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Research Article

Oscillation and Asymptotic Behavior of Three-Dimensional Third-Order Delay Systems

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In this paper, oscillation and asymptotic behavior of three-dimensional third-order delay systems are discussed. Some sufficient conditions are obtained to ensure that every solution of the system is either oscillatory or nonoscillatory and converges to zero or diverges as t goes to infinity. A special technique is adopted to include all possible cases for all nonoscillatory solutions (NOSs). The obtained results included illustrative examples.

1. Introduction

Differential equations are one of the most important topics in applied mathematics due to their multiple applications; for example, see [1–3]. Among these equations are delay differential equations (DDEs). DDE is an important type of differential equation in which the derivative of a function depends not only on the current value of the function but also on its past values with a finite time delay. Therefore, ordinary differential equations (ODEs) are a special case of DDEs. The effect of the presence of delay or not affects in one way or another the behavior of solving differential equations; for example, it is not possible to obtain an oscillating solution for ODEs of the first order, but in DDEs, this is possible.

Oscillation theory is an important branch of the applied theory of differential equations related to the study of oscillating phenomena in technology and the natural and social sciences. This interest is heightened by the existence of time delays. The presence or absence of oscillatory solutions is one of the most important topics in oscillatory theory for a given equation or system [4]. In the 1840s, the development of oscillation theory for ODEs began when Sturm's classic work appeared, in which oscillation comparison theorem were proved for solutions of homogeneous linear second-order ODE equations [5]. In 1921, the first paper on oscillating functional differential equations was written by Fite [6]. In 1987, Ladde et al. [7] presented their book's oscillation theory of differential equations with deviating arguments. In 1991, Győri et al. [8] presented one of the most important books on oscillation theory in DDE, which included many applications, followed by several books specializing in oscillation; for example, see Bainov and Mishev [9]. Numerous research studies and theses have been written about the oscillation and asymptotic behavior of DDEs with various orders. The reader can see these research studies in [10–18] and the references cited therein. However, there are few studies (books or papers) that discuss the concept of oscillation for solving delay equations such as Ladde et al. [7, 19], Foltynska [20], Agarwal et al. [21], Mohamad and Abdulkareem [22], Abdulkareem et al. [23], Akın-Bohner et al. [24], Špániková [25], and the references cited therein.

Up to our knowledge, there is no research published dealing with the study of almost oscillation and asymptotic behavior of three-dimensional delay system (3D-DS) of the third order; this is the reason why we entered into this type of research.

We consider the three-dimensional half-linear system as follows:

$$\begin{cases} \left(p_1(t)\left(y_1''(t)\right)^{\alpha_1}\right)' = \lambda q_1(t) y_2^{\alpha_1}\left(\tau_1(t)\right), \\ \left(p_2(t)\left(y_2''(t)\right)^{\alpha_2}\right)' = \lambda q_2(t) y_3^{\alpha_2}\left(\tau_2(t)\right), \quad t \geq t_0, \\ \left(p_3(t)\left(y_3''(t)\right)^{\alpha_3}\right)' = \lambda q_3(t) y_1^{\alpha_3}\left(\tau_3(t)\right). \end{cases} \tag{1}$$

The following hypotheses are assumed to be satisfied:

(i) $\lambda \in \{1, -1\}$,

(ii) $p_i, q_i \in C([t_0, \infty], \mathbb{R}^+)$ for large t,

(iii) $\tau_i, \sigma_i \in C([t_0, \infty], \mathbb{R})$, $\tau_i(t) \leq t$, and $\mathrm{Lim}_{t \longrightarrow \infty} \tau_i(t) = \infty$,

(iv) $\tau_i, \sigma_i \in C([t_0, \infty], \mathbb{R})$, $\tau_i(t) \leq t$, and $\mathrm{Lim}_{t \longrightarrow \infty} \tau_i(t) = \infty$,

(v) $\alpha_i > 0$ is the ratio of two odd integers,

(vi) $y_i(t) \in C^2([t_0, \infty]; \mathbb{R})$, $p_i(t)(y_i''(t))^{\alpha_i} \in C^1([t_0, \infty]; \mathbb{R}), i = 1, 2, 3$.

A solution $X(t) = (y_1(t), y_2(t), y_3(t))^T$ is said to oscillate if at least one component is oscillatory. Otherwise, the solution is called nonoscillatory.

This paper consists of five sections; in the second and third sections, the nonoscillatory solutions (NOSs) to the system (1) are studied with certain conditions. In the fourth section, the system (1) oscillation is studied with certain conditions. Finally, we give some examples that illustrate the results.

2. NOS of System (1), Case $\lambda = 1$

In this section, we study the asymptotic behavior of NOS with $\lambda = 1$, which we use in the following sections.

Lemma 1. *Suppose that $X(t)$ is a NOS to the system (1) with $\lambda = 1$, and*

$$\limsup_{t \longrightarrow \infty} \int_{t_2}^{t} \int_{s}^{\delta(s)} \left(\frac{1}{p_i(v)}\right)^{1/\alpha_i} \mathrm{d}v\, \mathrm{d}s = \infty, \quad i = 1, 2, 3. \tag{2}$$

Then there are only K_1–K_8 possible classes:

Proof. Suppose that $X(t)$ is an eventual positive solution to the system (1) (the case $X(t)$ is an eventually negative is similar). Then, from (1), it follows that

$$\begin{aligned} &\left(p_1(t)\left(y_1''(t)\right)^{\alpha_1}\right)' \geq 0, \left(p_2(t)\left(y_2''(t)\right)^{\alpha_2}\right)' \\ &\geq 0, \left(p_3(t)\left(y_3''(t)\right)^{\alpha_3}\right)' \geq 0. \end{aligned} \tag{3}$$

That means $p_1(t)(y_1''(t))^{\alpha_1}$, $p_2(t)(y_2''(t))^{\alpha_2}$, and $p_3(t)(y_3''(t))^{\alpha_3}$ are nondecreasing; hence, there exists $t_1 \geq t_0$ such that $p_1(t)(y_1''(t))^{\alpha_1}, p_2(t)(y_2''(t))^{\alpha_2}$, and $p_3(t)(y_3''(t))^{\alpha_3}$ are eventually positive or eventually negative. So, eight cases can be discussed, which are as follows:

Now, we discuss the cases in Table 1 successively:

(i) Since $p_i(t)(y_i''(t))^{\alpha_i} > 0$ and $(p_i(t)(y_i''(t))^{\alpha_i})' \geq 0, i = 1, 2, 3$, then $p_i(t)(y_i''(t))^{\alpha_i}$ is positive nondecreasing, then there exists $b_i > 0, t_2 \geq t_1$ such that $p_i(t)(y_i''(t))^{\alpha_i} \geq b_i$

$$y_i''(t) \geq b_i^{1/\alpha_i} \left(\frac{1}{p_i(t)}\right)^{1/\alpha_i}, \quad t \geq t_2. \tag{4}$$

