(u)=(u-1)^{+}-(u-1)^{-}, F=\left(F_{i}\right)_{i=1, \ldots, N}: \mathbb{R} \longrightarrow \mathbb{R}^{N} . \tag{81}
\end{gather*}
$$

Then, we have the problem

$$
\left\{\begin{array}{l}
l l(u-1)^{+}-(u-1)^{-}-\sum_{i=1}^{N} \frac{\partial}{\partial x_{i}}\left(\left|\frac{\partial u}{\partial x_{i}}\right|^{p_{i}(x)-2} \frac{\partial u}{\partial x_{i}}\right)+\operatorname{div} F(u)=f \text { in } \Omega  \tag{82}\\
u=0 \text { on } \partial \Omega .
\end{array}\right.
$$

$a_{i}(x, \xi)$ are Carathéodory functions satisfying the growth condition (4), the coercivity (6), and the monotonicity condition (5).

Since all the hypothesis of Theorem 2 are fulfilled, problem (82) has at least one solution for all $\in L^{1}(\Omega)$.

## 6. Conclusion

In the present study, we investigated the existence and uniqueness of solution of a class of anisotropic nonlinear elliptic problems defined with inclusion equation and Dirichlet boundary condition. Governing equations are solved by using the technic of monotone operators in Banach spaces and approximation methods. The novelty of this work relies on transposing nonlinear PDEs from classical (Lebesgue and Sobolev) spaces into generalized Lebesgue and Sobolev spaces with variable exponents. The main conclusions of this work are given as follows:
(1) When the components of the vector $\vec{p}()=.\left(p_{1}(),. \ldots, p_{N}().\right)$ are able to vary, the problem ( $E, f$ ) admits an unique renormalized solution in the anisotropic Sobolev space $W_{0}^{1, \vec{p}^{(.)}}(\Omega)$. Moreover if the graph is a nondecreasing function, the solution is unique.
(2) The main result of this study extends the previous works in $[25,32]$ in the context of anisotropic space with variable exponent.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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