Integrating (4) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, we obtain

$$y_i'(\delta(t)) - y_i'(t) \geq b_i^{1/\alpha_i} \int_t^{\delta(t)} \left(\frac{1}{p_i(s)}\right)^{1/\alpha_i} \mathrm{d}s. \tag{5}$$

We claim that $y_i'(t) > 0$ for $t \geq t_3 \geq t_2$, otherwise if $y_i'(t) < 0$. for $t \geq t_3 \geq t_2$, then (5) becomes

$$y_i'(t) \leq -b_i^{1/\alpha_i} \int_t^{\delta(t)} \left(\frac{1}{p_i(s)}\right)^{1/\alpha_i} \mathrm{d}s. \tag{6}$$

Integrating (6) from t_3 to t, we get

$$y_i(t) - y_i(t_3) \leq -b_i^{1/\alpha_i} \int_{t_3}^{t} \int_{s}^{\delta(s)} \left(\frac{1}{p_i(v)}\right)^{1/\alpha_i} \mathrm{d}v\, \mathrm{d}s. \tag{7}$$

Letting $t \longrightarrow \infty$ the last inequality leads to $\mathrm{Lim}_{t \longrightarrow \infty} y_i(t) = -\infty$, which is a contradiction. Hence the claim was verified and $y_i'(t) > 0$ and $y_i''(t) > 0$, this case leads to $\mathrm{Lim}_{t \longrightarrow \infty} y_i(t) = \infty$. That is $(y_1, y_2, y_3) \in K_1$.

(ii) Since $\zeta_i(t)(y_i''(t))^{\alpha_i} < 0$ and $(\zeta_i(t)(y_i''(t))^{\alpha_i})' \geq 0, i = 1, 2, 3$.

That is $\zeta_i(t)(y_i''(t))^{\alpha_i}$, is negative nondecreasing. So there are $\mathcal{O}_i \leq 0, i = 1, 2, 3$, such that $\mathrm{Lim}_{t \longrightarrow \infty} \zeta_i(t)(y_i''(t))^{\alpha_i} = \mathcal{O}_i \leq 0$, hence $\zeta_i(t)(y_i''(t))^{\alpha_1} \leq \mathcal{O}_i$, $t \geq t_2$ and so

$$y_i''(t) \leq \mathcal{O}_i^{1/\alpha_i} \left(\frac{1}{\zeta_i(t)}\right)^{1/\alpha_i}, \quad t \geq t_2. \tag{8}$$

Integrating (8) from t to $\delta(t)$ yields

$$y_i'(\delta(t)) - y_i'(t) \leq \mathcal{O}_i^{1/\alpha_i} \int_t^{\delta(t)} \left(\frac{1}{\zeta_i(s)}\right)^{1/\alpha_i} \mathrm{d}s. \tag{9}$$

We have two for $y_3'(t)$:

(a) If $y_i'(t) > 0$. For $t \geq t_3 \geq t_2$, Then the last inequality becomes

$$y_i'(t) \geq -\mathcal{O}_i^{1/\alpha_i} \int_t^{\delta(t)} \left(\frac{1}{\zeta_i(s)}\right)^{1/\alpha_i} \mathrm{d}s. \tag{10}$$

Integrating (10) from t_3 to t yields

$$y_i(t) - y_i(t_3) \geq -\mathcal{O}_i^{1/\alpha_i} \int_{t_3}^{t} \int_{t}^{\delta(s)} \left(\frac{1}{\zeta_i(v)}\right)^{1/\alpha_i} \mathrm{d}v\, \mathrm{d}s. \tag{11}$$

As $t \longrightarrow \infty$, it follows, either $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = \infty$, (if $\mathscr{E}_i < 0$) or $y_i(t)$ is bounded away from zero (if $\mathscr{E}_i = 0$).

(b) $y_i'(t) < 0$ and $y_i''(t) < 0$, leads to $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = -\infty$, which is a contradiction, which means $X(t) \in K_2$.

(iii) Since $\wp_1(t)(y_1''(t))^{\alpha_1} < 0, \wp_2(t)(y_2''(t))^{\alpha_2} < 0$ and $(\wp_i(t)(y_i''(t))^{\alpha_i})' \geq 0$, that is, $\wp_j(t)(y_j''(t))^{\alpha_j}$ are negative nondecreasing, $j = 1, 2$, there exists $l_j \leq 0$, such that $\mathrm{Lim}_{t\longrightarrow\infty} \wp_j(t)(y_j''(t))^{\alpha_j} = l_j \leq 0$. Then, $\wp_j(t)(y_j''(t))^{\alpha_j} \leq l_j, t \geq t_2$, thus

$$y_j''(t) \leq l_j^{1/\alpha_j}\left(\frac{1}{\wp_j(t)}\right)^{1/\alpha_j}, \quad t \geq t_2, j = 1, 2. \tag{12}$$

Integrating (10) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, we obtain

$$y_j'(\delta(t)) - y_j'(t) \leq l_j^{1/\alpha_j}\int_t^{\delta(t)}\left(\frac{1}{\wp_j(s)}\right)^{1/\alpha_j} \mathrm{d}s. \tag{13}$$

We claim that $y_j'(t) > 0, t \geq t_3 \geq t_2, j = 1, 2$, for otherwise if $y_j'(t) < 0, t \geq t_3 \geq t_2$ and $y_j''(t) < 0$ this implies to $y_j(t) < 0$ and $\mathrm{Lim}_{t\longrightarrow\infty} y_j(t) = -\infty$ which is a contradiction. Hence, $y_j(t) > 0, y_j'(t) > 0$, and $y_j''(t) < 0$. Now, $\wp_3(t)(y_3''(t))^{\alpha_3} > 0$ and $(\wp_3(t)(y_3''(t))^{\alpha_3})' \geq 0$, so $\wp_3(t)(y_3''(t))^{\alpha_3}$, is positive nondecreasing, then there exists $b_3 > 0$ and $t_2 \geq t_1$ such that

$$y_3''(t) \geq b_3^{1/\alpha_3}\left(\frac{1}{\wp_3(t)}\right)^{1/\alpha_3}, \quad t \geq t_2. \tag{14}$$

Integrating (14) from t to $\delta(t)$, we obtain

$$y_3'(\delta(t)) - y_3'(t) \geq b_3^{1/\alpha_3}\int_t^{\delta(t)}\left(\frac{1}{\wp_3(s)}\right)^{1/\alpha_3} \mathrm{d}s. \tag{15}$$

We claim that $y_3'(t) > 0$ for $t \geq t_3 \geq t_2$, otherwise if $y_3'(t) < 0$ for $t \geq t_3 \geq t_2$, then the last inequality becomes

$$y_3'(t) \leq -b_3^{1/\alpha_3}\int_t^{\delta(t)}\left(\frac{1}{\wp_3(s)}\right)^{1/\alpha_3} \mathrm{d}s. \tag{16}$$

Integrating (16) from t_3 to t

$$y_3(t) - y_3(t_3) \leq -b_3^{1/\alpha_3}\int_{t_3}^{t}\int_s^{\delta(s)}\left(\frac{1}{\wp_3(v)}\right)^{1/\alpha_3} \mathrm{d}v\,\mathrm{d}s. \tag{17}$$

Letting $t \longrightarrow \infty$, then inequality (17) leads to $\mathrm{Lim}_{t\longrightarrow\infty} y_3(t) = -\infty$, which is a contradiction. Hence $y_3'(t) > 0$ and $y_3''(t) > 0$, this case leads to $\mathrm{Lim}_{t\longrightarrow\infty} y_3(t) = \infty$, and so $X(t) \in K_3$. Analogously from the subcases (**iv-viii**), one can get $X(t) \in K_n, n = 4, 5, \ldots, 8$, respectively. □

3. NOS of the System (1), Case $\lambda = -1$

In this section, we study the asymptotic behavior of NOS with $\lambda = -1$, which we use in the following sections.

Lemma 2. *Assume that $X(t)$ is NOS of (1) with $\lambda = -1$ and let (2) hold. Then there are only L_1–L_8 possible classes.*

Proof. Suppose that $X(t)$ be an eventual positive solution of (1), then $(\wp_i(t)(y_i''(t))^{\alpha_i})' \leq 0, i = 1, 2, 3, t \geq t_1 \geq t_0$.

This means that $\wp_i(t)(y_i''(t))^{\alpha_i}$ is nonincreasing, so from Table 2, eight subcases can be discussed successively.

(i) $\wp_1(t)(y_1''(t))^{\alpha_1} > 0, \wp_2(t)\ (y_2''(t))^{\alpha_2} > 0, \wp_3(t)(y_3''(t))^{\alpha_3} > 0$.

Since $\wp_i(t)(y_i''(t))^{\alpha_i}$ is positively nonincreasing, there exists $l_i \geq 0, i = 1, 2, 3$ such that, $\mathrm{Lim}_{t\longrightarrow\infty} \wp_i(t)(y_i''(t))^{\alpha_i} = l_i \geq 0$, and then there exists $t_2 \geq t_1$ such that $\wp_i(t)(y_i''(t))^{\alpha_i} \geq l_i, t \geq t_2$, therefore,

$$y_i''(t) \geq l_i^{1/\alpha_i}\left(\frac{1}{\wp_i(t)}\right)^{1/\alpha_i}, \quad t \geq t_2. \tag{18}$$

Integrating (18) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, we obtain

$$y_i'(\delta(t)) - y_i'(t) \geq l_i^{1/\alpha_i}\int_t^{\delta(t)}\left(\frac{1}{\wp_i(s)}\right)^{1/\alpha_i} \mathrm{d}s. \tag{19}$$

We have two cases for $y_i'(t)$.

(a) If $y_i'(t) < 0$, for $t \geq t_3 \geq t_2$, in that case, we claim that $l_i = 0$ otherwise $l_i > 0$, then (19) becomes

$$y_i'(t) \leq -l_i^{1/\alpha_i}\int_t^{\delta(t)}\left(\frac{1}{\wp_i(s)}\right)^{1/\alpha_i} \mathrm{d}s. \tag{20}$$

Integrating (20) from t_3 to t

$$y_i(t) - y_i(t_3) \leq -l_i^{1/\alpha_i}\int_{t_3}^{t}\int_s^{\delta(s)}\left(\frac{1}{\wp_i(v)}\right)^{1/\alpha_i} \mathrm{d}v\,\mathrm{d}s. \tag{21}$$

As $\longrightarrow \infty$, it follows that $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = -\infty$, which is a contradiction, hence $l_i = 0$.

(b) If $y_i'(t) > 0$, and $y_i''(t) > 0$, it follows that $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = \infty$. Thus, $X(t) \in L_1$.

(ii) $\wp_1(t)(y_1''(t))^{\alpha_1} < 0, \wp_2(t)\ (y_2''(t))^{\alpha_2} < 0, \wp_3(t)(y_3''(t))^{\alpha_3} < 0$.

then $y_i''(t) < 0, i = 1, 2, 3$. Since $\wp_i(t)(y_i''(t))^{\alpha_i}$, is negative nonincreasing, then there exists $b_i < 0$, and $t_2 \geq t_1$ such that $\wp_i(t)(y_i''(t))^{\alpha_i} \leq b_i$ for $t \geq t_2$, therefore,

$$y_i''(t) \leq b_i^{1/\alpha_i}\left(\frac{1}{\wp_i(t)}\right)^{1/\alpha_i}, \quad t \geq t_2. \tag{22}$$

integrating (22) from t to $\delta(t)$, we obtain

$$y_i'(\delta(t)) - y_i'(t) \leq b_i^{1/\alpha_i}\int_t^{\delta(t)}\left(\frac{1}{\wp_i(s)}\right)^{1/\alpha_i} \mathrm{d}s, \tag{23}$$

We claim that $y_i'(t) > 0$ for $\geq t_3 \geq t_2$, otherwise if $y_i'(t) < 0$ for $t \geq t_3 \geq t_2$, and $y_i''(t) < 0$ implies that $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = -\infty$, which is a contradiction, thus $y_i'(t) > 0$ then (23) becomes

Table 1: The eight cases of $\wp_2(t)(y_2''(t))^{\alpha_2}, t \geq t_1, i = 1, 2, 3,$ can occur in (1).

(i)	$\wp_1(t)(y_1''(t))^{\alpha_1} > 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} > 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} > 0$
(ii)	$\wp_1(t)(y_1''(t))^{\alpha_1} < 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} < 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} < 0$
(iii)	$\wp_1(t)(y_1''(t))^{\alpha_1} < 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} < 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} > 0$
(iv)	$\wp_1(t)(y_1''(t))^{\alpha_1} < 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} > 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} < 0$
(v)	$\wp_1(t)(y_1''(t))^{\alpha_1} > 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} < 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} < 0$
(vi)	$\wp_1(t)(y_1''(t))^{\alpha_1} > 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} > 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} < 0$
(vii)	$\wp_1(t)(y_1''(t))^{\alpha_1} < 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} > 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} > 0$
(viii)	$\wp_1(t)(y_1''(t))^{\alpha_1} > 0$	$\wp_2(t)(y_2''(t))^{\alpha_2} < 0$	$\wp_3(t)(y_3''(t))^{\alpha_3} > 0$

$$y_i'(t) \geq -b_i^{1/\alpha_i} \int_t^{\delta(t)} \left(\frac{1}{\wp_i(s)}\right)^{1/\alpha_i} \mathrm{d}s. \tag{24}$$

Integrating (24) from t_3 to t

$$y_i(t) - y_i(t_3) \geq -b_i^{1/\alpha_i} \int_{t_3}^{t} \int_s^{\delta(s)} \left(\frac{1}{\wp_i(v)}\right)^{1/\alpha_i} \mathrm{d}v\, \mathrm{d}s. \tag{25}$$

As $\longrightarrow \infty$, it follows that $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = \infty$. Thus $X(t) \in L_2$.

(iii) $\wp_1(t)(y_1''(t))^{\alpha_1} < 0, \wp_2(t)(y_2'' \quad (t))^{\alpha_2} < 0, \wp_3(t)(y_3''(t))^{\alpha_3} > 0$.

Then, $y_j''(t) < 0, j = 1, 2$ and $y_3''(t) > 0$. Since $\wp_j(t)(y_j''(t))^{\alpha_i}$, are negative and nonincreasing, then there exists $b_j < 0$, and $t_2 \geq t_1$ such that $\wp_j(t)(y_j''(t))^{\alpha_i} \leq b_j$ for $t \geq t_2$, therefore,

$$y_j''(t) \leq b_j^{1/\alpha_i} \left(\frac{1}{\wp_j(t)}\right)^{1/\alpha_i}, \quad t \geq t_2. \tag{26}$$

Integrating (26) from t to $\delta(t)$, we obtain

$$y_j'(\delta(t)) - y_j'(t) \leq b_j^{1/\alpha_i} \int_t^{\delta(t)} \left(\frac{1}{\wp_j(s)}\right)^{1/\alpha_j} \mathrm{d}s, \tag{27}$$

and we claim that $y_j'(t) > 0$ for $\geq t_3 \geq t_2$, otherwise if $y_j'(t) < 0$ for $t \geq t_3 \geq t_2$, and $y_j''(t) < 0$ implies that $\mathrm{Lim}_{t\longrightarrow\infty} y_j(t) = -\infty$, which is a contradiction, thus $y_j'(t) > 0$ then (27) becomes

$$y_j'(t) \geq -b_j^{1/\alpha_i} \int_t^{\delta(t)} \left(\frac{1}{\wp_j(s)}\right)^{1/\alpha_j} \mathrm{d}s. \tag{28}$$

Integrating the last inequality from t_3 to t yields

$$y_j(t) - y_j(t_3) \geq -b_j^{1/\alpha_i} \int_{t_3}^{t} \int_t^{\delta(s)} \left(\frac{1}{\wp_j(v)}\right)^{1/\alpha_j} \mathrm{d}v\, \mathrm{d}s. \tag{29}$$

As $\longrightarrow\infty$, it follows that $\mathrm{Lim}_{t\longrightarrow\infty} y_j(t) = \infty, j = 1, 2$.

Concerning $\wp_3(t)(y_3''(t))^{\alpha_3} > 0$ and nonincreasing, so there exists $l_3 \geq 0$, such that, $\mathrm{Lim}_{t\longrightarrow\infty} \wp_3(t)(y_3''(t))^{\alpha_3} = l_3 \geq 0$, then $\wp_3(t)(y_3''(t))^{\alpha_3} \geq l_3, t \geq t_2 \geq t_1$ therefore

$$y_3''(t) \geq l_3^{1/\alpha_3} \left(\frac{1}{\wp_3(t)}\right)^{1/\alpha_3}, \quad t \geq t_2. \tag{30}$$

Integrating the last inequality from t to $\delta(t)$ leads to

$$y_3'(\delta(t)) - y_3'(t) \geq l_3^{1/\alpha_3} \int_t^{\delta(t)} \left(\frac{1}{\wp_3(s)}\right)^{1/\alpha_3} \mathrm{d}s, \tag{31}$$

we have two cases for $y_3'(t)$.

(a) If $y_3'(t) > 0$ for $t \geq t_3 \geq t_2$, and $y_3''(t) > 0$, it follows that $\mathrm{Lim}_{t\longrightarrow\infty} y_3(t) = \infty$.

(b) If $y_3'(t) < 0$ for $t \geq t_3 \geq t_2$, we claim that $l_3 = 0$, otherwise $l_3 > 0$, then (31) reduced to

$$y_3'(t) \leq -l_3^{1/\alpha_3} \int_t^{\delta(t)} \left(\frac{1}{\wp_3(s)}\right)^{1/\alpha_3} \mathrm{d}s. \tag{32}$$

Integrating (32) from t_3 to t

$$y_3(t) - y_3(t_3) \leq -l_3^{1/\alpha_3} \int_{t_3}^{t} \int_s^{\delta(s)} \left(\frac{1}{\wp_3(v)}\right)^{1/\alpha_3} \mathrm{d}v\, \mathrm{d}s. \tag{33}$$

As $\longrightarrow\infty$, it follows that $\mathrm{Lim}_{t\longrightarrow\infty} y_3(t) = -\infty$, which is a contradiction. Hence, $(y_1, y_2, y_3) \in L_3$. Analogously from the subcases (**iv**-**viii**), one can get $X(t) \in L_n, n = 4, 5, \ldots, 8$, respectively. □

4. Main Results of System (1)

In this section, some theorems and corollaries are established, which ensure that all bounded solutions of system (1) are either oscillatory or nonoscillatory and converge to zero as $t \longrightarrow \infty$. On the other hand, all unbounded solutions of system (1) are either oscillatory or nonoscillatory diverge to infinity when $t \longrightarrow \infty$.

Theorem 2. *Suppose that* $\lambda = 1$, *and (2) holds in addition to*

$$\limsup_{t\longrightarrow\infty} \int_{\mathrm{T}}^{t} \int_s^{\delta(s)} \left(\frac{1}{\wp_i(v)} \int_v^{\delta(v)} \varphi_i(w)\mathrm{d}w\right)^{1/\alpha_i} \mathrm{d}v\, \mathrm{d}s = \infty, \quad T \geq t_0, i = 1, 2, 3. \tag{34}$$

Then every bounded solution of system (1) oscillates.

Proof. Suppose that (1) has NOS $X(t)$, so by Lemma 1, from Table 3, there is only the class K_2, can occur for $t \geq t_1 \geq t_0$, that is,

$$y_i(t) > 0, y_i'(t) > 0, y_i''(t) < 0, \quad i = 1, 2, 3. \tag{35}$$

Table 2: The classes of all NOS of (1) with $\lambda = -\mathbf{1}$.

Classes	Sign of $y_i^{(j)}(t)$						Behavior when $t \longrightarrow \infty$
n	y_1'	y_2'	y_3'	y_1''	y_2''	y_3''	$y_i, i = 1,2,3$
L_1	$+$ $-$	$+$ $-$	$+$ $-$	$+$	$+$	$+$	$y_i \longrightarrow \infty$ $p_i(t)(y_i''(t))^{\alpha_i} \longrightarrow 0$
L_2	$+$	$+$	$+$	$-$	$-$	$-$	$y_i \longrightarrow \infty$
L_3	$+$ $+$	$+$ $+$	$+$ $-$	$-$	$-$	$+$	$y_i \longrightarrow \infty$ $y_j \longrightarrow \infty, j = 1,2,$ and $p_3(t)(y_3''(t))^{\alpha_3} \longrightarrow 0$
L_4	$+$ $+$	$+$ $-$	$+$ $+$	$-$	$+$	$-$	$y_i \longrightarrow \infty$ $y_j \longrightarrow \infty, j = 1,3,$ and $p_2(t)(y_2''(t))^{\alpha_2} \longrightarrow 0$
L_5	$+$ $-$	$+$ $+$	$+$ $+$	$+$	$-$	$-$	$y_i \longrightarrow \infty$ $y_j \longrightarrow \infty, j = 2,3,$ and $p_1(t)(y_1''(t))^{\alpha_1} \longrightarrow 0$
L_6	$+$ $-$	$+$ $-$	$+$ $+$	$+$	$+$	$-$	$y_i \longrightarrow \infty$ $y_3 \longrightarrow \infty\ p_{1,2}(t)(y_{1,2}''(t))^{\alpha_{1,2}} \longrightarrow 0$
L_7	$+$ $+$	$+$ $-$	$+$ $-$	$-$	$+$	$+$	$y_i \longrightarrow \infty$ $y_1 \longrightarrow \infty\ p_{2,3}(t)(y_{2,3}''(t))^{\alpha_{2,3}} \longrightarrow 0$
L_8	$+$ $-$	$+$ $+$	$+$ $-$	$+$	$-$	$+$	$y_i \longrightarrow \infty$ $y_2 \longrightarrow \infty\ p_{1,3}(t)(y_{1,3}''(t))^{\alpha_{1,3}} \longrightarrow 0$

Since $y_1(t), y_2(t), y_3(t)$, are increasing, so there exists $c_i > 0, i = 1,2,3$ and $t_2 \geq t_1$ such that $y_i(t) \geq c_i, t \geq t_2$. Integrating the first equation of system (1) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, leads to

$$p_1(\delta(t))\left(y_1''(\delta(t))\right)^{\alpha_1} - p_1(t)\left(y_1''(t)\right)^{\alpha_1} = \int_t^{\delta(t)} q_1(s) y_2^{\alpha_1}(\tau_1(s))\mathrm{d}s,$$
$$y_1''(t) \leq -c_2\left(\frac{1}{p_1(t)}\int_t^{\delta(t)} q_1(s)\mathrm{d}s\right)^{1/\alpha_1}. \tag{36}$$

Integrating (36) from t to $\delta(t)$ yields

$$y_1'(\delta(t)) - y_1'(t) \leq -c_2\int_t^{\delta(t)}\left(\frac{1}{p_1(s)}\int_s^{\delta(s)} q_1(v)\mathrm{d}v\right)^{1/\alpha_1}\mathrm{d}s. \tag{37}$$

Then,

$$y_1'(t) \geq c_2\int_t^{\delta(t)}\left(\frac{1}{p_1(s)}\int_s^{\delta(s)} q_1(v)\mathrm{d}v\right)^{1/\alpha_1}\mathrm{d}s. \tag{38}$$

Integrating the above inequality from t_2 to t, we get

$$y_1(t) - y_1(t_2) \geq c_2\int_{t_2}^{t}\int_s^{\delta(s)}\left(\frac{1}{p_1(v)}\int_v^{\delta(v)} q_1(w)\mathrm{d}w\right)^{1/\alpha_1}\mathrm{d}v\,\mathrm{d}s, \tag{39}$$

As $t \longrightarrow \infty$ concerning (34), it follows from (23) that $\mathrm{Lim}_{t\longrightarrow\infty} y_1(t) = \infty$, which is a contradiction. Similarly, it can be shown that $\mathrm{Lim}_{t\longrightarrow\infty} y_2(t) = \infty, \mathrm{Lim}_{t\longrightarrow\infty} y_3(t) = \infty$, which is a contradiction.

This leads to the solution $X(t)$ oscillates. □

Theorem 3. *Suppose $\lambda = -1$, (2) and (34) hold. Then every bounded solution of* (1) *oscillates or tends to zero as $t \longrightarrow \infty$.*

Proof. Suppose that system (1) has NOS $X(t)$ so by Lemma 1, Table 2, there is only the possible case $\mathbf{L}_1$ to consider for $t \geq t_1 \geq t_0$:

$$y_i(t) > 0, y_i'(t) < 0, y_i''(t) > 0, \quad i = 1,2,3. \tag{40}$$

Since $y_1(t), y_2(t), y_3(t)$, are positive and decreasing, so there exists $l_i \geq 0, i = 1,2,3$ such that $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = l_i$, we

claim that $l_i = 0$, otherwise $l_i > 0$ hence $y_i(t) \geq l_i > 0$ for $t \geq t_2 \geq t_1$.

Integrating the first equation of (1) from t to $\delta(t)$ yields:

$$\begin{aligned} p_1(\delta(t))\left(y_1''(\delta(t))\right)^{\alpha_1} - p_1(t)\left(y_1''(t)\right)^{\alpha_1} &= -\int_t^{\delta(t)} q_1(s) y_2^{\alpha_1}(\tau_1(s))\mathrm{d}s, \\ p_1(t)\left(y_1''(t)\right)^{\alpha_1} &\geq l_2^{\alpha_1}\int_t^{\delta(t)} q_1(s)\mathrm{d}s, \end{aligned} \tag{41}$$

$$y_1''(t) \geq l_2\left(\frac{1}{p_1(t)}\int_t^{\delta(t)} q_1(s)\mathrm{d}s\right)^{1/\alpha_1}. \tag{42}$$

Integrating (42) from t to $\delta(t)$, we get

$$y_1'(\delta(t)) - y_1'(t) \geq l_2\int_t^{\delta(t)}\left(\frac{1}{p_1(s)}\int_s^{\delta(s)} q_1(s)\mathrm{d}v\right)^{1/\alpha_1}\mathrm{d}s.$$

$$y_1'(t) \leq -l_2\int_t^{\delta(t)}\left(\frac{1}{p_1(s)}\int_s^{\delta(s)} q_1(s)\mathrm{d}v\right)^{1/\alpha_1}\mathrm{d}s. \tag{43}$$

Integrating (43) from t_2 to t, we get

$$y_1(t) - y_1(t_2) \leq -l_2\int_{t_2}^{t}\int_s^{\delta(s)}\left(\frac{1}{p_1(v)}\int_v^{\delta(v)} q_1(s)\mathrm{d}w\right)^{1/\alpha_1}\mathrm{d}v\,\mathrm{d}s. \tag{44}$$

As $t \longrightarrow \infty$ we get from (44) $\mathrm{Lim}_{t\longrightarrow\infty} y_1(t) = -\infty$, which is a contradiction. Similarly, $\mathrm{Lim}_{t\longrightarrow\infty} y_2(t) = -\infty, \mathrm{Lim}_{t\longrightarrow\infty} y_3(t) = -\infty$. Then $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = 0$, $i = 1, 2, 3$. □

Corollary 4. *Suppose that $\lambda = 1$, then (2) and (21) hold. Then every solution of system (1) is either oscillatory or $Lim_{t\longrightarrow\infty}|y_i(t)| = \infty, i = 1, 2, 3$.*

Proof. Suppose that system (1) has a nonoscillatory solution $X(t)$, let $y_i(t) > 0, t \geq t_0, i = 1, 2, 3$. So by Lemma 1 and Table 3, there are only the possible classes $K_1 - K_8$ to consider for $t \geq t_1 \geq t_0$. If $X(t)$ is bounded, then by Theorem 2, it follows that $X(t)$ is oscillatory. Otherwise, $X(t)$ is unbounded. □

Case 1. Suppose that $X(t) \in K_1$. By Lemma 1, it follows $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = \infty, i = 1, 2, 3$.

Case 2. Suppose that $X(t) \in K_2$. By Theorem 2, $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = \infty, i = 1, 2, 3..$

Case 3. Suppose that $X(t) \in K_3$. Since $y_1(t), y_2(t), y_3(t)$, are increasing, so there exists $c_i > 0, i = 1, 2, 3$ and $t_2 \geq t_1$ such that $y_i(t) \geq c_i, t \geq t_2$.

Integrating the first equation of system (1) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, leads to

$$\begin{aligned} p_1(\delta(t))\left(y_1''(\delta(t))\right)^{\alpha_1} - p_1(t)\left(y_1''(t)\right)^{\alpha_1} &= \int_t^{\delta(t)} q_1(s) y_2^{\alpha_1}(\tau_1(s))\mathrm{d}s, \\ y_1''(t) &\leq -c_2\left(\frac{1}{p_1(t)}\int_t^{\delta(t)} q_1(s)\mathrm{d}s\right)^{1/\alpha_1}. \end{aligned} \tag{45}$$

Integrating the (45) from t to $\delta(t)$, yields

$$y_1'(\delta(t)) - y_1'(t) \leq -c_2\int_t^{\delta(t)}\left(\frac{1}{p_1(s)}\int_s^{\delta(s)} q_1(v)\mathrm{d}v\right)^{1/\alpha_1}\mathrm{d}s. \tag{46}$$

Table 3: The classes of all possible NOS of system (1) with $\lambda = 1$.

Classes	Sign of $y_i^{(j)}$						Behavior as $\mathbf{t} \longrightarrow \infty$
n	y_1'	y_2'	y_3'	y_1''	y_2''	y_3''	y_i
K_1	+	+	+	+	+	+	$y_i \longrightarrow \infty$
K_2	+	+	+	−	−	−	$p_i(y_i''(t))^{\alpha_i} \longrightarrow 0$ $y_i \longrightarrow \infty$
K_3	+	+	+	−	−	+	$p_j(y_j''(t))^{\alpha_j} \longrightarrow 0,\ j = 1,2$ $y_3 \longrightarrow \infty$
K_4	+	+	+	−	+	−	$y_2 \longrightarrow \infty,\ p_j(y_j''(t))^{\alpha_j} \longrightarrow 0,\ j = 1,3$
K_5	+	+	+	+	−	−	$y_1 \longrightarrow \infty,\ p_j(y_j''(t))^{\alpha_j} \longrightarrow 0,\ j = 2,3$
K_6	+	+	+	+	+	−	$y_{1,2} \longrightarrow \infty$ $p_3(y_3''(t))^{\alpha_3} \longrightarrow 0,$
K_7	+	+	+	−	+	+	$y_{2,3} \longrightarrow \infty,\ p_1(y_1''(t))^{\alpha_1} \longrightarrow 0,$
K_8	+	+	+	+	−	+	$y_{1,3} \longrightarrow \infty,\ p_2(y_2''(t))^{\alpha_2} \longrightarrow 0,$

Then,

$$y_1'(t) \geq c_2 \int_t^{\delta(t)} \left(\frac{1}{p_1(s)} \int_s^{\delta(s)} q_1(v)\mathrm{d}v \right)^{1/\alpha_1} \mathrm{d}s. \tag{47}$$

Integrating the last inequality from t_2 to t, we get

$$y_1(t) - y_1(t_2) \geq c_2 \int_{t_2}^{t} \int_s^{\delta(s)} \left(\frac{1}{p_1(v)} \int_v^{\delta(v)} q_1(w)\mathrm{d}w \right)^{1/\alpha_1} \mathrm{d}v\,\mathrm{d}s, \tag{48}$$

As $t \longrightarrow \infty$ concerning (34), it follows from (30) that $\mathrm{Lim}_{t\longrightarrow\infty} y_1(t) = \infty$.

Now, similarly, it can be shown that $\mathrm{Lim}_{t\longrightarrow\infty} y_2(t) = \infty$, by Lemma 2.2, $\lim_{t\longrightarrow\infty} y_3(t) = \infty$.

Other cases can be handled in the same way**.** The proof is complete.

Corollary 5. *Suppose that $\lambda = -1, i = 1,2,3,$ and (2), (40) are held. Then every solution of system (1) is either oscillatory or converges to zero or tends to infinity as $t \longrightarrow \infty$.*

Proof. Suppose that system (1) has a NOS $X(t)$ so by Lemma 2 Table 2, there are only the possible cases $L_1 - L_8$ to consider for $t \geq t_1 \geq t_0$. If $X(t)$ is bounded, then by Theorem 2, it follows that $X(t)$ is either oscillatory or $X(t) \longrightarrow 0$ as $t \longrightarrow \infty$. If $X(t)$ is unbounded, then from Table 2, we conclude that $\mathrm{Lim}_{t\longrightarrow\infty} y_i(t) = \infty, i = 1,2,3.$ □

5. Examples

In this section, some examples illustrate the obtained results of the system (1).

Example 1. Consider the delay system as follows:

$$\begin{cases} y_1'''(t) = \dfrac{1}{2} y_2(t - 3\pi), \\ y_2'''(t) = \dfrac{1}{4} y_3(t - \pi) \quad t \geq 0, \\ y_3'''(t) = \dfrac{1}{8} y_1(t - 2\pi), \end{cases} \tag{49}$$

$$\alpha_i = 1, p_i(t) = 1, i = 1,2,3, \tau_1(t) = t - 3\pi, \tau_2(t) = t - \pi, \tau_3(t) = t - 2\pi,$$

$$q_1(t) = \frac{1}{2}, q_2(t) = \frac{1}{4}, q_3(t) = \frac{1}{4}, \mathrm{let}\delta(t) = 2t.$$

Obviously, to see that

$$\int_T^{\infty}\left(\frac{1}{\wp_i(t)}\right)^{1/\alpha_i}\mathrm{d}t=\int_T^{\infty}\mathrm{d}t=\infty, T\geq 0,$$

$$\begin{aligned}
&\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\wp_1(v)}\int_v^{\delta(v)}\mathcal{q}_1(w)\mathrm{d}w\right)^{1/\alpha_1}\mathrm{d}v\,\mathrm{d}s=\frac{1}{2}\operatorname*{Lim}_{t\longrightarrow\infty}\int_T^t\int_s^{2s}\int_v^{2v}\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}s=\infty,\\
&\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\wp_2(v)}\int_v^{\delta(v)}\mathcal{q}_2(w)\mathrm{d}w\right)^{1/\alpha_2}\mathrm{d}v\,\mathrm{d}s=\frac{1}{4}\operatorname*{Lim}_{t\longrightarrow\infty}\int_T^t\int_s^{2s}\int_v^{2v}\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}s=\infty,\\
&\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\wp_3(v)}\int_v^{\delta(v)}\mathcal{q}_3(w)\mathrm{d}w\right)^{1/\alpha_2}\mathrm{d}v\,\mathrm{d}s=\frac{1}{8}\operatorname*{Lim}_{t\longrightarrow\infty}\int_T^t\int_s^{2s}\int_v^{2v}\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}s=\infty.
\end{aligned}\tag{50}$$

Hence all conditions of Theorem 2 are satisfied, so according to Theorem 2, every bounded solution of system (49) is oscillatory. For instance, $(1/2\sin t/2, 1/4\cos t/2, 1/4\sin t/2)^{\mathrm{T}}$, has an oscillatory solution, as shown in Figure 1.

Example 2. Consider the delay system as follows:

$$\begin{cases}
\left[e^{-t/2}\left(y_1''(t)\right)^{1/3}\right]'=-\dfrac{11}{18\sqrt[3]{9}}e^{-t/2-2/9}{y_2}^{1/3}(t-2),\\
\left[e^{-t/3}\left(y_2''(t)\right)^{1/5}\right]'=-\dfrac{2}{5\sqrt[5]{9}}e^{-3/10t-3/10}y_3^{1/5}(t-3)\quad t\geq 0,\\
\left[e^{-t}\left(y_3''(t)\right)^{3/5}\right]'=-\dfrac{13}{10\sqrt[5]{64}}e^{-11/10t-1/5}y_1^{3/5}(t-1),
\end{cases}\tag{51}$$

$$\wp_1(t)=e^{-t/2},\wp_2(t)=e^{-t/3},\wp_3(t)=e^{-t},\tau_1(t)=t-2,\tau_2(t)=t-3,\tau_3(t)=t-1,$$

$$\mathcal{q}_1(t)=-\frac{11}{18\sqrt[3]{9}}e^{-t/2-2/9},\mathcal{q}_2(t)=-\frac{2}{5\sqrt[5]{9}}e^{-3/10t-3/10},\mathcal{q}_3(t)=-\frac{13}{10\sqrt[5]{64}}e^{-11/10t-1/5},$$

$$\alpha_1=\frac{1}{3},\alpha_2=\frac{1}{5},\alpha_3=\frac{3}{5}.$$

It is clear that

$$\int_T^{\infty}\left(\frac{1}{\wp_1(t)}\right)^3\mathrm{d}t=\int_T^{\infty}e^{t/6}\mathrm{d}t=\infty,\int_T^{\infty}\left(\frac{1}{\wp_2(t)}\right)^5\mathrm{d}t=\int_T^{\infty}e^{t/15}\mathrm{d}t=\infty,$$

$$\int_T^{\infty}\left(\frac{1}{\wp_3(t)}\right)^{5/3}\mathrm{d}t=\int_T^{\infty}e^{3t/5}\mathrm{d}t=\infty, T\geq 0.$$

$$\begin{aligned}
&\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\wp_1(v)}\int_v^{\delta(v)}\mathcal{q}_1(w)dw\right)^{1/\alpha_1}\mathrm{d}v\,\mathrm{d}s=\frac{11}{18\sqrt[3]{9}e^{(/)}}\operatorname*{Lim}_{t\longrightarrow\infty}\int_T^t\int_s^{2s}\left(e^{v/2}\int_v^{2v}e^{-1/2w}\mathrm{d}w\right)^3\mathrm{d}v\,\mathrm{d}s=\infty,\\
&\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\wp_2(v)}\int_v^{\delta(v)}\mathcal{q}_2(w)\mathrm{d}w\right)^{1/\alpha_2}\mathrm{d}v\,\mathrm{d}s=\infty,\\
&\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{\wp_3(v)}\int_v^{\delta(v)}\mathcal{q}_3(w)\mathrm{d}w\right)^{1/\alpha_3}\mathrm{d}v\,\mathrm{d}s=\infty.
\end{aligned}\tag{52}$$

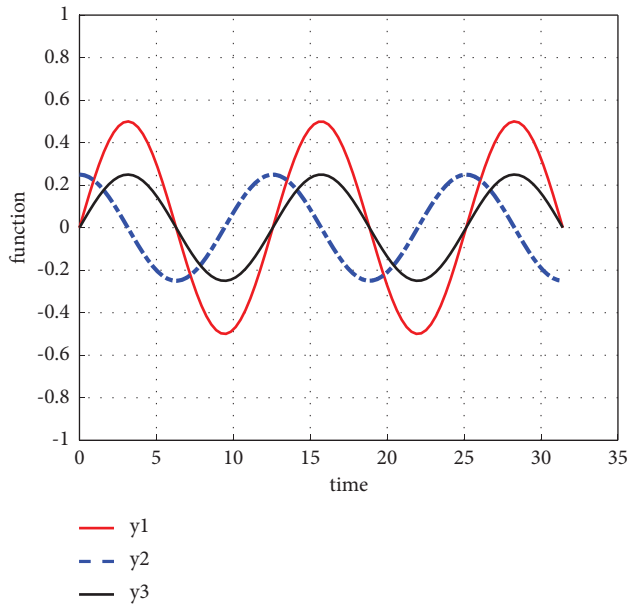

FIGURE 1: The solution of $(1/2 \sin t/2, 1/4 \cos t/2, 1/4 \sin t/2)^T$ of system (49).

Hence all conditions of Theorem 3 satisfies, so according to Theorem 3, every bounded solution of (1) oscillates or tends to zero as $t \longrightarrow \infty$. The solution $(e^{-t/3}, e^{-t/3}, e^{-t/2})^{\mathrm{T}}$ has an nonoscillatory solution tends to zero as $t \longrightarrow \infty$, as shown in Figure 2.

Example 3. Consider the delay system as follows:

$$\begin{cases} \left[\left(y_1''(t)\right)^{3/5}\right]' = \dfrac{3}{5\sqrt[5]{8}} e^{3/5} y_2{}^{3/5}(t-1), \\ \left[e^{-t/3}\left(y_2''(t)\right)\right]' = \dfrac{2}{3} e^{-1/3t} y_3(t-2) \quad t \geq 0, \\ \left[\left(y_3''(t)\right)^{1/3}\right]' = \dfrac{\sqrt[3]{3}}{3} e y_1^{1/3}(t-3), \end{cases} \tag{53}$$

$$p_1(t) = 1, p_2(t) = e^{-t/3}, p_3(t) = 1, \tau_1(t) = t-1, \tau_2(t) = t-2, \tau_3(t) = t-3,$$

$$q_1(t) = \frac{3}{5\sqrt[5]{8}} e^{3/5}, q_2(t) = \frac{2}{3} e^{-1/3t}, q_3(t) = \frac{\sqrt[3]{3}}{3} e,$$

$$\alpha_1 = \frac{3}{5}, \alpha_2 = 1, \alpha_3 = \frac{1}{3}.$$

It is clear that

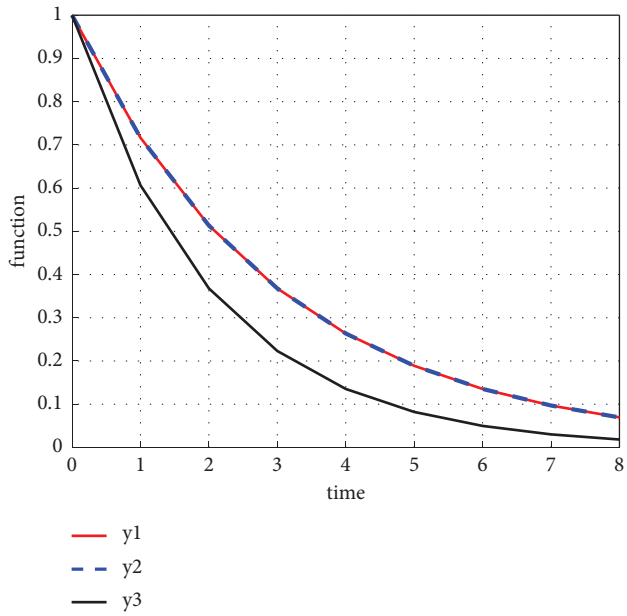

Figure 2: The solution of $(e^{-t/3}, e^{-t/3}, e^{-t/2})^T$ of system (51).

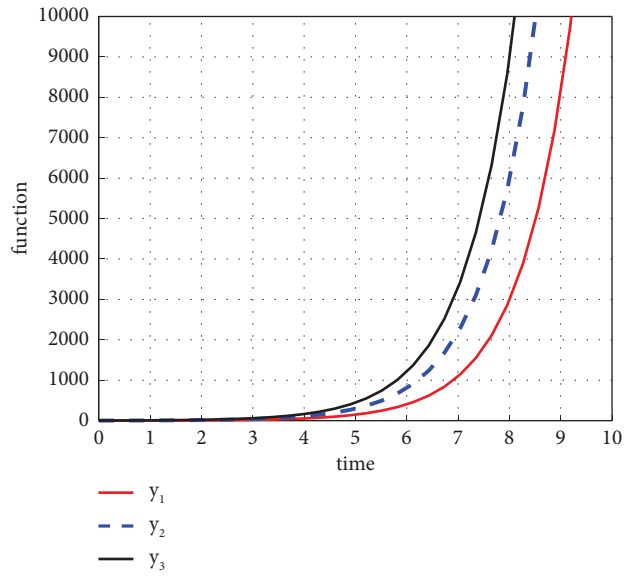

Figure 3: The solution of $(e^t, 2e^t, 3e^t)^T$ of system (53).

$$\int_0^{\infty}\left(\frac{1}{p_1(t)}\right)^{5/3}\mathrm{d}t=\int_0^{\infty}\left(\frac{1}{p_3(t)}\right)^{3}\mathrm{d}t=\int_0^{\infty}\mathrm{d}t=\infty,\int_0^{\infty}\left(\frac{1}{p_2(t)}\right)^{5}\mathrm{d}t=\int_0^{\infty}e^{t/3}\mathrm{d}t=\infty,$$

$$\operatorname*{Limsup}_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{p_1(v)}\int_v^{\delta(v)}q_1(w)dw\right)^{1/\alpha_1}\mathrm{d}v\,\mathrm{d}s=\left(\frac{3e^{3/5}}{5\sqrt[5]{8}}\right)^{5/3}\operatorname*{Lim}_{t\longrightarrow\infty}\int_T^t\int_s^{2s}\left(\int_v^{2v}\mathrm{d}w\,\mathrm{d}w\right)^{5/3}\mathrm{d}v\,\mathrm{d}s=\infty,$$

$$\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{p_2(v)}\int_v^{\delta(v)}q_2(w)\mathrm{d}w\right)^{1/\alpha_2}\mathrm{d}v\,\mathrm{d}s=\infty,$$

$$\limsup_{t\longrightarrow\infty}\int_T^t\int_s^{\delta(s)}\left(\frac{1}{p_3(v)}\int_v^{\delta(v)}q_3(w)\mathrm{d}w\right)^{1/\alpha_3}\mathrm{d}v\,\mathrm{d}s=\infty. \tag{54}$$

Hence all conditions of Corollary 5 satisfy, so according to Corollary 5, every solution of (1) oscillates or tends to zero or tends to infinity as $t\longrightarrow\infty$. The nonoscillatory solution $(e^t,2e^t,3e^t)^{\mathrm{T}}$ rose to infinity as $t\longrightarrow\infty$. as shown in Figure 3.

6. Conclusions

(i) Knowing and calculating all possible cases of positive solutions of the third-order three-dimensional half-linear system with delay equations.

(ii) Oscillation: this trend revolves around studying and obtaining the necessary and sufficient conditions for obtaining the oscillation of positive solutions for a three-dimensional half-linear system with delay equations of the third order.

(iii) Asymptotic behavior: the required sufficient conditions were drawn in this direction to obtain the convergence to zero or divergence of all NOS of the half-linear systems of DDEs in the third order when $t\longrightarrow\infty$. All the obtained results are included with illustrative examples.

Data Availability

The data used in this study are available upon reasonable request to the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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